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Cohomology over Fiber products of local rings ☆

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Let S and T be local rings with common residue field k, let R be the fiber product $S \times_k T$, and let M be an S-module. The Poincaré series P_M^R of M has been expressed in terms of P_M^S, P_k^S and P_k^T by Kostrikin and Shafarevich, and by Dress and Krämer. Here, an explicit minimal resolution, as well as theorems on the structure of $\operatorname{Ext}_R(k,k)$ and $\operatorname{Ext}_R(M,k)$ are given that illuminate these equalities. Structure theorems for the cohomology modules of fiber products of modules are also given. As an application of these results, we compute the depth of cohomology modules over a fiber product.

0. Introduction

In homological investigations one often has information on properties of a module over a certain ring, and wants to extract information on its properties over a different ring. In this paper we consider the following situation: $S \to k \leftarrow T$ are surjective homomorphisms of rings, k is a field, R is the fiber product $S \times_k T$, and M an S-module. We further assume that S and T are either local rings with common residue field k, or connected graded k-algebras.

The starting point of this paper is the construction of an explicit minimal free resolution of M, viewed as an R-module, from minimal resolutions of M and k over S and k over T. This is carried out in Section 1. The structure of the R-free resolution allows us to obtain precise information on the multiplicative structure of cohomology over R, given by composition products. Some of the results obtained in this work have been proved in the graded case by use of standard resolutions. However, no similar approach can be used in the local case.

The symbol $\sqcup$ denotes a coproduct, also known as a free product, of k-algebras.

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Theorem A. *The canonical homomorphism of graded k-algebras*

$$\mathrm{Ext}_S(k,k) \sqcup \mathrm{Ext}_T(k,k) \to \mathrm{Ext}_{S\times_k T}(k,k)$$

defined by the universal property of coproducts of k-algebras is bijective. For every S-module M, the canonical homomorphisms of graded left $\mathrm{Ext}_R(k,k)$*-modules*

$$\mathrm{Ext}_R(k,k) \otimes_{\mathrm{Ext}_S(k,k)} \mathrm{Ext}_S(M,k) \to \mathrm{Ext}_R(M,k)$$

defined by the multiplication map, is bijective.

The isomorphisms imply relations between the Poincaré series $P_M^R(t)$ of M over R to $P_M^S(t)$, $P_k^S(t)$ and $P_k^T(t)$; this relationship was proved for $M = k$ by Kostrikin and Shafarevich [6], and by Dress and Krämer [3, Theorem 1] in the present setting. In [7], Polishchuk and Positselski proved the preceding theorem, when S and T are connected k-algebras by using cobar constructions.

By combining Theorem A with an observation of Dress and Krämer concerning second syzygy modules over fiber products, we obtain the following corollary.

Corollary B. *Let L be an R-module. Then the second syzygy* $\Omega^2 L$ *decomposes as* $\Omega^2 L = M \oplus N$ *where M and N are S and T-modules, respectively. Furthermore, one has an exact sequence of graded left* $\mathcal{R}$*-modules*

$$0 \to \left(\Sigma^{-2}\mathcal{R} \otimes_{\mathcal{S}} \mathrm{Ext}_S(M,k)\right) \oplus \left(\Sigma^{-2}\mathcal{R} \otimes_{\mathcal{T}} \mathrm{Ext}_T(N,k)\right) \to \mathcal{L} \to \mathcal{L}/\mathcal{L}^{\geqslant 2} \to 0,$$

where $\mathcal{R} = \mathrm{Ext}_R(k,k)$, $\mathcal{S} = \mathrm{Ext}_S(k,k)$, $\mathcal{T} = \mathrm{Ext}_T(k,k)$ *and* $\mathcal{L} = \mathrm{Ext}_R(L,k)$.

Theorem A shows that $\mathrm{Ext}_{-}(k,k)$, as a functor in the ring argument, transforms products into coproducts. We prove that $\mathrm{Ext}_R(-,k)$, as a functor from R-modules to $\mathrm{Ext}_R(k,k)$-modules, has a similar property. In the graded setting, this was shown by Polishchuk and Positselski [7]. Their methods do not extend to the local case, where even the equality of Poincaré series given is new. To give the result, let M, N, and V be S, T, and k-modules respectively, that satisfy $M/\mathfrak{p}M \cong V \cong N/\mathfrak{q}M$, where $\mathfrak{p},\mathfrak{q}$ are the kernels of the surjections $S \to k$, $T \to k$, respectively. Define an R-module $M \times_V N$ by the exact sequence

$$0 \to M \times_V N \xrightarrow{\iota} M \times N \xrightarrow{\mu-\nu} V \to 0. \tag{1}$$

