Lecture #6: Examples of Regular Asymptotic Expansions

Example:

1. $f(\varepsilon) = \int_0^{\varepsilon} e^{-x^2}\,dx$ (let $u = x/\varepsilon$)

$$= \varepsilon \int_0^1 e^{-\varepsilon^2 u^2}\,du$$

$$= \varepsilon \int_0^1 \left(1 - \varepsilon^2 u^2 + \tfrac{1}{2}\varepsilon^4 u^4 + \dots\right) du$$

$$= \varepsilon - \tfrac{1}{3}\varepsilon^3 + \tfrac{1}{10}\varepsilon^5 + \dots$$

$$\Rightarrow f(\varepsilon) \sim \varepsilon - \tfrac{1}{3}\varepsilon^3 + \tfrac{1}{10}\varepsilon^5 + \dots$$

2. $g(\varepsilon) = \int_{1/\varepsilon}^{\infty} e^{-x^2}\,dx$ ($u = x^2$, $du = 2x\,dx = 2\sqrt{u}\,dx$)

$$= \int_{1/\varepsilon^2}^{\infty} \frac{1}{2u^{1/2}} e^{-u}\,du$$

$$= \frac{1}{2}\int_{1/\varepsilon^2}^{\infty} u^{-1/2} e^{-u}\,du$$

$$= \frac{1}{2}\left(-e^{-u}u^{-1/2}\Big|_{1/\varepsilon^2}^{\infty} - \frac{1}{2}\int_{1/\varepsilon^2}^{\infty} u^{-3/2} e^{-u}\,du \right)$$

$$= \frac{1}{2} e^{-1/\varepsilon^2}\varepsilon - \frac{1}{4}\int_{1/\varepsilon^2}^{\infty} u^{-3/2} e^{-u}\,du$$

$$= \frac{1}{2} e^{-1/\varepsilon^2}\cdot\varepsilon - \frac{1}{4}\left(-u^{-3/2}e^{-u}\Big|_{1/\varepsilon^2}^{\infty} - \frac{3}{2}\int_{1/\varepsilon^2}^{\infty} u^{-5/2} e^{-u}\,du \right)$$

$$= \frac{1}{2} e^{-1/\varepsilon^2}\varepsilon - \frac{1}{4} e^{-1/\varepsilon^2}\varepsilon^3 + \frac{3}{8}\int_{1/\varepsilon^2}^{\infty} u^{-5/2} e^{-u}\,du.$$

$$= \frac{\varepsilon}{2} e^{-1/\varepsilon^2}\left(1 - \frac{1}{2}\varepsilon^2 + \frac{3}{8}\varepsilon^4 + \dots\right)$$

$$\sim \frac{\varepsilon}{2} e^{-1/\varepsilon^2} \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^{2n}(2n-1)!!}{2^n}$$

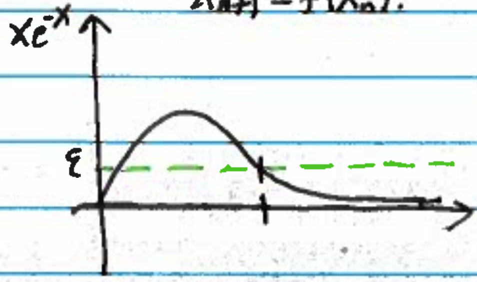
However, this series diverges for all ε!! However, truncating the series does yield a good approximation.

3. Find asymptotic approximations to the solutions of

$$xe^{-x} = \varepsilon$$
$$\Rightarrow x = \varepsilon e^{x} = f(x).$$

Essentially, we are looking for fixed points of the iterated map

$$x_{n+1} = f(x_n).$$



• First guess: $x_0 = 0$

$$\Rightarrow x_1 = f(x_0) = \varepsilon$$
$$\Rightarrow x_2 = f(x_1) = \varepsilon e^{\varepsilon} = \varepsilon\left(1+\varepsilon+\tfrac{1}{2}\varepsilon^2+\dots\right) = \varepsilon + \varepsilon^2 + \tfrac{1}{2}\varepsilon^3 + \dots$$
$$\Rightarrow x_3 = f(x_2) = \varepsilon e^{\varepsilon e^{\varepsilon}} = \varepsilon\left(1+\varepsilon e^{\varepsilon} + \tfrac{1}{2}\varepsilon^2 e^{2\varepsilon} + \dots\right)$$
$$= \varepsilon\left(1 + \varepsilon\left(1+\varepsilon+\tfrac{1}{2}\varepsilon^2+\dots\right) + \tfrac{1}{2}\varepsilon^2\left(1+2\varepsilon+\tfrac{1}{2}4\varepsilon^2+\dots\right)\right) + \dots$$
$$= \varepsilon + \varepsilon^2 + \tfrac{3}{2}\varepsilon^3 + \dots$$

