Lecture 5: Introduction to Asymptotics

Motivation: Many problems are nonlinear and need new techniques to solve them. For weakly nonlinear systems, perturbation theory is a powerful tool to study such problems.

The first nonlinear problems ever considered was solving polynomial equations.

Example:

Let $0 < \varepsilon \ll 1$. Solve the equation

$$x^2 - 2x + \varepsilon = 0$$

Idea 1:

We can solve exactly:

$$x = \frac{2 \pm \sqrt{4 - 4\varepsilon}}{2} = 1 \pm \sqrt{1-\varepsilon}$$

Taylor expansion of $\sqrt{1-\varepsilon}$

$$\Rightarrow x = 1 \pm \left(1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \ldots\right)$$

$$\Rightarrow x = 2 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \ldots, \quad x = \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2 + \ldots$$

Idea 2:

What if we did not know the solution beforehand? Assume a power series of the form:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

$$\Rightarrow (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^2 - 2(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots) + \varepsilon = 0$$

$$\Rightarrow x_0^2 + 2x_0 x_1 \varepsilon + (x_1^2 + 2x_0 x_2)\varepsilon^2 - 2x_0 - 2\varepsilon x_1 - 2\varepsilon^2 x_2 + \ldots + \varepsilon = 0$$

$$\Rightarrow x_0^2 - 2x_0 + (2x_0 x_1 - 2x_1 + 1)\varepsilon + (x_1^2 + 2x_0 x_2 - x_2)\varepsilon^2 = 0$$

Solve term by term:

ε^0 terms:

$$x_0^2 - 2x_0 = 0$$

$$\Rightarrow x_0 = 0, 2$$

ε^1 terms:

$$2x_1x_0 - 2x_1 + 1 = 0$$

If $x_0 = 0 \Rightarrow x_1 = 1/2$

If $x_0 = 2 \Rightarrow x_1 = -1/2$

ε^2 terms:

$$x_1^2 + 2x_0x_2 - x_2 = 0$$

If $x_0 = 0,\ x_1 = 1/2 \Rightarrow x_2 = 1/8$

If $x_0 = 2,\ x_1 = -1/2 \Rightarrow x_2 = -1/8$

We get back the correct series expansion

$$x = \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2 + \ldots,\quad 2 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \ldots$$

Remark:

This worked because the solution could be expressed as a power series in ε. This is an example of a regular perturbation problem.

Example:

For $0<\varepsilon<<1$, solve the equation

$$\varepsilon x^3 - x + 1 = 0$$

Idea #1

Guess $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

$$\Rightarrow \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^3 - x_0 - \varepsilon x_1 - \varepsilon^2 x_2 + \dots + 1 = 0$$

$$\Rightarrow \varepsilon x_0^3 + 3\varepsilon^2 x_0^2 x_1 + \dots - x_0 - \varepsilon x_1 - \varepsilon^2 x_2 + \dots + 1 = 0$$

ε^0:

$1 - x_0 = 0 \Rightarrow x_0 = 1$

ε^1:

$x_0^3 - x_1 = 0 \Rightarrow x_1 = 1$

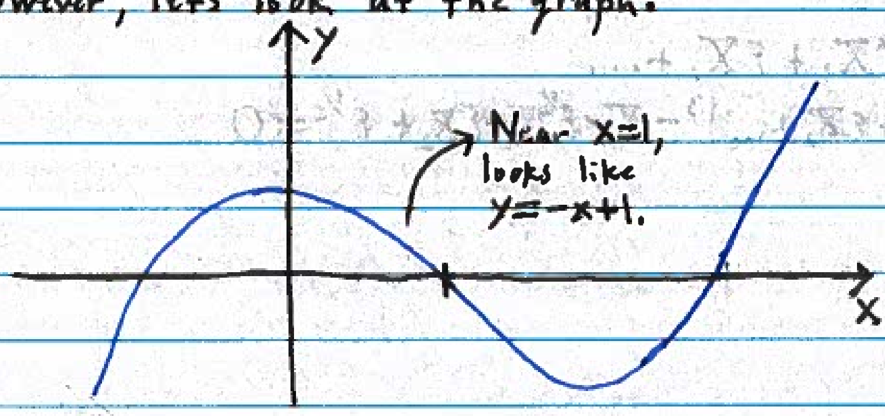
ε^2:

$3x_0 x_1 - x_2 = 0 \Rightarrow x_2 = 3$

The approximation for this solution is

$$x = 1 + \varepsilon + 3\varepsilon^2 + \dots$$

However, lets look at the graph:



How did we lose the other roots? We implicitely assumed that εx^3 is small near the roots. What if in fact εx^3 is large?

