

Moment angle complexes and big Cohen–Macaulayness

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Let $\mathcal{Z}_K \subset \mathbb{C}^m$ be the moment angle complex associated to a simplicial complex K on $[m]$, together with the natural action of the torus $T = U(1)^m$. Let $G \subset T$ be a (possibly disconnected) closed subgroup and $R := T/G$. Let $\mathbb{Z}[K]$ be the Stanley–Reisner ring of K and consider $\mathbb{Z}[R^*] := H^*(BR; \mathbb{Z})$ as a subring of $\mathbb{Z}[T^*] := H^*(BT; \mathbb{Z})$. We prove that $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is isomorphic to $\text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})$ as a graded module over $\mathbb{Z}[T^*]$. Based on this, we characterize the surjectivity of $H_T^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K; \mathbb{Z})$ (ie $H_G^{\text{odd}}(\mathcal{Z}_K; \mathbb{Z}) = 0$) in terms of the vanishing of $\text{Tor}_{\mathbb{Z}[R^*]}^{\mathbb{Z}[K]}(\mathbb{Z}[K], \mathbb{Z})$ and discuss its relation to the freeness and the torsion-freeness of $\mathbb{Z}[K]$ over $\mathbb{Z}[R^*]$. For various toric orbifolds \mathcal{X} , by which we mean quasitoric orbifolds or toric Deligne–Mumford stacks, the cohomology of \mathcal{X} can be identified with $H_G^*(\mathcal{Z}_K)$ with appropriate K and G and the above results mean that $H^*(\mathcal{X}; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})$ and that $H^{\text{odd}}(\mathcal{X}; \mathbb{Z}) = 0$ if and only if $H^*(\mathcal{X}; \mathbb{Z})$ is the quotient $H_R^*(\mathcal{X}; \mathbb{Z})$.

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1 Introduction

The equivariant cohomology and the ordinary cohomology with \mathbb{Z} -coefficients of a “compact smooth toric space” (including quasitoric manifolds, complete smooth toric varieties) is known by the work of Danilov [8], Jurkiewicz [19] and Davis and Januszkiewicz [9]: the equivariant cohomology is the Stanley–Reisner ring of the associated simplicial complex and the ordinary cohomology is the quotient of the equivariant cohomology by linear relations.

The orbifold analogue of these spaces has also been introduced and studied by several people, for example, Lerman and Tolman [21], Borisov, Chen and Smith [3] and Poddar and Sarkar [25]. The equivariant cohomology of these toric orbifolds with \mathbb{Z} -coefficients is also known to be the associated Stanley–Reisner rings and the ordinary cohomology is the quotient of the equivariant cohomology over \mathbb{Q} -coefficients; cf [3;

8; 19; 25]. However the ordinary cohomology with \mathbb{Z} -coefficients is hard to compute because it is not the quotient of the equivariant cohomology in general (for example in the case of a direct product of weighted projective spaces). The main theme of this paper is to characterize when the ordinary cohomology is the quotient of the equivariant cohomology.

Our approach is to view previously mentioned toric orbifolds as *quotient stacks* given by partial quotients of moment angle complexes and consider the *cohomology of stacks* in the sense of Edidin [10] (see also Tolman [28] and Edidin and Graham [11]). The *moment-angle complex* \mathcal{Z}_K was introduced by Buchstaber and Panov in [6] as a disc-circle decomposition of the Davis–Januszkiewicz universal space associated to a simplicial complex K [9] where they introduced a quasitoric manifold as a partial quotient of the moment angle manifold associated to a simple polytope.

If K is a simplicial complex on $[m] := \{1, \dots, m\}$, then \mathcal{Z}_K carries a natural action of the torus $T := U(1)^m$. The quotient stack $[\mathcal{Z}_K/G]$ with an appropriate choice of the subgroup $G \subset T$ can be used as a topological model to compute the cohomology of quasitoric orbifolds [9; 25], symplectic toric orbifolds [21] and toric Deligne–Mumford stacks [3]. In other words, *the ordinary cohomology of these toric orbifolds as stacks can be defined as the G -equivariant cohomology $H_G^*(\mathcal{Z}_K; \mathbb{Z})$* . Similarly the equivariant cohomology can be defined as $H_T^*(\mathcal{Z}_K; \mathbb{Z})$ which is isomorphic to the Stanley–Reisner ring $\mathbb{Z}[K]$ as quotient rings of $\mathbb{Z}[T^*] := H^*(BT; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_m]$. In Section 2, we recall the constructions of those toric orbifolds and the relation to moment angle complexes to motivate our readers.

In Section 3, we start with proving the following.

Theorem A *Let $G \subset T$ be a (possibly disconnected) closed subgroup and $R := T/G$. There is an isomorphism of graded \mathbb{Z} -modules:*

$$H_G^*(\mathcal{Z}_K; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K]; \mathbb{Z})$$

Furthermore there is a natural commutative diagram:

$$\begin{array}{ccc} H_T^*(\mathcal{Z}_K; \mathbb{Z}) & \longrightarrow & \mathbb{Z}[K] \\ \downarrow & & \downarrow \\ H_G^*(\mathcal{Z}_K; \mathbb{Z}) & \longrightarrow & \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}) \end{array}$$

Here $\mathbb{Z}[R^*] := H^*(BR; \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n]$ is considered as a subring of $\mathbb{Z}[T^*]$ so that the u_i are linear combinations of the x_j . It is worth noting that this theorem holds more generally. Namely, the theorem holds for any topological space X with T -action

such that $C^*(ET \times_{\mathbb{T}} X; \mathbb{Z})$ is formal in the category of $H^*(BT; \mathbb{Z})$ –modules up to homotopy in the sense of Franz [13]. See Section 3 for the details.

Based on this isomorphism, we prove our characterization theorem.

Theorem B *The following conditions are equivalent:*

- (i) $\text{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$
- (ii) $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is isomorphic to the quotient of $\mathbb{Z}[K]$ by linear forms
- (iii) $H_G^{\text{odd}}(\mathcal{Z}_K; \mathbb{Z}) = 0$

We will explain in Section 4 that, even though $\mathbb{Z}[K]$ might not be finitely generated over $\mathbb{Z}[\mathbb{R}^*]$, the vanishing of $\text{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z})$ has the usual meaning in terms of regular sequences, ie $\text{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$ if and only if u_1, \dots, u_n form a $\mathbb{Z}[K]$ –regular sequence. Thus we say $\mathbb{Z}[K]$ is big Cohen–Macaulay over $\mathbb{Z}[\mathbb{R}^*]$ if (i) is satisfied.

By presenting a toric orbifold \mathcal{X} as $[\mathcal{Z}_K/G]$, we obtain the following immediate corollary.

Corollary C *If \mathcal{X} is a toric orbifold stack presented as $[\mathcal{Z}_K/G]$, then*

$$H^*(\mathcal{X}; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[\mathbb{R}^*]}^*(\mathbb{Z}[K]; \mathbb{Z}).$$

Also, $H^(\mathcal{X}; \mathbb{Z})$ is the quotient of Stanley–Reisner ring $\mathbb{Z}[K]$ if and only if one of the following equivalent conditions holds: (i) $H^{\text{odd}}(\mathcal{X}; \mathbb{Z}) = 0$; (ii) $\text{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K]; \mathbb{Z}) = 0$.*

For example, the cohomology of weighted projective spaces as stacks are shown to be the quotient of its equivariant cohomology, based on the computation exhibited in Holm [17]. On the other hand, the cohomology of a direct product of weighted projective spaces is not the quotient of its equivariant cohomology. See Section 6 for the details and more examples.

In Section 5, we will discuss the freeness and the torsion-freeness of $\mathbb{Z}[K]$ over $\mathbb{Z}[\mathbb{R}^*]$. In particular, we show that the equivariant cohomology of a toric orbifold is torsion-free over $\mathbb{Z}[\mathbb{R}^*]$. We also give an injectivity theorem on the equivariant cohomology of \mathcal{Z}_K (Theorem 5.14) for a symplectic toric orbifold $[\mathcal{Z}_K/G]$, which gives a sufficient condition that $\mathbb{Z}[K]$ is free over a subring of $\mathbb{Z}[\mathbb{R}^*]$.

Finally, in Section 7, in light of Theorem 3.3, we construct an algebraic Gysin sequence for Tor of $\mathbb{Z}[K]$ in analogy with the Gysin sequence of S^1 –fibration over a toric manifold.

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2 Moment angle complexes and toric orbifolds

In this section, we review the basic facts about moment angle complexes and various toric orbifolds to motivate our results.

2.1 Moment angle complexes

The *moment angle complex* \mathcal{Z}_K associated to a simplicial complex K was introduced by Buchstaber and Panov in [6] as a disc-circle decomposition of the Davis–Januszkiewicz universal space associated to a simplicial complex K [9] and it has been actively studied in *toric topology* and its connections to symplectic and algebraic geometry and combinatorics. For convenience, we use the following notation for the rest of the paper.

Notation 2.1 Let X, Y be subsets of a set Z . For a subset $\sigma \subset [m]$, $X^\sigma \times Y^{[m] \setminus \sigma} \subset Z^m$ is the direct product of X 's and Y 's where i^{th} component is X if $i \in \sigma$ and Y if $i \notin \sigma$.

For a simplicial complex K on vertices $[m] := \{1, \dots, m\}$, the moment angle complex $\mathcal{Z}_K \subset \mathbb{C}^m$ is defined as $\mathcal{Z}_K = \bigcup_{\sigma \in K} D^\sigma \times (\partial D)^{[m] \setminus \sigma}$, where $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the unit disk and ∂D is its boundary circle. This space \mathcal{Z}_K carries a natural action of $T = U(1)^m$. It is originally proved in [9] that

$$(1) \quad H_T^*(\mathcal{Z}_K; \mathbb{Z}) \cong \mathbb{Z}[K] \quad \text{as graded } \mathbb{Z}[T^*]\text{-algebras.}$$

Here, $\mathbb{Z}[K] = \mathbb{Z}[x_1, \dots, x_m] / \langle x_\sigma, \sigma \notin K \rangle$ is the *Stanley–Reisner (face) ring*, where $x_\sigma := \prod_{i \in \sigma} x_i$. With the identification $\mathbb{Z}[T^*] := H^*(BT; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_m]$, the isomorphism is as graded algebras over the polynomial ring with $\deg x_i = 2$. For the details, we refer to Buchstaber and Panov [5, Chapter 6].

