Lecture #7: Method of Multiple Scales

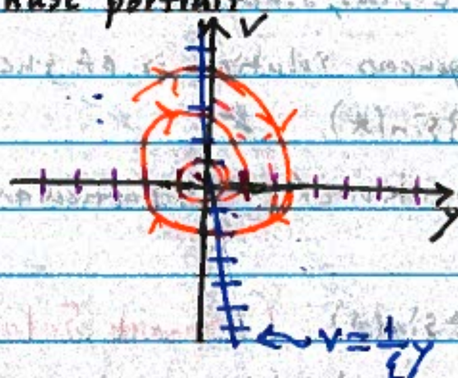
Example:

$$\ddot{y} + \varepsilon\dot{y} + y = 0$$
$$y(0) = 0$$
$$\dot{y}(0) = 1$$

We can draw a phase portrait

$$\dot{y} = v$$
$$\dot{v} = -\varepsilon v - y$$
$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix}$$

$\Rightarrow$ stable spirals



The exact solution is given by

$$y(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2} \sin\left(t\sqrt{1-\varepsilon^2/4}\right)$$

period $= 2\pi/\sqrt{1-\varepsilon^2/4} \sim 2\pi\left(1 - \varepsilon^2/8 + \ldots\right)$

Damping Scaling $= 1/\varepsilon$

Lets try a naive solution

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots$$

$O(1)$:

$$\ddot{y}_0 + y_0 = 0$$
$$y_0(0) = 0, \ \dot{y}_0(0) = 1$$
$$\Rightarrow y_0(t) = \sin(t)$$

$\underline{O(\varepsilon)}$:

$$\ddot{y}_1 + \dot{y}_0 + y = 0 \quad , \quad y_1(0) = 0$$
$$\Rightarrow \ddot{y}_1 + y_1 = -\cos(t) \quad \dot{y}_1(0) = 0$$

Let $\mathcal{L} = \frac{d^2}{dt^2} + 1$, we have

$$\mathcal{L} y_1 = -\cos(t).$$

Therefore,

$$\operatorname{Ker}(\mathcal{L}) = \operatorname{span}\{\cos(t), \sin(t)\}.$$

Consequently the homogeneous solution is of the form

$$y_1^H = A\cos(t) + B\sin(t).$$

However, since $-\cos(t) \in \operatorname{Ker}(\mathcal{L})$ the particular solution will be of the form

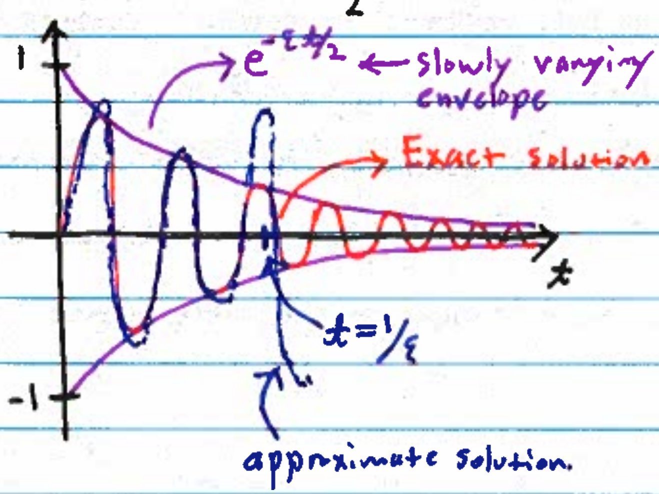
$$y_1^P = Ct\cos(t) + Dt\sin(t) \quad \text{(Resonant Solutions)}$$

Using the initial conditions and solving we have

$$y_1 = y_1^H + y_1^P = \frac{1}{2} t \sin(t)$$

Therefore, to two terms we have that

$$y \sim \sin(t) + \frac{1}{2}\varepsilon t \sin(t).$$



The asymptotic expansion works for fixed t and decreasing ε. However, what we want is a <u>uniform expansion</u> that works for abitrarily large times.