Theorem C. *In the notation above, the short exact sequence* (1) *induces a short exact sequence of graded left* $\mathrm{Ext}_R(k,k)$*-modules*

$$0 \to \mathrm{Ext}_R(V,k) \xrightarrow{\binom{\mu^*}{-\nu^*}} \mathrm{Ext}_R(M,k) \oplus \mathrm{Ext}_R(N,k) \xrightarrow{\iota^*} \mathrm{Ext}_R(M\times_V N,k) \to 0.$$

In Section 4 we study the depth of $\mathrm{Ext}_R(M,k)$ over $\mathrm{Ext}_R(k,k)$ for an R-module M. The notion of depth was used in [4] to study the homotopy Lie algebras of simply connected CW complexes, and of local rings. More recently, Avramov and Veliche [1] have shown that small depth of $\mathrm{Ext}_R(M,k)$ over $\mathrm{Ext}_R(k,k)$ is responsible for significant complications in the structure of the stable cohomology of M over R. For cohomology modules of R-modules, large depth is impossible:

Theorem D. *Let M be an R-module. Then one has an inequality*

$$\mathrm{depth}_{\mathrm{Ext}_R(k,k)} \mathrm{Ext}_R(M,k) \leqslant 1,$$

with equality if M is an S or T-module. In particular, one has $\mathrm{depth}\,\mathrm{Ext}_R(k,k) = 1$.

1. Resolutions over a fiber product

In this section, we set notation and define a complex that will be used throughout the article. Let k be a field.

1.1. A k-algebra A is *graded* if there is a decomposition of $A = \bigoplus_{i \in \mathbb{Z}} A_i$ as k-vector spaces, and for all $i, j \in \mathbb{Z}$, one has $A_i A_j \subseteq A_{i+j}$. We use both upper and lower indexed graded objects and adopt the notation $A^i = A_{-i}$. One says that A is *connected* if $A_0 = k$ and $A_i = 0$ for $i < 0$ (or equivalently, $A^i = 0$ for $i > 0$).

1.2. A left module M over a graded k-algebra A is *graded* if there is a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as k-vector spaces, and for all $i, j \in \mathbb{Z}$, one has $A_i M_j \subseteq M_{i+j}$. For a homogeneous element $m \in M_i$, we denote its degree by $|m|$.

For graded left A-modules M and N and an integer j, the degree j homomorphisms from M to N form the abelian group

$$\operatorname{Homgr}_A(M, N)^j = \{\phi : M \to N \mid \phi(M_i) \subseteq N_{i-j} \text{ for each } i\} \subseteq \operatorname{Hom}_A(M, N).$$

The collection of graded homomorphisms from M to N form the graded group

$$\operatorname{Homgr}_A(M, N) = \bigoplus_{j \in \mathbb{Z}} \operatorname{Homgr}_A(M, N)^j.$$

1.3. In the sequel we assume that we are in one of the following situations:

(1) A is a commutative noetherian local ring, $\mathfrak{m}$ denotes its unique maximal ideal, and $k = A/\mathfrak{m}$ its residue field.
(2) A is a non-negatively graded, connected, degree-wise finite k-algebra, and $\mathfrak{m}$ denotes the unique graded maximal ideal A_+; in this case, the differentials in a complex of graded R-modules preserve degrees.

We say that an A-module M is *convenient* provided it is finitely generated in case (1), and if it is bounded below, degree-wise finite, and graded in case (2).

One says that a complex of free A-modules F is *minimal* if each module in the complex is convenient, and $\partial(F) \subseteq \mathfrak{m}F$. When such a resolution exists, it is unique up to isomorphism. If M is a convenient A-module, then M has a minimal free resolution, and such a resolution is unique.

1.4. In case 1.3(2), the A-module $\operatorname{Ext}_A^i(M, k)$ can be computed by taking a minimal graded free resolution F of M over A, and computing $H^i(\operatorname{Homgr}_A(M, k))$. This provides $\operatorname{Ext}_A^i(M, k)$ the structure of a graded A-module for each i; we denote the jth graded piece of $\operatorname{Ext}_A^i(M, k)$ by $\operatorname{Ext}_A^i(M, k)^j$.