Baskakov, Buchstaber and Panov [2] also computed the ordinary cohomology of \mathcal{Z}_K ,

$$(2) \quad H^*(\mathcal{Z}_K; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[T^*]}^*(\mathbb{Z}[K], \mathbb{Z}) \quad \text{as graded rings.}$$

Here the grading on the right hand side is the total degree of bidegree coming from the (co)homological degree of Koszul complex and the degree of $\mathbb{Z}[K]$, as is explained in the following definition.

Definition 2.2 Let M be a graded $\mathbb{Z}[R^*] := \mathbb{Z}[u_1, \dots, u_n]$ -module. Let Λ be the exterior algebra generated by η_1, \dots, η_n with $\deg \eta_i = 1$. Let $M \otimes^R \Lambda$ be the corresponding Koszul complex. Its homology is $\text{Tor}_*^{\mathbb{Z}[R^*]}(M, \mathbb{Z})$ where the degree is given by $\deg \eta_i = 1$. The complex $M \otimes^R \Lambda$ is also a bigraded differential complex with $\text{bideg}(\alpha \otimes \xi_i) = [-1, 2] + [0, |\alpha|]$, where $\alpha \in M^{|\alpha|}$. The cohomological degree of Tor is defined to be the total degree of this bidegree and is denoted by the superscript as in $\text{Tor}_{\mathbb{Z}[R^*]}^*(M, \mathbb{Z})$.

It is natural to ask if $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ can be computed by $\text{Tor}_{\mathbb{Z}[R^*]}^*(\mathcal{Z}_K, \mathbb{Z})$, where $G \subset T$ is a (possibly disconnected) closed subgroup, $R := T/G$ and $\mathbb{Z}[R^*] := H^*(BR; \mathbb{Z}) \subset \mathbb{Z}[T^*]$. In Section 3, we show that

$$H_G^*(\mathcal{Z}_K; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathcal{Z}_K, \mathbb{Z}) \quad \text{as graded } \mathbb{Z}[T^*]\text{-modules (Theorem 3.3).}$$

2.2 Partial quotient of moment angle complexes

Various toric “spaces,” such as *quasitoric orbifolds* defined by Poddar and Sarkar [25], symplectic compact toric orbifolds defined by Lerman and Tolman [21] and *algebraic toric orbifolds* defined by Borisov, Chen and Smith [3], can be topologically regarded as the quotient stack $[\mathcal{Z}_K/G]$ with an appropriate choice of subgroup $G \subset T$ that acts on \mathcal{Z}_K locally freely. In the next section, we recall the construction of those spaces and see that the cohomology rings of all of these toric spaces in nice cases are computed as the quotient of the Stanley–Reisner ring $\mathbb{Z}[K]$. In this section, we just give a criteria for the local freeness of the action of a closed subgroup G of T on \mathcal{Z}_K and a remark about the cohomology of orbifolds as stacks.

Lemma 2.3 *Let n be the largest cardinality of a face in a pure simplicial complex K . If a closed subgroup $G \subset T$ acts on \mathcal{Z}_K locally freely, then $\dim G \leq m - n$. Furthermore, if n is the cardinality of maximal faces in K and $\dim G = m - n$, then G acts on \mathcal{Z}_K locally freely if and only if $T_\sigma := U(1)^\sigma \times \{1\}^{[m] \setminus \sigma} \subset T = U(1)^m$ surjects to $R := T/G$ for all maximal faces σ .*

Proof Let $\sigma \in K$ such that $|\sigma| = n$. Let $0_\sigma \in D^\sigma \times (\partial D)^{[m] \setminus \sigma}$ such that i^{th} component of 0_σ for $i \in \sigma$ is $0 \in D$. Then the stabilizer of 0_σ in T is $T_\sigma := U(1)^\sigma \times \{1\}^{[m] \setminus \sigma} \subset T = U(1)^m$. Consider the map $\theta: G \times T_\sigma \rightarrow T, (g, t) \mapsto gt$. The kernel of this map θ is finite if and only if the stabilizer of 0_σ in G is finite. Therefore the local freeness of

the G -action implies that the dimension of the image of θ is $\dim G + |\sigma| = \dim G + n$. Thus $\dim G \leq m - n$ since $\dim T = m$. To prove the latter claim, note that the local freeness of the G -action is equivalent to the stabilizer of 0_σ in G being finite for each maximal face σ . Since $G = m - n$ and $\dim T_\sigma$, $G \cap T_\sigma$ is zero-dimensional if and only if $T_\sigma \rightarrow R$ is surjective. □

Remark 2.4 In [10], Edidin defined the integral cohomology of a stack and showed that if the stack is given as a global quotient stack $[M/G]$, then $H^*([M/G]; \mathbb{Z})$ is canonically isomorphic to $H_G^*(M; \mathbb{Z})$. Therefore when $[M/G]$ is an orbifold, the cohomology of the orbifold $[M/G]$ as a stack is also $H_G^*(M; \mathbb{Z})$. On the other hand, the projection map $BG \times_G M \rightarrow M/G$, where M/G is the quotient topological space, induces an isomorphism $H_G^*(M; \mathbb{Q}) \cong H^*(M/G; \mathbb{Q})$ since the fiber is “ \mathbb{Q} -acyclic”. If G acts freely on M , then $H^*([M/G]; \mathbb{Z}) \cong H_G^*(M; \mathbb{Z}) \cong H^*(M/G; \mathbb{Z})$. If L acts on M and G is a subgroup of L that acts on M locally freely, we have the action of $K := L/G$ on the orbifold $[M/G]$. In this case, there is an isomorphism of stacks $[[M/G]/K] \cong [M/L]$ and we can define $H_K^*([M/G]; \mathbb{Z}) := H^*([M/L]; \mathbb{Z}) = H_L^*(M; \mathbb{Z})$; cf Romagny [26] and Lerman and Malkin [20].

Remark 2.5 The following is a useful criteria for the connectedness of G . Let B be the integer matrix induced from the quotient map $T \rightarrow R$. Then G is connected if and only if $B: \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ is surjective.

2.3 Quasitoric orbifolds

Quasitoric manifolds are introduced and studied by Davis and Januszkiewicz [9] and its orbifold counterpart is studied by Poddar and Sarkar [25]. Let Δ be a simple polytope of dimensional n in R^n . Let H_1, \dots, H_m be the facets of Δ and for a face $F_\sigma = \bigcap_{i \in \sigma} H_i$, let $T_\sigma := U(1)^\sigma \times \{1\}^{[m] \setminus \sigma} \subset T = U(1)^m$. Define $\mathcal{Z}_\Delta := T \times \Delta / \sim$, where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ is contained in a relative interior of F_σ and $t_1 t_2^{-1} \in T_\sigma$; cf [5, Definition 6.1]. It is known that \mathcal{Z}_Δ is a smooth manifold; cf [5, Lemma 6.2]. Let B be an integer $n \times m$ matrix such that for each vertex $H_{i_1} \cap \dots \cap H_{i_n}$ of Δ , the corresponding columns $\lambda_{i_1}, \dots, \lambda_{i_n}$ form a basis of \mathbb{Q}^n . By the assumption, B defines a surjective map $T \twoheadrightarrow R$ also denoted by B . Let G be the kernel of B . A *quasitoric orbifold* for the pair (Δ, B) is defined as the quotient stack $[\mathcal{Z}_\Delta/G]$. Here note that the assumption on B is equivalent to the local freeness of the G -action on \mathcal{Z}_Δ (see Lemma 2.3). Since \mathcal{Z}_Δ is T -equivariantly homeomorphic to \mathcal{Z}_{K_Δ} , where K_Δ is the simplicial complex associated to Δ (see [5, Section 6.2]), the quasitoric orbifold $[\mathcal{Z}_\Delta/G]$ is topologically the quotient of the moment angle complex \mathcal{Z}_{K_Δ} by G .

As a consequence and by Remark 2.4, the \mathbb{R} -equivariant cohomology of the quasitoric orbifold $[\mathcal{Z}_\Delta/\mathbb{G}]$ is

$$H_{\mathbb{R}}^*([\mathcal{Z}_\Delta/\mathbb{G}]; \mathbb{Z}) \cong H_{\mathbb{T}}^*(\mathcal{Z}_\Delta; \mathbb{Z}) \cong H_{\mathbb{T}}^*(\mathcal{Z}_{K_\Delta}; \mathbb{Z}) \cong \mathbb{Z}[K_\Delta].$$

The rational cohomology ring of the quasitoric orbifold is computed by Poddar and Sarkar [25].

Theorem 2.6 (Poddar–Sarkar) *If $[\mathcal{Z}_\Delta/\mathbb{G}]$ is a quasitoric orbifold,*

$$H^*([\mathcal{Z}_\Delta/\mathbb{G}]; \mathbb{Q}) \cong \mathbb{Q}[K_\Delta]/\langle u_1, \dots, u_n \rangle.$$

Here $u_j = \sum_{i=1}^m B_{ji}x_i \in \mathbb{Z}[x_1, \dots, x_m]$ and we identify $H^*(BR; \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n]$. A quasitoric orbifold given by (Δ, B) is a *quasitoric manifold* if and only if for each vertex $H_{i_1} \cap \dots \cap H_{i_n}$ of Δ , the corresponding columns $\lambda_{i_1}, \dots, \lambda_{i_n}$ of B form a \mathbb{Z} -basis. In this case, the isomorphism holds with \mathbb{Z} -coefficients.

Theorem 2.7 (Davis–Januszkiewicz [9]) *If $[\mathcal{Z}_\Delta/\mathbb{G}]$ is a quasitoric manifold,*

$$H^*([\mathcal{Z}_\Delta/\mathbb{G}]; \mathbb{Z}) \cong \mathbb{Z}[K_\Delta]/\langle u_1, \dots, u_n \rangle.$$

Moreover $H^*([\mathcal{Z}_\Delta/\mathbb{G}], \mathbb{Z})$ has no \mathbb{Z} -torison (which follows from [9, Theorem 3.1] and the fact that a quasitoric manifold is closed and orientable).

Since \mathcal{Z}_Δ is \mathbb{T} -equivariantly homeomorphic to \mathcal{Z}_{K_Δ} , we have the following.