Idea #2

Lets rescale $x = \varepsilon^{\alpha} X$

$\Rightarrow \varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$

The key idea is that in order for a solution to exist in a series form dominant terms must "balance"

Possible Balances:

1. $\varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$

$\alpha = 0 \Rightarrow 1 + 3\alpha > 0$

$\Rightarrow -X + 1 = 0$ (already discussed)

2. $\varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$

$1 + 3\alpha = 0 \Rightarrow \alpha = -1/3$

$\Rightarrow X^3 - \underbrace{\varepsilon^{-1/3}}_{\text{largest term}} X + 1 = 0$

largest term so this can't be useful.

3. $\varepsilon^{1+3\alpha} X^3 - \varepsilon^{\alpha} X + 1 = 0$

$1 + 3\alpha = \alpha \Rightarrow \alpha = -1/2.$

$\Rightarrow \varepsilon^{-1/2} X^3 - \varepsilon^{-1/2} X + 1 = 0$

$\Rightarrow X^3 - X + \varepsilon^{1/2} = 0$

This is a correct rescaling as now there are two dominant terms.

Try

$X = X_0 + \varepsilon^{1/2} X_1 + \varepsilon X_2 + \ldots$

$\Rightarrow (X_0 + \varepsilon^{1/2} X_1 + \varepsilon X_2 + \ldots)^3 - X_0 - \varepsilon^{1/2} X_1 - \varepsilon X_2 + \varepsilon^{1/2} = 0$

ε^0 terms:

$X_0^3 - X_0 = 0$

$\Rightarrow X_0 = 0, \pm 1$

$\varepsilon^{1/2}$ terms:

$3 X_0^2 X_1 - X_1 + 1 = 0$

If $X_0 = 0 \Rightarrow X_1 = 1$.

If $X_0 = 1 \Rightarrow X_1 = -\frac{1}{2}$

If $X_0 = -1 \Rightarrow X_1 = \frac{1}{4}$

$$\Rightarrow X = \varepsilon^{1/2} + \dots,\ 1 - \tfrac{1}{2}\varepsilon^{-1/2} + \dots,\ -1 + \tfrac{1}{4}\varepsilon^{-1/2} + \dots$$

$$\Rightarrow x = \varepsilon^{\alpha} X = \varepsilon^{-1/2} X = 1 + \dots,\ \varepsilon^{-1/2} - \tfrac{1}{2} + \dots,\ -\varepsilon^{-1/2} + \tfrac{1}{4} + \dots$$

That is the solutions are approximated by:

$$x \approx \underbrace{1 + \varepsilon + 3\varepsilon^2}_{\text{Taylor series}},\ \underbrace{\varepsilon^{-1/2} - \tfrac{1}{2},\ \varepsilon^{-1/2} + \tfrac{1}{4}}_{\text{Something else.}}$$

<u>Remark:</u> This calculation was formal and is an example of a singular perturbation problem. Can we generalize and rigorously justify this process?

Order Symbols:

Suppose we want to approximate $1/1-\varepsilon$ for $0 \le \varepsilon \ll 1$.

We can expand in powers of ε:

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon + \frac{\varepsilon^2}{2} + \dots$$

An approximation is then given by

$$\frac{1}{1-\varepsilon} \approx 1 + \varepsilon$$

The error in this approximation is bounded by

$$E = \left| \underbrace{\frac{1}{1-\varepsilon}}_{\text{Exact}} - \underbrace{(1+\varepsilon)}_{\text{Approximation}} \right| \le \underbrace{C\varepsilon^2}_{\text{Scaling law in error.}} \quad (0 < \varepsilon < \tfrac{1}{2})$$

The error in this approximation "scales like ε^2" and this result follows from Taylor's theorem.

Theorem - Let $f(\varepsilon)$ be a function that is $(n+1)$-times differentiable for $\varepsilon_1 < \varepsilon < \varepsilon_2$. If $\varepsilon_0, \varepsilon \in (\varepsilon_1, \varepsilon_2)$ then there exists $\xi \in (\varepsilon_1, \varepsilon_2)$ such that

$$f(\varepsilon) = f(\varepsilon_0) + f'(\varepsilon_0)(\varepsilon - \varepsilon_0) + \frac{1}{2} f''(\varepsilon_0)(\varepsilon - \varepsilon_0)^2 + \dots + \frac{1}{n!} f^{(n)}(\varepsilon_0)(\varepsilon - \varepsilon_0)^n + \frac{1}{(n+1)!} f^{(n+1)}(\xi)(\varepsilon - \varepsilon_0)^{n+1}.$$

We need to define what is meant by "scaling".