Continuing we would get one root satisfies

$$x^* \sim \varepsilon + \varepsilon^2 + \tfrac{3}{2}\varepsilon^3 + \dots$$

We could also solve for x using logarithms:

$$\ln(x) - x = \ln(\varepsilon)$$
$$\Rightarrow x = \ln(x) - \ln(\varepsilon) = \ln(x) + \ln(1/\varepsilon)$$
$$\Rightarrow x_{n+1} = f(x_n) = \ln(x_n) + \ln(1/\varepsilon)$$

• Second guess: $x_0 = \ln(1/\varepsilon)$

$$\Rightarrow x_1 = \ln(1/\varepsilon) + \ln(\ln(1/\varepsilon))$$
$$\Rightarrow x_2 = \ln(1/\varepsilon) + \ln\left(\ln(1/\varepsilon) + \ln(\ln(1/\varepsilon))\right)$$
$$= \ln(1/\varepsilon) + \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\left(1 + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)}\right)\right)$$
$$= \ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right) + \ln\left(1 + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)}\right)$$

$$\Rightarrow x_2 \sim \ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right) + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)}.$$

Example:

$$m\frac{dv}{dt} = -av + bv^2, \quad v(0) = V_0$$

$$\Rightarrow \frac{dv}{dt} = -\tilde{a}v + \tilde{b}v^2,$$

where $\tilde{a} = a/m$, $\tilde{b} = b/m$. Through dimensional analysis we have:

$$[\tilde{a}] = \frac{1}{\text{time}}, \quad [\tilde{b}] = \frac{1}{\text{time}}\,\frac{1}{[v]}, \quad [V_0] = [v].$$

We can rescale by

$$\tau = at, \quad y = v/V_0.$$

In this case

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = a\frac{d}{d\tau}$$

$$\Rightarrow \frac{dv}{dt} = V_0\frac{dy}{dt} = aV_0\frac{dy}{d\tau} = -aV_0 y + bV_0^2 y^2$$

$$\Rightarrow \boxed{\begin{aligned} &\frac{dy}{d\tau} = y + \frac{bV_0}{a}y^2 = -y + \varepsilon y^2, \quad (\varepsilon = bV_0/a) \\ &y(0) = 1. \end{aligned}}$$

To solve we make a guess of the form

$$y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots$$

$$\Rightarrow \frac{dy_0}{d\tau} + \varepsilon\frac{dy_1}{d\tau} + \varepsilon^2\frac{dy_2}{d\tau} + \ldots = -y_0 - \varepsilon y_1 - \varepsilon^2 y_2 + \ldots + \varepsilon(y_0 + \varepsilon y_1 + \ldots)^2$$

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \ldots = 1$$

Gathering terms order by order, we have that

$\underline{O(1)}$:

$$\frac{dy_0}{d\tau} = -y_0$$

$$y_0(0) = 1$$

$$\Rightarrow y_0(\tau) = e^{-\tau}$$

$\underline{O(\varepsilon)}$:

$$\frac{dy_1}{d\tau} = -y_1 + \boxed{y_0^2}$$

nonlinear interaction term

$$y_1(0) = 0$$

$$\Rightarrow \frac{dy_1}{d\tau} = -y_1 + e^{-2\tau}$$

$$y_1(0) = 0$$

We know that y_1 is the sum of a homogeneous and particular solution

$$\Rightarrow y_1(\tau) = Ae^{-\tau} + \frac{1}{2}e^{-2\tau}$$

Initial conditions imply $A = -\frac{1}{2}$

$$\Rightarrow y_1(\tau) = -\frac{1}{2}e^{-\tau} + \frac{1}{2}e^{-2\tau}$$

$\underline{O(\varepsilon^2)}$:

$$\frac{dy_2}{d\tau} = -y_2 + \boxed{2y_0y_1}$$

Nonlinear interaction

$$y_2(0) = 0$$

$$\Rightarrow \frac{dy_2}{d\tau} = -y_2 + e^{-3\tau}$$

$$\Rightarrow y_2(\tau) = -\frac{1}{3}e^{-\tau} + \frac{1}{3}e^{-3\tau}$$

Putting it all together

$$y(\tau) = e^{-\tau} + \varepsilon\left(-\frac{1}{2}e^{-\tau} + \frac{1}{2}e^{-2\tau}\right) + \varepsilon^2\left(-\frac{1}{3}e^{-\tau} + \frac{1}{3}e^{-3\tau}\right) + \dots$$

Decaying on different time scales

The nonlinear terms give feedback terms that introduce small multiple time scale effects.

This is an example of a uniform asymptotic expansion since for all T and $t \in [0,T]$ there exists uniform bounds on the time dependent coefficients in the asymptotic expansion.