Corollary 2.8 *If K_Δ and B give a quasitoric orbifold, then*

$$H_{\mathbb{G}}^*(\mathcal{Z}_{K_\Delta}; \mathbb{Q}) \cong \mathbb{Q}[K_\Delta]/\langle u_1, \dots, u_n \rangle.$$

If K_Δ and B give a quasitoric manifold, then $H_{\mathbb{G}}^(\mathcal{Z}_{K_\Delta}; \mathbb{Z}) \cong \mathbb{Z}[K_\Delta]/\langle u_1, \dots, u_n \rangle.$*

2.4 Compact symplectic toric orbifolds

Lerman and Tolman [21] classified compact symplectic toric (effective) orbifolds in terms of *labeled polytopes*. A labeled polytope (Δ, b) is a rational simple polytope Δ in \mathbb{R}^n with each facet $H_i, i = 1, \dots, m$ is labeled by a positive integer b_i . If β_i is the integral primitive inward normal vector to each facet H_i , then by assigning the integer matrix $B = [b_1\beta_1, \dots, b_m\beta_m]$, we obtain a quasitoric orbifold given by (Δ, B) . Here the symplectic structure on $[\mathcal{Z}_\Delta/\mathbb{G}]$ comes from identifying \mathcal{Z}_Δ with the level set for the reduction of \mathbb{C}^m by the action of \mathbb{G} . A *compact symplectic toric manifold* is given by the labeled polytope such that $b_i = 1$, for all $i = 1, \dots, m$ and such that for each vertex $H_{i_1} \cap \dots \cap H_{i_n}$ of Δ , the corresponding primitive normal vectors $\beta_{i_1}, \dots, \beta_{i_n}$ of B form a \mathbb{Z} -basis. This is exactly the Delzant condition in the classification of compact symplectic manifolds.

2.5 Algebraic toric (effective) orbifold (also known as toric Deligne–Mumford stack)

Let K be a pure simplicial complex on $[m]$. Define a fan Σ_K in \mathbb{R}^m where each cone is generated by the part of standard basis $e_i, i \in \sigma$ for each $\sigma \in K$. The corresponding toric variety X_{Σ_K} is a smooth open subvariety of \mathbb{C}^m that is exactly the complement of subspace arrangements given by K ; cf [5, Chapter 8]. There is a natural embedding of \mathcal{Z}_K into X_{Σ_K} .

Proposition 2.9 [5, Proposition 8.9] *There is a \mathbb{T} -equivariant deformation retract for $\mathcal{Z}_K \subset X_{\Sigma_K}$ and, in particular, $H_{\mathbb{C}}^*(\mathcal{Z}_K; \mathbb{Z}) \cong H_{G_{\mathbb{C}}}^*(X_{\Sigma_K}; \mathbb{Z})$ for each subgroup $G \subset \mathbb{T}$ and its complexification $G_{\mathbb{C}}$.*

The algebraic toric orbifolds studied by Borisov, Chen and Smith [3] are defined by the *stacky fan*. There they consider possibly noneffective orbifolds. In this paper, since we are interested in the effective case, we simplify the stacky fan and call it the *labeled fan*. A *labeled fan* (Σ, b) is a rational simplicial fan Σ in \mathbb{R}^n with each ray ρ_i labeled by a positive integer b_i , where $i = 1, \dots, m$. Let K be the simplicial complex associated to Σ . Let β_i be the integral primitive generator of each ray ρ_i , define an integral $n \times m$ matrix $B := [b_1\beta_1, \dots, b_m\beta_m]$, and let G be the kernel of the induced map of tori $B: \mathbb{T} \rightarrow \mathbb{R}$. The *toric Deligne–Mumford (DM) stack* associated to a labeled fan (Σ, b) is defined as the quotient stack $\mathcal{X}_{\Sigma, b} := [X_{\Sigma_K}/G_{\mathbb{C}}]$, where $G_{\mathbb{C}}$ is the complexification of G .

A toric DM stack $\mathcal{X}_{\Sigma, b}$ (or its labeled fan (Σ, b)) is *complete* if the fan is complete ie the union of cones is \mathbb{R}^n . A toric DM stack $\mathcal{X}_{\Sigma, b}$ is a nonsingular toric variety if and only if the labels $b_i = 1$ and the fan Σ is nonsingular, ie for each maximal cone, the primitive generators of the rays in the cone, $\beta_{i_1}, \dots, \beta_{i_n}$, form a \mathbb{Z} -basis. In this case, we call (Σ, b) nonsingular. For a nonsingular and complete labeled fan, we have the following classical result.

Theorem 2.10 (Danilov [8], Jurkiewicz [19]) *If (Σ, b) is nonsingular and complete, then $H^*([\mathcal{X}_{\Sigma_K}/G_{\mathbb{C}}]; \mathbb{Z}) \cong \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle$. Furthermore, it has no \mathbb{Z} -torison [8, Theorem 10.8].*

In [3, Proposition 3.7], it is proved that the *coarse moduli space* (underlying algebraic variety) for $[X_{\Sigma_K}/G_{\mathbb{C}}]$ is exactly the toric variety X_{Σ} given by the fan Σ ; see also Cox [7].¹ Thus for more general cases, one still has an isomorphism with \mathbb{Q} -coefficients.

¹Even if the toric orbifold is a nontrivial orbifold, its coarse moduli space could be a nonsingular toric variety. The simplest example of such a case may be the weighted projective space $[\mathbb{C}P^1_{12}] = [\mathbb{C}^2 \setminus \{(0, 0)\}/\mathbb{C}^{\times}]$ where the action of \mathbb{C}^{\times} is weighted by $(1, 2)$. Its coarse moduli space is simply $\mathbb{C}P^1$.

Theorem 2.11 (Danilov [8]) *If (Σ, \mathfrak{b}) is complete, then*

$$H^*([X_{\Sigma_K}/G_{\mathbb{C}}]; \mathbb{Q}) \cong \mathbb{Q}[K]/\langle u_1, \dots, u_n \rangle,$$

where $u_j = \sum_{i=1}^m B_{ji}x_i$.

Proposition 2.9 and Remark 2.4 therefore imply the following corollary.

Corollary 2.12 *If K and B are given by a complete labeled fan, then $H_G^*(\mathcal{Z}_K; \mathbb{Q}) \cong \mathbb{Q}[K]/\langle u_1, \dots, u_n \rangle$. If K and B are given by a complete and nonsingular labeled fan, then $H_G^*(\mathcal{Z}_K; \mathbb{Z}) \cong \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle$.*

Note that the underlying combinatorial structures for quasitoric orbifolds and toric DM stacks are both simplicial complexes. Any symplectic toric orbifold can be made into an algebraic one by taking the normal fans to the polytopes. However, not all quasitoric orbifolds can be made algebraic. A toric DM stack associated to a *polytopal fan* can be made into a symplectic toric orbifold but there is a toric DM stack associated to a *nonpolytopal fan*. Such a toric DM stack can not be realized even as a quasitoric orbifold.

In light of Theorem 3.3, and Corollaries 2.8 and 2.12, it is natural to ask the following question.

Question 2.13 When is $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ a quotient of the Stanley–Reisner ring for a general subgroup G ?

Our answer to this question is Theorem 5.1. Also we will see in Section 5 that when the G –action is locally free and $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ is a quotient of Stanley–Reisner ring, then the dimension of G must be maximal.

3 The proof that $H_G^*(\mathcal{Z}_K; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[\mathbb{R}^*]}^*(\mathbb{Z}[K], \mathbb{Z})$

In this section, we prove that $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is isomorphic to $\text{Tor}_{\mathbb{Z}[\mathbb{R}^*]}^*(\mathbb{Z}[K], \mathbb{Z})$ as a graded module over $\mathbb{Z}[\mathbb{T}^*]$. The idea of the proof, especially to use the homological machinery developed in [12], was communicated to us by Franz. Throughout, we will use terminology found in [12].

We will use the following notation consistently throughout this paper unless otherwise specified.

Notation 3.1 Let K be a simplicial complex on $[m] := \{1, \dots, m\}$ possibly with ghost vertices, ie there may be $i \in [m]$ such that $\{i\} \notin K$. Let \mathcal{Z}_K be the associated moment angle complex with the standard torus $T := U(1)^m$ -action. Let $\mathfrak{t} := \text{Lie}(T) = \mathbb{R}^m$, and let $N_T = \mathbb{Z}^m$ be the kernel of the exponential map $\mathfrak{t} \rightarrow T$. Let $G \subset T$ be a (possibly disconnected) closed subgroup of dimension $m - n$ and let $R := T/G$ be the quotient torus. We identify $R \cong U(1)^n$ so that the quotient map $T \rightarrow R$ defines an integral $n \times m$ matrix B which is viewed as the surjective linear map $\mathfrak{t} \rightarrow \mathfrak{r} := \text{Lie } R$.

Let $\mathbb{Z}[T^*] := H^*(BT; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_m]$ where $\{x_j\}$ is the standard basis of N_T^* and let $\mathbb{Z}[R^*] := H^*(BR; \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n]$ where $\{u_i\}$ is the standard basis of N_R^* . We regard $\mathbb{Z}[R^*]$ as a subring of $\mathbb{Z}[T^*]$ so that $u_i := \sum_{j=1}^m B_{ij}x_j$. The Stanley–Reisner ring $\mathbb{Z}[K]$ is defined as the quotient of $\mathbb{Z}[T^*]$ by the monomials corresponding to nonfaces of K .

Definition 3.2 M is an $H^*(BR; \mathbb{Z})$ -module up to homotopy if it is a module over the reduced cobar construction of $H^*(R; \mathbb{Z})$ [13, Section 4]. Note that $H^*(R; \mathbb{Z})$ is regarded as a coalgebra dual to the algebra $H_*(R; \mathbb{Z})$ which is canonically isomorphic to the exterior algebra of N_R .