Definition - $f = O(\phi)$ as $\varepsilon \to \varepsilon_0$ means there exists $C, M > 0$ so that $|\varepsilon - \varepsilon_0| < M$ implies

$$|f(\varepsilon)| \le C |\phi(\varepsilon)|.$$

We say "f is big Oh of ϕ" as $\varepsilon \to \varepsilon_0$.

Example:

1. $\sin(\varepsilon) - \varepsilon = O(\varepsilon^2)$, as $\varepsilon \to 0$,
2. $\sin(\varepsilon) - \varepsilon \ne O(\varepsilon^3)$, as $\varepsilon \to 0$,
3. $\sin(\varepsilon) - \varepsilon = O(1)$, as $\varepsilon \to 0$.

Definition - $f = o(\phi)$ as $\varepsilon \to \varepsilon_0$ means for all $\delta > 0$ there exists $M(\delta)$ such that $|\varepsilon - \varepsilon_0| < M(\delta)$ implies

$$|f(\varepsilon)| \le \delta |\phi(\varepsilon)|.$$

We say "f is little Oh of ϕ" as $\varepsilon \to \varepsilon_0$.

Example:

1. $\sin(\varepsilon) - \varepsilon \ne o(\varepsilon^2)$
2. $\sin(\varepsilon) - \varepsilon \ne o(\varepsilon^3)$
3. $\sin(\varepsilon) - \varepsilon = o(\varepsilon)$
4. $\sin(\varepsilon) - \varepsilon = o(1)$

Remark:

1. Big Oh means "roughly the same size as" near ε_0.
2. Little oh means "much smaller than" near ε_0.

Theorem - If

$$\lim_{\varepsilon\to\varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = L,$$

then $f = \mathcal{O}(\phi)$ as $\varepsilon \to \varepsilon_0$

proof

If $\lim_{\varepsilon\to\varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = L$ then for all $\eta > 0$, there exists $\delta(\eta)$ such that $|\varepsilon - \varepsilon_0| < \delta(\eta)$ implies $|f(\varepsilon)/\phi(\varepsilon) - L| \leq \eta$. Therefore,

$$-\eta \leq f(\varepsilon)/\phi(\varepsilon) - L \leq \eta$$

$$\Rightarrow (L-\eta)\phi(\varepsilon) \leq f(\varepsilon) \leq (L+\eta)\phi(\varepsilon).$$

Setting $\eta = |L/2|$, then $|\varepsilon - \varepsilon_0| < \delta(|L/2|)$ implies

$$|f(\varepsilon)| \leq \max\{|L + |L/2||, |L - |L/2||\}\phi(\varepsilon).$$

$$= C\phi(\varepsilon).$$

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Theorem - If

$$\lim_{\varepsilon\to\varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = 0$$

then $f = o(\phi)$ as $\varepsilon \to \varepsilon_0$.

proof:

If $\lim_{\varepsilon\to\varepsilon_0} f(\varepsilon)/\phi(\varepsilon) = 0$ then for all $\delta > 0$ there exists $M(\delta)$ such that $|\varepsilon - \varepsilon_0| < M(\delta)$ implies

$$|f(\varepsilon)/\phi(\varepsilon)| \leq \delta$$

$$\Rightarrow |f(\varepsilon)| \leq \delta|\phi(\varepsilon)|.$$

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Notation-

1. $f \ll \phi$ means $f = o(\phi)$
2. $\varepsilon \ll 1$ means $\varepsilon \to 0$.

Properties:

1. $f = O(1) \Leftrightarrow f$ is bounded as $\varepsilon \to \varepsilon_0$
2. $f = o(1) \Leftrightarrow f \to 0$ as $\varepsilon \to \varepsilon_0$
3. $f = o(\phi) \Rightarrow f = O(\phi)$

Definition- We say $f = O_s(\varepsilon^\alpha)$ if $f = O(\varepsilon^\alpha)$ but $f \neq o(\varepsilon^\alpha)$. We say "$f$ is strictly order ε^α".

Example:

1. $\sin(\varepsilon) = O_s(\varepsilon)$, as $\varepsilon \to 0$ since

$$\lim_{\varepsilon \to 0} \frac{\sin(\varepsilon)}{\varepsilon} = 1 \neq 0.$$

2. What about $e^{-1/\varepsilon}$ as $\varepsilon \to 0$?

$$\lim_{\varepsilon \to 0} \frac{e^{-1/\varepsilon}}{\varepsilon^N} = \lim_{x \to \infty} \frac{e^{-x}}{(\frac{1}{x})^N} = \lim_{x \to \infty} \frac{x^N}{e^x} = 0 \quad \text{(By L'Hospital's rule)}$$

Therefore, it can be shown

$$e^{-1/\varepsilon} = o(\varepsilon^N),$$

i.e. $e^{-1/\varepsilon}$ is smaller than any power of ε. This is the definition of a transcendentally small terms.