The key observation is that there is a second time scale εt that slightly distorts the intrinsic period from being resonant. A useful expansion has to account for both of these scales. A better ansatz assumes the solution depends on multiple scales:

$$t_1 = t,\ t_2 = \varepsilon t,\ t_3 = \varepsilon^2 t, \ldots$$

$$y(t,\varepsilon) = y(t_1, t_2, t_3, \ldots, \varepsilon)$$

Now,

$$\frac{d}{dt} = \frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1} + \frac{\partial t_2}{\partial t}\frac{\partial}{\partial t_2} + \frac{\partial t_3}{\partial t}\frac{\partial}{\partial t_3} + \ldots$$

$$= \frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2} + \varepsilon^2\frac{\partial}{\partial t_3} + \ldots$$

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2} + \varepsilon^2\frac{\partial}{\partial t_3} + \ldots\right)\left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2} + \varepsilon^2\frac{\partial}{\partial t_3} + \ldots\right)$$

$$= \frac{\partial^2}{\partial t_1^2} + \varepsilon\left(\frac{\partial^2}{\partial t_1 \partial t_2} + \frac{\partial^2}{\partial t_2 \partial t_1}\right) + \varepsilon^2\left(\frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial t_1 \partial t_3} + \frac{\partial^2}{\partial t_3 \partial t_1}\right) + \ldots$$

can't always assume mixed partials commute, but we will and see if things go wrong.

The ODE up to ε^2 becomes

$$\frac{\partial^2 y}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 y}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2 y}{\partial t_2^2} + 2\varepsilon^2\frac{\partial^2 y}{\partial t_1 \partial t_3} + \varepsilon\frac{\partial y}{\partial t_1} + \varepsilon^2\frac{\partial y}{\partial t_2} + y = 0$$

$$y(0) = 0$$

$$\dot{y}(0) = 1$$

We now expand

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots$$

$\underline{O(1):}$

$$\frac{\partial^2 y_0}{\partial t_1^2} + y_0 = 0\ , \quad y_0(0) = 0,$$

$$\Rightarrow y_0(t_1, t_2) = A(t_2)\sin(t_1)$$

$\underline{O(\varepsilon):}$

$$\frac{\partial^2 y_1}{\partial t_1^2} + 2\frac{\partial^2 y_0}{\partial t_1 \partial t_2} + \frac{\partial y_0}{\partial t_1} + y_1 = 0$$

$$\Rightarrow \frac{\partial^2 y_1}{\partial t_1^2} + 2\frac{\partial A}{\partial t_2}\cos(t_1) + A(t_2)\cos(t_1) + y_1 = 0$$

$$\Rightarrow \frac{\partial^2 y_1}{\partial t_1^2} + y_1 = -2\frac{\partial A}{\partial t_2}\cos(t_1) + A(t_2)\cos(t_1)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial t_1^2} + 1\right) y_1 = -2\frac{\partial A}{\partial t_2}\cos(t_1) + A(t_2)\cos(t_1)$$

To remove the secular terms we need

$$-2\frac{\partial A}{\partial t_2} + A(t_2) = 0$$

$$\Rightarrow A(t_2) = e^{-t_2/2}$$

Therefore, to lowest order

$$y(t) \sim e^{-\varepsilon/2\, t}\sin(t)$$

$\underline{\text{Remark:}}$

We derived an <u>amplitude equation</u> and a <u>wave equation</u>:

$$\frac{\partial A}{\partial t_2} = -\frac{1}{2}A(t_2) \quad \text{and} \quad \frac{\partial^2 y}{\partial t_1^2} + y_0 = 0$$

Example:

$$\ddot{y} + \varepsilon \dot{y}^3 + y = 0$$

Arbitrary initial conditions.

Naive Expansion

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$$

$O(1)$:

$$\ddot{y}_0 + y_0 = 0$$

$$\Rightarrow y_0 = \underbrace{A}_{\text{complex amplitude}} e^{it} + A^{*} e^{-it}.$$

$\xrightarrow{}$ complex conjugate

$O(\varepsilon)$:

$$\ddot{y}_1 + \dot{y}_0^3 + y_1 = 0$$

$$\Rightarrow \ddot{y}_1 + y_1 = -(A i e^{it} - A^{*} i e^{-it})^3$$

$$\Rightarrow \ddot{y}_1 + y_1 = -i(A e^{it} - A^{*} e^{-it})^3$$

$$\Rightarrow \ddot{y}_1 + y_1 = -i(A^3 e^{3it} \underbrace{- 3A^2 e^{it} + 3A(A^{*})^2 e^{-it}}_{\text{These terms are resonant}} - (A^{*})^3 e^{-3it})$$

Multiple Scale Expansion

Let $t_1 = t$, $t_2 = \varepsilon t$.