Theorem 3.3 Under Notation 3.1, there is an isomorphism of graded \mathbb{Z} -modules

$$\Theta_R: H_G^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}),$$

where the cohomological grading on the right hand side is given in Definition 2.2. Furthermore there is a commutative diagram of graded \mathbb{Z} -modules

$$(3) \quad \begin{array}{ccc} H_T^*(\mathcal{Z}_K; \mathbb{Z}) & \xrightarrow{\Theta} & \mathbb{Z}[K] \\ \iota_R^* \downarrow & & \downarrow \phi_R \\ H_G^*(\mathcal{Z}_K; \mathbb{Z}) & \xrightarrow{\Theta_R} & \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}), \end{array}$$

where ϕ_R is the obvious map induced from the inclusion of Koszul complexes, ι_R^* is the pullback of the quotient map $\iota_R: ET \times_G \mathcal{Z}_K \rightarrow ET \times_T \mathcal{Z}_K$ and Θ is the isomorphism (1) of Section 2. In particular, Θ_R is an isomorphism of graded rings if ι_R^* is surjective.

Proof We suppress the coefficient ring \mathbb{Z} . Given a map $p: Y \rightarrow BR$, the twisted tensor product $C^*(Y) \otimes^R H^*(R)$ is defined by [12, (5.7)]. Franz [12, Proposition 5.2] states that there is a quasi-isomorphism of differential graded (dg) $H_*(R)$ -modules

$$\Phi_Y^*: C^*(Y) \otimes^R H^*(R) \rightarrow C^*(Y \times^{BR} ER),$$

where $Y \times^{BR} ER$ is a pullback of $ER \rightarrow BR$ along $p: Y \rightarrow BR$.

Let $Y := ER \times_R (ET \times_G \mathcal{Z}_K)$ and $p: Y \rightarrow BR$ be defined as the composition of

$$ER \times_R (ET \times_G \mathcal{Z}_K) \xrightarrow{q_R} ET \times_T \mathcal{Z}_K \longrightarrow BT \longrightarrow BR,$$

where q_R is the projection to the 2nd and 3rd components, the second map is the projection to the 1st component and the last map is a classifying map for $B: T \rightarrow R$. We observe that p is obtained by taking the quotient of

$$ER \times (ET \times_G \mathcal{Z}_K) \rightarrow ET \times_G \mathcal{Z}_K \rightarrow ET/G \rightarrow ER$$

by the free actions of R on each space. This implies that $Y \times^{BR} ER = ER \times (ET \times_G \mathcal{Z}_K)$.

Now [12, Proposition 5.2] states that we have the quasi-isomorphism of dg $H_*(R)$ –modules:

$$\Phi_Y^*: C^*(ER \times_R (ET \times_G \mathcal{Z}_K)) \otimes^R H^*(R) \rightarrow C^*(ER \times (ET \times_G \mathcal{Z}_K))$$

The homology of the right hand side is $H_G^*(\mathcal{Z}_K)$. On the left hand side, since R acts on $ET \times_G \mathcal{Z}_K$ freely, the fibers of q_R are ER and therefore it induces a quasi-isomorphism of $H^*(BT)$ –modules up to homotopy

$$q_R^*: C^*(ET \times_T \mathcal{Z}_K) \rightarrow C^*(ER \times_R (ET \times_G \mathcal{Z}_K)),$$

ie it is a homomorphism of dg $C^*(BR)$ –modules such that after taking homology, it becomes an isomorphism of $H^*(BR)$ –modules. Franz [13, Theorem 1.1] implies that $C^*(ET \times_T \mathcal{Z}_K)$ is formal as a $H^*(BT)$ –module up to homotopy, ie there is a sequence of quasi-isomorphisms connecting $C^*(ET \times_T \mathcal{Z}_K)$ to $H^*(ET \times_T \mathcal{Z}_K)$ as dg modules over the reduced cobar construction of $H^*(T)$, and therefore as dg modules over the reduced cobar construction of $H^*(R)$. Since the operation to take the twisted tensor product $\otimes^R H^*(BR)$ and the homology of it preserves quasi-isomorphisms of $H^*(BR)$ –modules up to homotopy (cf McCleary [24, Theorem 8.20]), the map q_R^* induces a quasi-isomorphism

$$\tilde{q}_R^*: C^*(ER \times_R (ET \times_G \mathcal{Z}_K)) \otimes^R H^*(R) \rightarrow C^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R),$$

and there is a sequence of quasi-isomorphisms connecting $C^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R)$ and $H^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R)$. Since the homology of $H^*(ET \times_T \mathcal{Z}_K) \otimes^R H^*(R)$ is $\text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z})$, we obtain the isomorphism Θ_R . We summarize all in the diagram:

$$\begin{array}{ccc}
 H_G^*(Z_K) & \xleftarrow{\text{homology}} & C^*(ER \times (ET \times_G Z_K)) \\
 \downarrow \Theta_R \cong & & \uparrow \Phi_Y^* \\
 & & C^*(ER \times_R (ET \times_G Z_K)) \otimes^R H^*(R) \\
 & & \downarrow \bar{q}_R^* \\
 & & C^*(ET \times_T Z_K) \otimes^R H^*(R) \\
 & & \uparrow \text{sequence of quasi-isomorphisms} \\
 \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}) & \xleftarrow{\text{homology}} & H^*(ET \times_T Z_K) \otimes^R H^*(R)
 \end{array}$$

(4)

The right vertical maps gives a sequence of quasi-isomorphisms of dg $H_*(R)$ -modules and at the both ends, we have the desired graded \mathbb{Z} -modules after taking homology.

To prove the commutativity of diagram (3), first note that Θ coincides with Θ_R when $R = \mathbb{1}$, where $\mathbb{1}$ is the trivial group. Consider the map $\varphi: R \rightarrow \mathbb{1}$. Since it satisfies the condition in [12, Proposition 4.11], the naturality stated in [12, Proposition 5.2] implies that the following diagram commutes:

$$\begin{array}{ccc}
 C^*(ET \times_T Z_K) & \xleftarrow{\quad} & C^*(ET \times_T Z_K) \otimes^{\mathbb{1}} H^*(\mathbb{1}) \\
 \bar{q}_R^* \downarrow & & \bar{\phi}_R \downarrow \\
 C^*(ER \times (ET \times_G Z_K)) & \xleftarrow{\Phi_Y^*} & C^*(ER \times_R (ET \times_G Z_K)) \otimes^R H^*(R),
 \end{array}$$

where \bar{q}_R^* is the pullback of the projection $\bar{q}_R: ER \times (ET \times_G Z_K) \rightarrow ET \times_T Z_K$ and $\bar{\phi}_R$ is the map induced by q_R and $\varphi: R \rightarrow \mathbb{1}$. After taking the homology, \bar{q}_R^* and $\bar{\phi}_R$ naturally coincide with ι_R^* and ϕ_R respectively and the resulting diagram extends to the commutative diagram (3) since the rest of the right vertical maps in the diagram (4) are also natural with respect to $\varphi: R \rightarrow \mathbb{1}$.

Lastly, if ι_R^* is surjective, the commutative diagram (3) makes Θ_R an isomorphism of rings since the other three maps are ring homomorphism. □

Remark 3.4 Let X be any topological space with a T -action where $C^*(ET \times_T X; \mathbb{Z})$ is formal as a $H^*(BT; \mathbb{Z})$ -module up to homotopy. Then Theorem 3.3 also holds for X . Namely, the above proof can be identically applied to this case and gives the isomorphism $\Theta_R: H_G^*(X; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(H_T^*(X; \mathbb{Z}), \mathbb{Z})$.

In terms of toric orbifolds discussed in Section 2, we have the following corollary.

Corollary 3.5 *Let \mathcal{X} be a quasitoric orbifold or an effective toric Deligne–Mumford stack with associated simplicial complex K , and let R and T be the tori given in the notation of Section 2. Then there is an isomorphism of graded modules over $\mathbb{Z}[T^*]$*

$$H^*(\mathcal{X}; \mathbb{Z}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}).$$

Remark 3.6 Theorem 3.3 (or Corollary 3.5) generalizes several results that have been proved.

- (i) In case $G = T$, Theorem 3.3 states that the T -equivariant cohomology of \mathcal{Z}_K is the Stanley–Reisner ring of K , which is well-known; cf [5]. In case $G = 1$, then one recovers that the ordinary cohomology of \mathcal{Z}_K is the Tor–algebra of $\mathbb{Z}[K]$ over $\mathbb{Z}[T^*]$ [5, Theorem 7.6]. One may therefore view this result as interpolating between these extreme cases.
- (ii) If \mathcal{Z}_K/G is a quasitoric manifold, then one recovers [5, Theorem 7.37].
- (iii) When $X_\Sigma := X_{\Sigma_K}/G_{\mathbb{C}}$ is the coarse moduli for a toric orbifold, one recovers [13, Theorem 1.2]:

$$H^*(X_\Sigma; \mathbb{Q}) \cong H_{G_{\mathbb{C}}}^*(X_{\Sigma_K}; \mathbb{Q}) = H_G^*(\mathcal{Z}_K; \mathbb{Q}) \cong \text{Tor}_{\mathbb{Z}[R^*]}^*(\mathbb{Z}[K], \mathbb{Z}) \otimes \mathbb{Q}$$

4 Basics from commutative algebra

In this section, we collect some definitions and basic properties of graded modules over a polynomial ring and discuss the relations among them. Throughout this section, R will be a polynomial ring in variables u_1, \dots, u_n (generated in degree 2) over $k = \mathbb{Z}$ or \mathbb{Q} , and M will be a graded R -module (though not necessarily finitely generated). We will denote the ideal of R generated by polynomials of positive degree by R_+ .

Below we give brief names to several properties of R -modules so that we can refer to them later.

Definition 4.1 For M and R as above, one says that

- (k1) M is *free over R* if $M \cong \bigoplus_{e \in E} R \cdot e$ and $R \cdot e \cong R$ as graded R -modules where E is a subset of M ,
- (k2) M is *flat over R* if $\text{Tor}_{>0}^R(M, N) = 0$ for any (finitely generated) module N ,
- (k3) M is *torsion-free over R* if there is no torsion over R ($x \in M$ is a torsion element over R if x is in the kernel of the multiplication map $\cdot r: M \rightarrow M$ for some nonzero $r \in R$),
- (k4) M is a *big Cohen–Macaulay R -module* if $\text{Tor}_1^R(M, k) = 0$.

In general, (k1) implies (k2), and (k2) implies both (k3) and (k4).

We will see that in the case of interest to us, (k4) has the usual meaning in terms of regular sequences; see Proposition 4.6.