Asymptotic Expansions

Definition - Given $f(\varepsilon)$ and $\phi(\varepsilon)$ we say $\phi(\varepsilon)$ is an asymptotic approximation if

$$f(\varepsilon) - \phi(\varepsilon) = o(\phi)$$

In this case we write $f \sim \phi$.

* Note the above definition means the error goes to zero faster than the approximation, i.e.,

$$\frac{|f(\varepsilon) - \phi(\varepsilon)|}{|\phi(\varepsilon)|} \ll 1.$$

Example:

1. $\frac{1}{1-\varepsilon} \sim 1 + \varepsilon$
2. $\frac{1}{1-\varepsilon}$ is not asymptotic to $1 + 2\varepsilon$.
3. $\sin(\varepsilon) \sim \varepsilon$
4. $\sin(\varepsilon) \sim \varepsilon + \sqrt{50}\,\varepsilon^2$ (This is a problem)

Remark: Asymptotic approximations are not unique and give little information about accuracy.

Definition - The functions $\phi_1, \phi_2, \ldots$ form an asymptotic sequence or are well ordered if and only if $\phi_{n+1} = o(\phi_n)$. The functions ϕ_n are called gauge functions.

Definition - If $\phi_1(\varepsilon), \phi_2(\varepsilon), \ldots$ is an asymptotic sequence then f has an asymptotic expansion to n terms if and only if

$$f - \sum_{k=1}^{m} a_k \phi_k = o(\phi_m)$$

In this case we write

$$f \sim a_1 \phi_1(\varepsilon) + \ldots + a_m \phi_m(\varepsilon).$$

Remark: Note the coefficients can be determined by

$$a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\phi_1(\varepsilon)},$$

$$a_2 = \lim_{\varepsilon \to 0} \frac{f - a_1 \phi_1(\varepsilon)}{\phi_2(\varepsilon)}.$$

Examples:

1. $1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$ is an asymptotic sequence
2. $\varepsilon^{1/2}, \varepsilon^{6/2}, \varepsilon^2, \ldots$ is an asymptotic sequence
3. $\varepsilon, \varepsilon^2 \sin(t), \varepsilon^3, \ldots$ is an asymptotic sequence.

Example:

1. $\frac{\sin(\varepsilon)}{\varepsilon^{3/2}} = \frac{\varepsilon - \frac{1}{6}\varepsilon^3 + 1/5!\, \varepsilon^5 + \ldots}{\varepsilon^{3/2}} = \varepsilon^{-1/2} - \frac{1}{6}\varepsilon^{3/2} + 1/5!\, \varepsilon^{7/2} + \ldots$

$$\frac{\sin(\varepsilon)}{\varepsilon^{3/2}} \sim \varepsilon^{-1/2} - 1/6\, \varepsilon^{3/2} + 1/5!\, \varepsilon^{7/2} + \ldots$$

2. $\frac{\sqrt{1+\varepsilon}}{\sin(\varepsilon^{1/2})} = \frac{1 + 1/2\, \varepsilon - 1/8\, \varepsilon^2 + \ldots}{\varepsilon^{1/2} - 1/6\, \varepsilon^{3/2} + 1/5!\, \varepsilon^{5/2} + \ldots} = \frac{1 + \frac{1}{2}\varepsilon - 1/8\, \varepsilon^2 + \ldots}{\varepsilon^{1/2}(1 - 1/6\, \varepsilon^1 + 1/5!\, \varepsilon^3 + \ldots)}$

$$\Rightarrow \frac{\sqrt{1+\varepsilon}}{\sin(\varepsilon^{1/2})} = \varepsilon^{-1/2}(1 + \tfrac{1}{2}\varepsilon - 1/8\, \varepsilon^2 + \ldots)(1 + (1/6\, \varepsilon^1 - 1/5!\, \varepsilon^3 + \ldots) + (1/6\, \varepsilon^1 - 1/5!\, \varepsilon^3 + \ldots)^2 + \ldots)$$

$$\Rightarrow \frac{\sqrt{1+\varepsilon}}{\sin(\varepsilon^{1/2})} \sim \varepsilon^{-1/2} + 2/3\, \varepsilon^{1/2}$$