Definition 1.5. Let A be a ring and X a graded set, $\mathsf{X} = \bigsqcup_{n \geqslant 0} \mathsf{X}_n$. We let ${}^A\mathsf{X}$ denote the graded free left A-module with basis X_n in degree n, and set ${}^A\mathsf{X}_n = 0$ when $\mathsf{X}_n = \emptyset$. We call ${}^A\mathsf{X}$ a *graded based module* over A with basis X. Homomorphisms of based modules are identified with their matrices in the chosen bases.

For a based module ${}^A\mathsf{X}$, we identify $A \otimes_A {}^A\mathsf{X}$ and ${}^A\mathsf{X}$ by means of the canonical isomorphism. We use ${}^A[\mathsf{XY}]$ to denote the graded based A-module ${}^A\mathsf{X} \otimes_A {}^A\mathsf{Y}$ with graded basis $\mathsf{XY} = \bigsqcup_n [\mathsf{XY}]_n$, where $[\mathsf{XY}]_n$ is the set of symbols

$$\{xy \mid x \in \mathsf{X}_i,\ y \in \mathsf{Y}_j,\ i + j = n\}.$$

Notation 1.6. We consider a diagram of homomorphisms of rings,

$$\begin{tikzcd} S \times_k T \arrow[r, "\tau"] \arrow[d, "\sigma"'] & T \arrow[d, "\pi_T"] \\ S \arrow[r, "\pi_S"'] & k \end{tikzcd} \tag{1.6.1}$$

where π_S and π_T are surjective, and $R := S \times_k T$ is the fiber product:

$$S \times_k T = \{(s,t) \in S \times T\colon\ \pi_S(s) = \pi_T(t)\}.$$

If S and T are as in either case of 1.3, then so is R, and its maximal ideal is $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$; we thus identify $\mathfrak{p}$ and $\mathfrak{q}$ with subsets of R. Every S-module is considered an R-module via σ, and similarly for T-modules.

Construction 1.7. Let M be an S-module. Let $P \to M$ and $E \to k$ be minimal free resolutions of M, respectively k, over S, and let $F \to k$ be a free resolution of k over T such that $E_0 = S$ and $F_0 = T$. Choose bases P, E, and F of the graded modules P, E, and F over S, S and T, respectively, so that $\mathsf{E}_0 = \{1_S\}$ and $\mathsf{F}_0 = \{1_T\}$. Consider the elements of P, $\mathsf{E}_{\geqslant 1}$ and $\mathsf{F}_{\geqslant 1}$ as letters of an alphabet. The degree of a word in this alphabet is defined to be the sum of the degrees of its letters. The length of a word w is defined to be the number of letters in w.

Let G be the set of all words of the form

$$\{f_1 e_2 f_3 \cdots e_{2l-2} f_{2l-1} p_{2l}\} \quad \text{and} \quad \{e_1 f_2 e_3 \cdots e_{2l-1} f_{2l} p_{2l+1}\},$$

where e_i, f_i and p_i range over $\mathsf{E}_{\geqslant 1}$, $\mathsf{F}_{\geqslant 1}$ and P respectively, and $l \geqslant 0$. Form the free graded R-module $G = {}^R\mathsf{G}$.

Every word $w \in \mathsf{G}$ has the form xw' for some letter x and a (possibly empty) word w'. Assume that one has $\partial(E) \subseteq \mathfrak{p}E$, $\partial(P) \subseteq \mathfrak{p}P$, $\partial(F) \subseteq \mathfrak{q}F$, and set

$$\partial^G(w) = \begin{cases} \partial^P(x) & \text{for } x \in \mathsf{P}, \\ \partial^E(x)w' & \text{for } x \in \mathsf{E}, \\ \partial^F(x)w' & \text{for } x \in \mathsf{F}, \end{cases}$$

and extend ∂^G to an endomorphism of G by R-linearity. Set $\partial_i^G = \partial^G|_{G_i}$. We remark that a matrix φ with entries in $\mathfrak{p}$ defines a homomorphism ${}^S\varphi$ of free S-modules, as well as a homomorphism ${}^R\varphi$ of free R-modules.

Theorem 1.8. *The maps of free modules ∂_i^G defined in Construction* 1.7 *give a minimal free resolution*

$$G\colon \cdots \to G_i \xrightarrow{\partial_i^G} G_{i-1} \to \cdots \to G_1 \xrightarrow{\partial_1^G} G_0 \to 0,$$

of the R-module M.