Definition 4.2 A nonzero element $r \in R$ is M -regular if $0 \rightarrow M \xrightarrow{r} M$ is exact. A sequence of elements $f_1, \dots, f_c \subset R$ is an M -regular sequence if, for each $i \leq c$, f_i is $(M/(f_1, \dots, f_{i-1})M)$ -regular.

Remark 4.3 We call the condition (k4) in Definition 4.1 *big Cohen–Macaulay* because $\text{Tor}_1^R(M, k) = 0$ is the same as saying that there exists a system of parameters of R_+ (namely, the variables of R ; see Corollary 4.12) that is an M -regular sequence. The ‘big’ terminology is a reference to the fact that M need not be finitely generated; see Bruns and Herzog [4, Chapter 8].²

Definition 4.4 Let I be an ideal of R such that $M \neq IM$. One defines the *grade of I on M* by

$$(5) \quad \text{grade}(I, M) := \min\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}.$$

If $M = IM$, one sets $\text{grade}(I, M) = \infty$. When M is finitely generated over R , this definition is the usual definition of the grade of I on M and it is the length of maximal M -regular sequence in I [4].

If $I = R_+$ is the ideal generated by the positive degree elements of R , the grade of I on M is called the *depth* of M over R , and is denoted $\text{depth}_R M := \text{grade}(I, M)$.

Our goal for this section is to show that $\text{depth}_R(M)$ is the length of the longest M -regular sequence in R_+ when M is only assumed to be finitely generated over S , where $f: R \rightarrow S$ is a ring homomorphism and the R -action on M is induced by f . This is recorded in the following proposition.

Proposition 4.5 Let $S = k[x_1, \dots, x_m]$, M be a finitely generated graded S -module, and $\varphi: R \rightarrow S$ be a graded ring homomorphism (so that M is hence a graded R -module via φ). Then all maximal M -regular sequences in R_+ have the same length $\text{depth}_R(M)$.

This proposition is a special case of the following proposition, whose proof will come after some lemmas.

²In the reference [4], it is mentioned that the existence of a big Cohen–Macaulay module over a local ring is an open problem. This question is not interesting for R , since it is a (nonlocal) Cohen–Macaulay ring.

Proposition 4.6 *Let $\varphi: R \rightarrow S$ be a homomorphism of Noetherian rings, M a finitely generated S -module and I an ideal of R with $IM \neq M$. Then $\text{grade}(I, M)$ agrees with the length of all maximal M -regular sequences in I .*

The fact that allows one to extend the usual finitely generated setup to the generality above is the following lemma.

Lemma 4.7 *Let $\varphi: R \rightarrow S$ be a homomorphism of graded Noetherian rings, M a finitely generated S -module and N a finitely generated R -module. Then we have $\text{Hom}_R(N, M) = 0$ if and only if $\text{Ann}_R(N)$ contains an M -regular element.*

Remark 4.8 Before starting on the proof, let us remark that in the setup of the lemma the set of associated primes of M over R is finite, even though M may not be finitely generated over R ; see Matsumura [23, Exercise 6.7]. Indeed, one sees this by taking a primary decomposition of M over S , which is also a primary decomposition over R , since the R -annihilator of an S -module is the preimage of the annihilator in S (and the inverse image of a primary ideal is primary).

Proof of Lemma 4.7 Suppose that $x \in \text{Ann}_R(N)$ is M -regular. Then for any $\psi \in \text{Hom}_R(N, M)$, we have

$$x\psi(n) = \psi(xn) = \psi(0) = 0 \quad \text{for all } n \in N.$$

Since x is M -regular, we have $\psi(n) = 0$.

Now assume that $\text{Ann}_R(N)$ does not contain an M -regular element. That is, all elements of $\text{Ann}_R(N)$ are zero divisors on M . As mentioned in Remark 4.8, the set of associated primes of M over R is finite. Since $\text{Ann}_R(N)$ consists of zero divisors, it is contained in the (finite) union of all associated primes of M . Therefore, we can apply the Prime Avoidance Lemma to get $\text{Ann}_R(N) \subset \mathfrak{p}$ for some associated prime \mathfrak{p} of M over R . We then have the nontrivial map

$$N_{\mathfrak{p}} \twoheadrightarrow N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}} \twoheadrightarrow k(\mathfrak{p}) \hookrightarrow M_{\mathfrak{p}},$$

where $k(\mathfrak{p})$ denotes the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Note that the first map is the canonical projection, the second is projection onto a one-dimensional subspace of $N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}}$, and the inclusion comes from localizing the inclusion $R/\mathfrak{p} \hookrightarrow M$ (which exists because \mathfrak{p} is an associated prime of M) at \mathfrak{p} . Thus, since N is finitely generated over R , $\text{Hom}_R(N, M)_{\mathfrak{p}} = \text{Hom}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$, which gives $\text{Hom}_R(N, M) \neq 0$. \square

We also record without proof a basic fact from the homological algebra of commutative rings.

Lemma 4.9 [4, Lemma 1.2.4] *Let $\varphi: R \rightarrow S$ be a homomorphism of Noetherian rings, M a finitely generated S -module and N a finitely generated R -module. If (x_1, \dots, x_r) is a regular sequence in $\text{Ann}_R(N)$ for M , then*

$$\text{Hom}_R(N, M/(x_1, \dots, x_r)M) = \text{Ext}_R^r(N, M). \quad \square$$

Proof of Proposition 4.6 Let (x_1, \dots, x_r) be a maximal M -regular sequence in I . By Lemma 4.9,

$$\text{Ext}_R^i(R/I, M) \cong \text{Hom}_R(R/I, M/(x_1, \dots, x_i)M).$$

If $i < r$, then x_{i+1} is $M/(x_1, \dots, x_i)M$ -regular. Therefore $\text{Ext}_R^i(R/I, M) = 0$ by Lemma 4.7. Since (x_1, \dots, x_r) is maximal, I doesn't contain any $M/(x_1, \dots, x_r)M$ -regular elements. Thus $\text{Ext}_R^r(R/I, M) \neq 0$. This proves the claim. \square

One has the following well known characterization of (k4) in terms of M -regular sequences due to Serre [27, Chapter IV.A].

Proposition 4.10 *Let u_1, \dots, u_n be a homogeneous minimal generating set of R_+ . Suppose that R and M satisfy the hypotheses of Proposition 4.5. Then the following properties are equivalent:*

- (a) $H_p(\mathbf{u}, M) = 0$ for $p \geq 1$.
- (b) $H_1(\mathbf{u}, M) = 0$.
- (c) The sequence u_1, \dots, u_n is M -regular.

Here, $H_p(\mathbf{u}, M)$ denotes the p^{th} Koszul homology of the sequence u_1, \dots, u_n with coefficients in M .

Proof The proof that appears in Serre [27, Chapter IV.A] uses standard techniques of the Koszul complex which hold even for modules which are not finitely generated over R , together with Nakayama's lemma for finitely generated modules over a Noetherian local ring. Since a version of Nakayama's lemma holds for graded modules that satisfy our hypothesis, Serre's original argument remains valid. \square

Corollary 4.11 *Let u_1, \dots, u_n be a homogeneous minimal generating set of R_+ , and suppose that R and M satisfy the hypothesis of Proposition 4.6. Then $\text{Tor}_1^R(M, k) = 0$ if and only if (u_1, \dots, u_n) is a regular sequence for M .*

Proof The Koszul complex on u_1, \dots, u_n resolves k over R , and hence one can use its homology to compute the Tor modules. Now appeal to the previous proposition. \square

Propositions 4.11 and 4.5 show that in the setup of Proposition 4.5, big Cohen–Macaulayness of M has the usual meaning in terms of the maximal length of an M –regular sequence.

Corollary 4.12 *In the setup of Proposition 4.5, one has that $\text{Tor}_1^R(M, k) = 0$ if and only if $\text{depth}_R(M) = n$.*

5 Properties of $\mathbb{Z}[K]$ as an algebra over $\mathbb{Z}[\mathbb{R}^*]$

In this section, we start with the characterization of big Cohen–Macaulayness (k4), and then discuss freeness (k1) and torsion-freeness (k3), of $\mathbb{Z}[K]$ as a ring over $\mathbb{Z}[\mathbb{R}^*]$. In the rest of the paper, we use Notation 3.1 unless otherwise specified.

5.1 Big Cohen–Macaulayness

The following theorem is a variant of Franz and Puppe [15, Theorem 1.1] and Franz [14, Lemma 5.1]. The differences are that the T–CW complex $EG \times_G \mathcal{Z}_K$ is not finite and that we consider the cohomology of the quotient stack $[\mathcal{Z}_K/G]$ instead of the one of the underlying topological space \mathcal{Z}_K/G .

Theorem 5.1 *Let $\iota_R: ET \times_G \mathcal{Z}_K \rightarrow ET \times_T \mathcal{Z}_K$ be the quotient map by the R –action. The following are equivalent:*

- (i) $\mathbb{Z}[K]$ is big Cohen–Macaulay over $\mathbb{Z}[\mathbb{R}^*]$, ie $\text{Tor}_1^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$
- (ii) $\iota_R^*: H_T^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is surjective
- (iii) $H_G^{\text{odd}}(\mathcal{Z}_K; \mathbb{Z}) = 0$

Proof Since $\text{Tor}_0^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle$, the commutative diagram (3) in Theorem 3.3 implies that (ii) is equivalent to the vanishing of $\text{Tor}_{i>0}^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z})$, which is actually equivalent to (i) by [15, Proposition 2.3]. (ii) implies (iii) because $H_T^*(\mathcal{Z}_K; \mathbb{Z})$ has only even degree classes. Now (iii) implies that the Serre spectral sequence for the Borel construction for the residual R –action for $ET \times_G \mathcal{Z}_K$ degenerates at E_2 level and hence the pullback of the fiber inclusion $ET \times_G \mathcal{Z}_K \hookrightarrow ER \times_R (ET \times_G \mathcal{Z}_K)$ is surjective. This pullback can be identified as ι_R^* and thus (iii) implies (ii); see also [14, Lemma 5.1] and its proof. \square

Again it is worth noting that Theorem 5.1 holds for any T–space X that satisfies the formality hypothesis in Remark 3.4, as well as $H_T^{\text{odd}}(X; \mathbb{Z})$.