$$\Rightarrow \frac{d}{dt} = \frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1} + \frac{\partial t_2}{\partial t}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}$$

$$\Rightarrow \frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}\right)\left(\frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}\right)$$

$$= \frac{\partial^2}{\partial t_1^2} + 2\varepsilon \frac{\partial^2}{\partial t_1 \partial t_2} + \varepsilon^2 \frac{\partial^2}{\partial t_2^2}$$

Assume $y(t_1, t_2, \varepsilon) = y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \dots$

The ODE becomes the following PDE:

$$\frac{\partial^2 y}{\partial t_1^2} + 2\varepsilon \frac{\partial^2 y}{\partial t_1 \partial t_2} + \varepsilon^2 \frac{\partial^2 y}{\partial t_2^2} + \varepsilon\left(\frac{\partial y}{\partial t_1} + \varepsilon \frac{\partial y}{\partial t_2}\right)^3 + y = 0.$$

The order by order equations are given by

$\underline{O(1)}$:

$$\frac{\partial^2 y_0}{\partial t_1} + y_0 = 0$$

$$\Rightarrow y_0 = A(t_2)e^{it_1} + A^*(t_2)e^{-it_1}.$$

$\underline{O(\varepsilon)}$:

$$\frac{\partial^2 y_1}{\partial t_1^2} + 2\frac{\partial^2 y_0}{\partial t_1 \partial t_2} + \left(\frac{\partial y_0}{\partial t_1}\right)^3 + y_1 = 0$$

$$\Rightarrow \frac{\partial^2 y_1}{\partial t_1^2} + y_1 = -2i\left(\frac{\partial A}{\partial t_2}e^{it_1} - \frac{\partial A^*}{\partial t_2}e^{-it_1}\right) + \left(iAe^{it_1} - iA^*e^{-it_1}\right)^3$$

$$\Rightarrow \frac{\partial^2 y_1}{\partial t_1^2} + y_1 = -2i\left(\frac{\partial A}{\partial t_2}e^{it_1} - \frac{\partial A^*}{\partial t_2}e^{-it_1}\right) + iA^3e^{3it_1} - 3iA^2A^*e^{it_1}$$

$$+ 3iA(A^*)^2e^{-it_1} - i(A^*)^3e^{3it_1}.$$

To remove resonant terms we need

$$2i\frac{\partial A}{\partial t_2} + 3iA^2A^* = 0$$

$$2i\frac{\partial A^*}{\partial t_2} + 3i(A^*)^2A = 0$$

These equations are complex conjugates of each other so we really only need:

$$\frac{\partial A}{\partial t_2} = -\frac{3}{2}A^2A^*$$

How do we solve complex valued ODEs??

Let $A = r(t_2)e^{i\theta(t_2)}$

$$\Rightarrow \frac{\partial A}{\partial t_2} = r' e^{i\theta(t_2)} + r i \theta' e^{i\theta} = -\frac{3}{2} r^3 e^{i\theta}$$

$$\Rightarrow r' + i r \theta' = -\frac{3}{2} r^3.$$

Comparing real and imaginary parts we have:

$$r' = -\frac{3}{2} r^3 \ , \quad \theta' = 0$$

$$\Rightarrow \int_{r_0}^{r} \frac{1}{r^3}\, dr = \int_0^{t_2} -\frac{3}{2}\, dt_2$$

$$\Rightarrow -\frac{1}{2} r^{-2} + \frac{1}{2} r_0^{-2} = -\frac{3}{2} t_2$$

$$\Rightarrow r^{-2} = 3t_2 + r_0^{-2}$$

$$\Rightarrow r = \frac{r_0}{\sqrt{1+3r_0^2 t_2}}$$

The second differential equation in θ' yields $\theta = \theta_0$.

Putting it all together:

$$y \sim y_0 = A(t_2)e^{it_1} + A^*(t_2)e^{-it_1}$$

$$= \frac{r_0}{\sqrt{1+3r_0^2 t_2}} e^{i\theta_0} e^{it_1} + \frac{r_0}{\sqrt{1+3r_0^2 t_2}} e^{-i\theta_0} e^{-it_1}$$

$$\Rightarrow y \sim y_0 = \frac{r_0}{\sqrt{1+3r_0^2 \varepsilon t}} \left(e^{i(\theta_0 + t)} + e^{-i(\theta_0 + t)} \right)$$

$$\Rightarrow \boxed{y \sim y_0 = \frac{2r_0}{\sqrt{1+3r_0^2 \varepsilon t}} \cos(\theta_0 + t)}$$