Remark 5.2 By Theorem 3.3, ι_R^* is surjective if and only if

$$\Theta: H_G^*(\mathcal{Z}_K, \mathbb{Z}) \cong \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle.$$

Remark 5.3 Theorem 5.1 holds after replacing \mathbb{Z} by \mathbb{Q} .

Corollary 5.4 *If $\mathcal{X} = [\mathcal{Z}_K/G]$ is a quasitoric orbifold or effective toric Deligne–Mumford stack, then the following are equivalent:*

- (i) $\text{Tor}_1^{\mathbb{Z}[R^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$
- (ii) $H^*(\mathcal{X}; \mathbb{Z})$ is the quotient of $\mathbb{Z}[K]$ by u_1, \dots, u_n
- (iii) $H^{\text{odd}}(\mathcal{X}; \mathbb{Z}) = 0$

5.2 Freeness

The following theorem is analogous to [14, Lemma 6.1].

Proposition 5.5 $\mathbb{Z}[K]$ is free over $\mathbb{Z}[R^*]$ if and only if $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ has no \mathbb{Z} –torison and is concentrated in even degree.

Proof If $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is concentrated in even degree, then by Theorem 5.1 we have that $\iota_R^*: H_T^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is surjective. The surjectivity implies that $H_G^r(\mathcal{Z}_K; \mathbb{Z})$ has finite rank for each r and is a finitely generated free \mathbb{Z} –module if it has no \mathbb{Z} –torison. The Leray–Hirsch theorem (cf Hatcher [16, Theorem 4D.1]) can be applied to the fiber bundle $ER \times_R (ET \times_G \mathcal{Z}_K) \rightarrow BR$ where the pullback along the fiber $ET \times_G \mathcal{Z}_K$ can be identified with ι_R^* and therefore we have an isomorphism $\mathbb{Z}[R^*] \otimes_{\mathbb{Z}} H_G^*(\mathcal{Z}_K; \mathbb{Z}) \cong \mathbb{Z}[K]$ of $\mathbb{Z}[R^*]$ –modules. Since $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is a free \mathbb{Z} –module, $\mathbb{Z}[K]$ is a free $\mathbb{Z}[R^*]$ –module.

On the other hand, freeness of $\mathbb{Z}[K]$ over $\mathbb{Z}[R^*]$ implies $\text{Tor}_1^{\mathbb{Z}[R^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0$ and so ι_R^* is surjective by Theorem 5.1. Thus $H_G^*(\mathcal{Z}_K, \mathbb{Z}) \cong H_T^*(\mathcal{Z}_K, \mathbb{Z}) \otimes_{\mathbb{Z}[R^*]} \mathbb{Z}$. By freeness, we can write $H_T^*(\mathcal{Z}_K; \mathbb{Z}) \cong \bigoplus_{e \in E} \mathbb{Z}[R^*]e$ as a free $\mathbb{Z}[R^*]$ –module, where the e are even degree classes. Then $H_T^*(\mathcal{Z}_K; \mathbb{Z}) \otimes_{\mathbb{Z}[R^*]} \mathbb{Z} \cong \bigoplus_{e \in E} \mathbb{Z}e$. Thus there is no \mathbb{Z} –torison and no odd degree classes. □

The same proof as above proves the following theorem; see also Remark 5.3.

Proposition 5.6 $\mathbb{Q}[K]$ is free over $\mathbb{Q}[R^*]$ if and only if $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ has no odd degree classes.

With this proposition, together with the local freeness of the G -action, we can also prove the following lemma.

Lemma 5.7 *Suppose that the G -action on \mathcal{Z}_K is locally free. If $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ is concentrated in even degree, then $\mathbb{Q}[K]$ is finitely generated over $\mathbb{Q}[\mathbb{R}^*]$.*

Proof Since the G -action on the smooth variety X_{Σ_K} defined in Section 2.5 is locally free, we have the differentiable orbifold $[X_{\Sigma_K}/G]$. By the construction of de Rham cohomology for differentiable orbifolds, cf Adem, Leida and Ruan [1, Section 2.1], $H^*([X_{\Sigma_K}/G]; \mathbb{R})$ is finite-dimensional. Since $\mathcal{Z}_K \hookrightarrow X_{\Sigma_K}$ is a T -equivariant deformation retract, $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ is also finite-dimensional. On the other hand, by Proposition 5.6, if $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ has no odd degree, then $\mathbb{Q}[K]$ is free over $\mathbb{Q}[\mathbb{R}^*]$. Since $H_G^*(\mathcal{Z}_K; \mathbb{Q}) \cong H_T^*(\mathcal{Z}_K; \mathbb{Q}) \otimes_{\mathbb{Q}[\mathbb{R}^*]} \mathbb{Q}$, the finiteness of $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ implies that $H_T^*(\mathcal{Z}_K; \mathbb{Q})$ is finitely generated over $\mathbb{Q}[\mathbb{R}^*]$. \square

5.3 Torsion-freeness

First we observe the following equivalence.

Lemma 5.8 *$\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[\mathbb{R}^*]$ if and only if $\mathbb{Z}[K] \otimes \mathbb{Q}$ is torsion-free over $\mathbb{Q}[\mathbb{R}^*]$.*

Proof Suppose that $f \neq 0$ is a torsion element in $\mathbb{Z}[K]$ over $\mathbb{Z}[\mathbb{R}^*]$, ie there is a nonzero element $g \in \mathbb{Z}[\mathbb{R}^*]$ such that $fg = 0$ in $\mathbb{Z}[K]$. Since $\mathbb{Z}[K]$ is free over \mathbb{Z} , $f \neq 0$ in $\mathbb{Q}[K]$. Therefore f is also a torsion in $\mathbb{Q}[K]$ over $\mathbb{Q}[\mathbb{R}^*]$. On the other hand, suppose that $f \neq 0$ is a torsion element of $\mathbb{Q}[K]$ over $\mathbb{Q}[\mathbb{R}^*]$. Let $g \in \mathbb{Q}[\mathbb{R}^*]$ be a nonzero element such that $fg = 0$ in $\mathbb{Q}[K]$. Let a be the product of denominators of the coefficients of f and b be the product of denominator of coefficients of g . Then the pair of $af \in \mathbb{Z}[K]$ and $bg \in \mathbb{Z}[\mathbb{R}^*]$ gives a torsion element of $\mathbb{Z}[K]$ over $\mathbb{Z}[\mathbb{R}^*]$. \square

Theorem 5.9 *If $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ is concentrated in even degree, then $\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[\mathbb{R}^*]$.*

Proof By Proposition 5.6, $\mathbb{Q}[K]$ is free over $\mathbb{Q}[\mathbb{R}^*]$, therefore it is torsion-free over $\mathbb{Q}[\mathbb{R}^*]$. We conclude that $\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[\mathbb{R}^*]$ by Lemma 5.8. \square

If $[\mathcal{Z}_K/G]$ is a quasitoric orbifold or a complete toric DM stack (Section 2), $H_G^*(\mathcal{Z}_K; \mathbb{Q})$ is concentrated in even degree by Theorems 2.6 and 2.11. Thus we have the following:

Corollary 5.10 *If $[\mathcal{Z}_K/G]$ is a quasitoric orbifold or a complete toric DM stack, then $\mathbb{Z}[K]$ is torsion-free over $\mathbb{Z}[\mathbb{R}^*]$.*

Remark 5.11 The converse of Theorem 5.9 is not true. The direct product of weighted projective spaces is a complete toric DM stack and its cohomology has odd degree classes. See Example 6.2.

The following proposition shows that the vanishing of odd classes in $H_G^*(Z_K; \mathbb{Q})$ implies that the size of G is maximal and it is analogous to [14, Proposition 5.2].

Proposition 5.12 *Let n' be the largest cardinality of a face of K . Suppose that G acts on Z_K locally freely. If $H_G^*(Z_K; \mathbb{Q})$ is concentrated in even degree, then $\dim R = n'$. Furthermore for a closed subgroup $G_1 \subset G$ such that $\dim G_1 < \dim G = m - n$, $\mathbb{Q}[K]$ has a torsion over $\mathbb{Q}[R_1^*]$, where $R_1 := T/G_1$.*

Proof By Lemma 5.7, $\mathbb{Q}[K]$ is finitely generated over $\mathbb{Q}[R^*]$ and hence over $\mathbb{Q}[R_1^*]$. Moreover $\mathbb{Q}[K]$ is free over $\mathbb{Q}[R^*]$ by Proposition 5.6 and so $\text{Ann}_{\mathbb{Q}[R_1^*]} \mathbb{Q}[K] = 0$. Thus we have

$$n' = \dim \mathbb{Q}[K] = \dim \mathbb{Q}[R^*] = \dim \frac{\mathbb{Q}[R_1^*]}{\text{Ann}_{\mathbb{Q}[R_1^*]} \mathbb{Q}[K]}.$$

Thus $\dim R = \dim \mathbb{Q}[R^*] = n'$ and $\dim \mathbb{Q}[R_1^*] = \dim R_1 > \dim R$, which implies $\text{Ann}_{\mathbb{Q}[R_1^*]} \mathbb{Q}[K] \neq 0$. □

Example 5.13 Let Δ be a n -dimensional rational bounded simple polytope in $\mathbb{R}^n = \mathfrak{r}^*$, where \mathfrak{r} is the Lie algebra of R and K_Δ the corresponding simplicial complex. Let $\rho_1, \dots, \rho_m \in \mathfrak{r}$ be the inward primitive normal vectors of the facets H_1, \dots, H_m of Δ and let $B = [\rho_1 \cdots \rho_m]$ be an integral $n \times m$ matrix. Adopt the notation from Notation 3.1. The GKM description of the rational equivariant cohomology of the corresponding toric orbifold is essentially the description of the piecewise polynomial functions on the corresponding normal fan: Let v_1, \dots, v_r be the vertices of Δ and denote the edge between v_i and v_j by (i, j) . Then there is an isomorphism of $\mathbb{Q}[u_1, \dots, u_n]$ -modules

$$\Phi : \mathbb{Q}[K_\Delta] \cong \{(f_1, \dots, f_r) \mid \lambda_{ij} \text{ divides } f_i - f_j \text{ for all edges } (i, j) \text{ of } \Delta\} \subset \bigoplus_{i=1}^r \mathbb{Q}[u],$$

where $\mathbb{Q}[u] := \mathbb{Q}[u_1, \dots, u_n]$ and $\lambda_{ij} \in \mathfrak{r}^*$ is a vector generating the edge (i, j) of Δ . It is well-known that $\Phi(x_i)_k = 0$ unless v_k lies on H_i . Therefore, for each vertex $v_k = H_{i_1} \cap \cdots \cap H_{i_n}$, $\Phi(x_{i_1} \cdots x_{i_n})_l = 0$ unless $k = l$. Now suppose that we take a subgroup $G_1 \subset G$ with $\dim G_1 < m - n$ and let $R_1 := T/G_1$. Let \mathfrak{r}_1 be the Lie algebra of R_1 and $u \in \mathfrak{r}_1^*$ a rational element such that $u \notin \mathfrak{r}^*$. Let $\Phi(u)_k = \sum_{i=1}^n a_i u_i$ so that $\Phi(u - \sum_{i=1}^n a_i u_i)_k = 0$ and $u - \sum_{i=1}^n a_i u_i \neq 0$ since $u \notin \mathfrak{r}^*$. Therefore $(u - \sum_{i=1}^n a_i u_i) \cdot x_{i_1} \cdots x_{i_n} = 0$, ie $x_{i_1} \cdots x_{i_n}$ is a torsion element of $\mathbb{Q}[K_\Delta]$ as a module over $\mathbb{Q}[R_1^*]$.

5.4 An injectivity theorem and freeness

Suppose that a subtorus G of T acts on \mathcal{Z}_K locally freely and consider a torus W such that $G \subset W \subset T$ and $m' := \dim W$. Let $F \subset \mathcal{Z}_K$ be the set of $(m - n)$ -dimensional W -orbits, where $\dim G = m - n$. We have the following injectivity result for the W -action on \mathcal{Z}_K when $[\mathcal{Z}_K/G]$ is a symplectic compact toric orbifold.

Theorem 5.14 *Suppose that $[\mathcal{Z}_K/G]$ is a symplectic compact toric orbifold corresponding to a labeled polytope (Δ, b) . Suppose that the stabilizer of any point $x \in \mathcal{Z}_K$ in W is connected. Then the pullback $H_W^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_W^*(F; \mathbb{Z})$ is injective. In particular, $H_T^*(\mathcal{Z}_K; \mathbb{Z})$ is free over $\mathbb{Z}[(T/W)^*]$.*

Proof Let $\{H_1, \dots, H_m\}$ be the set of all facets of Δ . The simplicial complex K is the one associated to Δ and $\tau \in K$ if and only if $\Delta_\tau := \bigcap_{i \in \tau} H_i \neq \emptyset$. Let F_a be a connected component of F . Then by Holm and the second author [18, Lemma 3.4] and our assumption of connected isotropy groups, the stabilizers of every $x \in F_a$ in W coincide. Let W_a be the stabilizer of points in F_a . Let $\mu: \mathcal{Z}_K \rightarrow \Delta$ be the moment map. Note that this is the quotient map by the action of T . First we show the following lemma.

Lemma 5.15 *$F_a = \mu^{-1}(\Delta_\sigma)$ for some $\sigma \in K$. In particular, the stabilizer of each point $x \in F_a$ is $W_\sigma := W \cap T_\sigma$.*

Proof of Lemma 5.15 For $x \in F_a$, let $\sigma_x := \{i \in [m], x_i = 0\}$. Then $\mu^{-1}(\Delta_{\sigma_x}) \subset F_a$ and the unique stabilizer for F_a is given by $W_a = W \cap T_{\sigma_x}$. Note that $\sigma_x \neq \emptyset$ since $m > n$. It suffices to show that there is an element $x \in F_a$ such that σ_x is the unique minimal subset among the collection of subsets, $\{\sigma_y \mid y \in F_a\}$. Let σ_x and σ_y be minimal for some $x, y \in F_a$. Suppose that $\sigma_x \neq \sigma_y$ and consider $z \in \mu^{-1}(\Delta_{\sigma_x \cap \sigma_y})$ such that $\sigma_z = \sigma_x \cap \sigma_y$. Since $W_a = W \cap T_{\sigma_x} \cap T_{\sigma_y} = W_z$, $z \in F$ by dimension counting. The connectivity of F_a then implies that $z \in F_a$. This contradicts the assumption that σ_x and σ_y are minimal, so $\sigma_x = \sigma_y$. \square

Now let $\{F_\sigma\}$ be the set of connected components of F where $F_\sigma = \mu^{-1}(\Delta_\sigma)$. For each σ , choose a splitting $W = W_\sigma \times (W/W_\sigma)$. The target of the injectivity map is computed as follows:

$$\begin{aligned} H_W^*(F; \mathbb{Z}) &= \bigoplus_{\sigma} H_W^*(F_\sigma; \mathbb{Z}) = \bigoplus_{\sigma} H^*(BW \times_W F_\sigma; \mathbb{Z}) \\ &= \bigoplus_{\sigma} H^*(BW_\sigma; \mathbb{Z}) \otimes H^*(F_\sigma/(W/W_\sigma); \mathbb{Z}) \end{aligned}$$

Now we show that $F_\sigma/(W/W_\sigma)$ is a compact toric symplectic manifold so that $H^*(F_\sigma/(W/W_\sigma); \mathbb{Z})$ has only even degree and has no \mathbb{Z} -torison. Since $[F_\sigma/G]$ is the suborbifold of (W/G) -fixed orbifold points, it is a symplectic orbifold (cf [21, page 4210, Corollary 3.8]), which is compact. Since the unique stabilizer of points of F_σ in G is given by $G_\sigma = G \cap W_\sigma$, the quotient $F_\sigma/(G/G_\sigma)$ is a compact toric manifold with the effective Hamiltonian action of $(T/G)/(T_\sigma/G_\sigma)$. On the other hand, $F_\sigma/(G/G_\sigma)$ is exactly $F_\sigma/W = F_\sigma/(W/W_\sigma)$. Thus $H^*(F_\sigma/(W/W_\sigma); \mathbb{Z})$ has only even degree and has no \mathbb{Z} -torison. Therefore, if we show the injectivity of $H_{\mathbb{W}}^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_{\mathbb{W}}^*(F; \mathbb{Z})$, it follows that $H_{\mathbb{W}}^*(\mathcal{Z}_K; \mathbb{Z})$ has no \mathbb{Z} -torison and has only even degree. Then we also have freeness of $H_{\mathbb{T}}^*(\mathcal{Z}_K; \mathbb{Z})$ over $\mathbb{Z}[(T/W)^*]$ by Proposition 5.5.

To show injectivity of $H_{\mathbb{W}}^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_{\mathbb{W}}^*(F; \mathbb{Z})$, we apply [18, Theorem 4.10]. It suffices to show that W_σ is connected and the weights of the action on the (negative) normal bundle are all primitive for each connected component of F_σ . The former is true by the assumption. For the latter, look at the normal bundle of F_σ in \mathcal{Z}_K which is given by $\bigoplus_{i \in \sigma} \mathbb{C}(\partial/\partial z_i)$. The weights of the T_σ -equivariant normal bundle are the standard \mathbb{Z} -basis $\{\lambda_i, i \in \sigma\}$ of $N_{T_\sigma}^*$. We need to check that the induced W_σ -weights $\tilde{\lambda}_i := A_\sigma^*(\lambda_i) \in N_{W_\sigma}^*$ are nonzero and primitive where $A_\sigma : W_\sigma \hookrightarrow T_\sigma$ is the restriction of the natural inclusion $A : W \rightarrow T$. It is easy to see that $\tilde{\lambda}_i$ is nonzero, since, if otherwise, the normal direction $\mathbb{C}(\partial/\partial z_i)$ is also contained in F_a . Finally the proof is completed by the following lemma.

Lemma 5.16 $A_\sigma^*(\lambda_i) \in N_{W_\sigma}^*$ is primitive.

Proof of Lemma 5.16 Consider the following commutative diagram of tori and its dual for the weight lattices:

$$(6) \quad \begin{array}{ccc} W_{\sigma \setminus \{i\}} & \longrightarrow & T_{\sigma \setminus \{i\}} \\ \downarrow & & \downarrow \\ W_\sigma & \xrightarrow{A_\sigma} & T_\sigma \\ & \searrow f_i & \downarrow g_i \\ & & T_{\{i\}} \end{array} \quad \begin{array}{ccc} N_{W_{\sigma \setminus \{i\}}}^* & \longleftarrow & N_{T_{\sigma \setminus \{i\}}}^* \\ \uparrow & & \uparrow \\ N_{W_\sigma}^* & \xleftarrow{A_\sigma^*} & N_{T_\sigma}^* \\ & \swarrow f_i^* & \uparrow g_i^* \\ & & N_{T_{\{i\}}}^* \end{array}$$

Here g_i is the canonical projection. The map $f_i = g_i \circ A_\sigma$ must be surjective since T_i is one-dimensional and $\tilde{\lambda}_i$ is nonzero. Also we have $W_{\sigma \setminus \{i\}} = \ker f_i$ which must be connected by assumption. Therefore we have a short exact sequence of tori $0 \rightarrow W_{\sigma \setminus \{i\}} \rightarrow W_\sigma \rightarrow T_{\{i\}} \rightarrow 0$ which implies that f_i^* maps $N_{T_i}^*$ to a direct summand. Thus $\tilde{\lambda}_i$ must be primitive since λ_i is a basis of $N_{T_i}^*$. \square

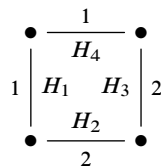
This completes the proof of Theorem 5.14. □

Remark 5.17 To give some ideas of how W can be chosen, please see Example 6.1. In the extreme case when $W = G$, the assumption implies that $F = \mathcal{Z}_K$ and G acts on \mathcal{Z}_K freely. Thus Theorem 5.14 is nothing but the fact that the equivariant cohomology of toric manifolds (or smooth toric varieties) is free over $\mathbb{Z}[R^*]$.

6 Examples

Example 6.1 (Effective weighted projective spaces) Let $a := (a_1, \dots, a_m)$ be a sequence of positive integers with $\gcd(a_1, \dots, a_m) = 1$ and let $[\mathbb{C}\mathbb{P}_a^{m-1}]$ be the corresponding effective weighed projective space. As in Section 2, $H^*([\mathbb{C}\mathbb{P}_a^{m-1}]; \mathbb{Z}) = H_G^*(\mathcal{Z}_K; \mathbb{Z})$, where $G = \{(t^{a_1}, \dots, t^{a_m})\} \subset T$ and K the boundary of an $(m - 1)$ -simplex. The corresponding Stanley–Reisner ring is $\mathbb{Z}[x_1, \dots, x_m]/\langle x_1 x_3 \cdots x_m \rangle$. In [17], the ordinary cohomology is computed as $H_G^*(\mathcal{Z}_K, \mathbb{Z}) \cong \mathbb{Z}[y]/\langle a_1 \cdots a_m y^m \rangle$. It has only even degree, so $H_T^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K; \mathbb{Z})$ is surjective. However $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ has \mathbb{Z} -torison, and so $H_T^*(\mathcal{Z}_K, \mathbb{Z})$ is not free over $\mathbb{Z}[R^*]$. The simplest such example is $[\mathbb{C}\mathbb{P}_{12}^1]$. In light of Theorem 5.14, we can find a subtorus $W \subset T$ such that $G \subset W$ and $\mathbb{Z}[K]$ is free over $\mathbb{Z}[(T/W)^*]$ as follows. Define $W \subset T$ by $W = \{(t_1, \dots, t_m) \in T \mid t_i = t_j \text{ if } a_i = a_j\}$. Such W contains G and the stabilizer of any point $x \in \mathcal{Z}_K$ in W is connected. Therefore by Theorem 5.14, $\mathbb{Z}[K]$ is free over $\mathbb{Z}[(T/W)^*]$.

Example 6.2 As we saw in the previous examples, $H_G^*(\mathcal{Z}_K; \mathbb{Z})$ has \mathbb{Z} -torison (infinitely many) in even degree for $[\mathcal{Z}_K/G] = \mathbb{C}\mathbb{P}_a^{m-1}$. The direct product of such toric orbifolds is also a toric orbifold and the Künneth theorem provides the \mathbb{Z} -torisons in odd degree. More concretely, take the labeled polytope



which gives $B = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$. This defines the direct product $[\mathbb{C}\mathbb{P}_{12}^1 \times \mathbb{C}\mathbb{P}_{12}^1]$ which gives odd degree elements in $H_T^*(\mathcal{Z}_K, \mathbb{Z})$. Thus $H_T^*(\mathcal{Z}_K, \mathbb{Z}) \rightarrow H_G^*(\mathcal{Z}_K, \mathbb{Z})$ is not surjective. We can also see this by determining whether $(x_1 - 2x_3, 2x_2 - x_4)$ is a regular sequence of $\mathbb{Z}[K] = \mathbb{Z}[x_1, \dots, x_4]/\langle x_1 x_3, x_2 x_4 \rangle$ as a module over $\mathbb{Z}[x_1 - 2x_3, 2x_2 - x_4]$. Indeed, it is not a regular sequence: $x_1 - 2x_3$ is a nonzero divisor in $\mathbb{Z}[K]$ but $2x_2 - x_4$ is a zero divisor in $\mathbb{Z}[K]/(x_1 - 2x_3)$ since $(2x_2 - x_4)x_2x_3^2 = 0$ and $x_2x_3^2 \neq 0$ in $\mathbb{Z}[K]/(x_1 - 2x_3)$.

7 Algebraic Gysin sequence

Let U be a subgroup of G such that $L := G/U$ is a 1–dimensional torus. By choosing an identification of L with a standard circle $U(1)$, we have an oriented principal $U(1)$ –bundle $\pi: ET \times_U \mathcal{Z}_K \rightarrow ET \times_G \mathcal{Z}_K$ and the corresponding Gysin sequence

$$\begin{aligned} & \dots \longrightarrow H_G^{i-1}(\mathcal{Z}_K, \mathbb{Z}) \\ & \xrightarrow{\cup e} H_G^{i+1}(\mathcal{Z}_K, \mathbb{Z}) \xrightarrow{\pi_*} H_U^{i+1}(\mathcal{Z}_K, \mathbb{Z}) \xrightarrow{\pi_*} H_G^i(\mathcal{Z}_K, \mathbb{Z}) \\ & \xrightarrow{\cup e} H_G^{i+2}(\mathcal{Z}_K, \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

where e is the Euler class of the bundle, π_* is the push forward map and π^* is the pullback map. In light of Theorem 3.3, it is natural to ask if there is a purely algebraic construction of a long exact sequence of Tor’s corresponding the Gysin sequence. It turns out that it comes from the long exact sequence associated to the iterative construction of the commutative Koszul complex. We describe the construction in the following.

Construction 7.1 (Algebraic Gysin sequence) Let $\tilde{R} := T/U$ and make the identification $H^*(B\tilde{R}, \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_n, u_{n+1}] = \mathbb{Z}[\tilde{R}^*]$, where $\mathbb{Z}[u_1, \dots, u_n] = H^*(BR, \mathbb{Z})$. Consider the short exact sequence of Koszul complexes (as modules over $\mathbb{Z}[\tilde{R}^*]$)

$$(7) \quad 0 \rightarrow K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) \xrightarrow{\tau^*} K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n, \xi_{n+1}) \xrightarrow{\tau_*} K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) \rightarrow 0,$$

where the map τ^* is the obvious inclusion and the map τ_* is *getting rid of ξ_{n+1}* , lowering the homological degree by 1. Note that

$$K^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) = \mathbb{Z}[u_1, \dots, u_n, u_{n+1}] \langle \xi_1, \dots, \xi_n \rangle$$

and the differential is given by extending $\partial \xi_i = u_i$ as a differential algebra, where $\langle \rangle$ denotes the exterior algebra.

Let M be a graded $\mathbb{Z}[x_1, \dots, x_m]$ –module. After tensoring with M over $\mathbb{Z}[\tilde{R}^*]$, we obtain the long exact sequence of Tor modules over $\mathbb{Z}[u_1, \dots, u_n, u_{n+1}]$:

$$(8) \quad \dots \xrightarrow{\delta} \text{Tor}_{i+1}^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \xrightarrow{\tau^*} \text{Tor}_{i+1}^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \xrightarrow{\tau_*} \text{Tor}_i^{\mathbb{Z}[\tilde{R}^*]}(M, \mathbb{Z}) \xrightarrow{\delta} \dots$$

We call this the *algebraic (homological) Gysin sequence*.

Proposition 7.2 *The connecting map δ is a multiplication by u_{n+1} .*

Proof It follows from a diagram chase. Consider the part of the map (7) of complexes:

$$\begin{array}{ccccc}
 K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) & \longrightarrow & K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_{n+1}) & \longrightarrow & K_i^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) \\
 \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
 K_i^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n) & \longrightarrow & K_i^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_{n+1}) & \longrightarrow & K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n)
 \end{array}$$

Let z be a cycle in $K_{i-1}^{\mathbb{Z}[\tilde{R}^*]}(\xi_1, \dots, \xi_n)$ and lift it to top left corner:

$$\begin{array}{ccccc}
 u_i \cdot z & \longrightarrow & \partial(\xi_{n+1} \wedge z) = u_i \cdot z & \longrightarrow & 0 \\
 & & \uparrow \partial & & \uparrow \partial \\
 & & \xi_{n+1} \wedge z & \longrightarrow & z
 \end{array}$$

This concludes the proof. □

Definition 7.3 (Cohomological algebraic Gysin sequence) As in Definition 2.2, we can assign the cohomological degree and turn the sequence (8) into a cohomological sequence:

$$\begin{aligned}
 (9) \quad \dots & \xrightarrow{u_{n+1}} \text{Tor}_{\mathbb{Z}[\mathbb{R}^*]}^{i+1}(M, \mathbb{Z}) \xrightarrow{\tau^*} \text{Tor}_{\mathbb{Z}[\tilde{R}^*]}^{i+1}(M, \mathbb{Z}) \\
 & \xrightarrow{\tau_*} \text{Tor}_{\mathbb{Z}[\mathbb{R}^*]}^i(M, \mathbb{Z}) \xrightarrow{u_{n+1}} \dots
 \end{aligned}$$

We call this the *cohomological algebraic Gysin sequence*.

Remark 7.4 We remark here that the above sequence depends on the choice of u_{n+1} , which is unique up to a sign and a choice of an element of $\mathbb{Z}[u_1, \dots, u_n] = \mathbb{Z}[\mathbb{R}^*]$. This is because the choice of u_{n+1} is nothing more than a choice of complement of \mathbb{R} in $\tilde{\mathbb{R}}$. The choice of sign persists to (9), but the element of $\mathbb{Z}[\mathbb{R}^*]$ vanishes upon taking cohomology, since elements of $\mathbb{Z}[\mathbb{R}^*]$ are boundaries in the image. On the topological side, this choice of sign amounts to the choice of orientation needed to define the Euler class of the $U(1)$ -bundle π .

Remark 7.5 In the special case when K is the simplicial complex associated to a Delzant polytope Δ , the corresponding G -action on \mathcal{Z}_K is free and \mathcal{Z}_K/G is homeomorphic to the corresponding symplectic toric manifold. We have

$$\mathrm{Tor}_i^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0 \quad \text{for all } i \geq 1$$

since $\mathbb{Z}[K]$ is free over $\mathbb{Z}[\mathbb{R}^*]$. Thus the long exact sequence in Construction 7.1 implies that

$$\mathrm{Tor}_i^{\mathbb{Z}[\tilde{\mathbb{R}}^*]}(\mathbb{Z}[K], \mathbb{Z}) = 0 \quad \text{for all } i \geq 2.$$

Hence the only nonzero part of the long exact sequence is:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}[\tilde{\mathbb{R}}^*]}(\mathbb{Z}[K], \mathbb{Z}) & \xrightarrow{\tau_*} & \mathrm{Tor}_0^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) & & \\ & & & \nearrow \delta & & & \\ \mathrm{Tor}_0^{\mathbb{Z}[\mathbb{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) & \xrightarrow{\tau^*} & \mathrm{Tor}_0^{\mathbb{Z}[\tilde{\mathbb{R}}^*]}(\mathbb{Z}[K], \mathbb{Z}) & \longrightarrow & 0 & & \end{array}$$

This sequence, together with the identification of Tor algebras and cohomology rings of toric manifolds, gives the Gysin sequence used in the first author's paper [22] to compute the cohomology ring of a good contact toric manifold.

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