Remark 1.9. The first few degrees of the complex G in Construction 1.7 look as follows. Note that each map in the diagram acts on the leftmost letter of a word.

$$\begin{tikzcd}[row sep=small]
{}^R\mathsf{P}_3 \arrow[dr] & & & \\
& {}^R\mathsf{P}_2 \arrow[ddr] & & \\
{}^R[\mathsf{F}_1\mathsf{P}_2] \arrow[ur] & & & \\
& & {}^R\mathsf{P}_1 \arrow[ddddr] & \\
{}^R[\mathsf{E}_1\mathsf{F}_1\mathsf{P}_1] \arrow[dr] & & & \\
& {}^R[\mathsf{F}_1\mathsf{P}_1] \arrow[uur] & & \\
{}^R[\mathsf{F}_2\mathsf{P}_1] \arrow[ur] & & & \\
& & & {}^R\mathsf{P}_0 \\
{}^R[\mathsf{E}_2\mathsf{F}_1\mathsf{P}_0] \arrow[dr] & & & \\
& {}^R[\mathsf{E}_1\mathsf{F}_1\mathsf{P}_0] \arrow[ddr] & & \\
{}^R[\mathsf{F}_1\mathsf{E}_1\mathsf{F}_1\mathsf{P}_0] \arrow[ur] & & & \\
& & {}^R[\mathsf{F}_1\mathsf{P}_0] \arrow[uuuur] & \\
{}^R[\mathsf{E}_1\mathsf{F}_2\mathsf{P}_0] \arrow[dr] & & & \\
& {}^R[\mathsf{F}_2\mathsf{P}_0] \arrow[uur] & & \\
{}^R[\mathsf{F}_3\mathsf{P}_0] \arrow[ur] & & &
\end{tikzcd}$$

To explain the notation used, let ${}^R[\mathsf{E}_{\geqslant 2}\mathsf{G}]$ denote the R-linear span of words whose first letter is in E_i for some $i \geqslant 2$, see Definition 1.5. Let ${}^R[\mathsf{F}_1\mathsf{E}_{\geqslant 1}\mathsf{G}]$ denote the span of words starting with a letter from F_1, followed by a letter from E. Symbols such as ${}^R[\mathsf{F}_{\geqslant 2}\mathsf{G}]$, ${}^R[\mathsf{E}_1\mathsf{F}_{\geqslant 1}\mathsf{G}]$, etc. are defined similarly.

Proof of Theorem 1.8. To show that G is a complex, let w be a word of degree i, with $i \geqslant 2$. Suppose $w = xyw'$ where w' is a word, x is a letter of degree 1 and y is an arbitrary letter. For $x \in \mathsf{E}_1$ and $y \in \mathsf{F}$ one has

$$\partial^2(w) = \partial\big(\partial^E(x)yw'\big) \in \partial(\mathfrak{p}yw') = \mathfrak{p}\partial(yw') = \mathfrak{p}\partial^F(y)w' \subseteq \mathfrak{p}\mathfrak{q}w' = 0.$$

The cases with $x \in \mathsf{F}_1$ and $y \in \mathsf{P}$, and with $x \in \mathsf{F}_1$ and $y \in \mathsf{E}$ are similar. If $w = xw'$ where x is a letter of degree greater than or equal to 2, then $\partial^2(w) = 0$, since ${}^R\mathsf{P}$, ${}^R\mathsf{E}$, and ${}^R\mathsf{F}$ are complexes of R-modules, and hence one has $\partial^2(w) = 0$.

The proof that $H(G) = M$, proceeds in several steps. First, we describe a decomposition of an arbitrary cycle into a sum of cycles of a special type. Then we show that each summand is in fact an element of $\mathfrak{m}G$. Finally, we prove that each summand is a boundary through a computation that uses the special form of these cycles and the exactness of the complexes E, F and P

Let a be an element of G_i, with $i \geqslant 1$. It has a unique expression (cf. Remark 1.9 for notation)

$$\begin{aligned}
&a = (x + x') + (y + y') + (z + z') \quad \text{where} \\
&\quad x \in {}^R[\mathsf{E}_{\geqslant 2}\mathsf{G}], \ \ y \in {}^R[\mathsf{F}_{\geqslant 2}\mathsf{G}], \ \ z \in {}^R\mathsf{P}, \\
&\quad x' \in {}^R[\mathsf{F}_1\mathsf{E}_{\geqslant 1}\mathsf{G}], \ \ y' \in {}^R[\mathsf{E}_1\mathsf{F}_{\geqslant 1}\mathsf{G}], \ \ z' \in {}^R[\mathsf{F}_1\mathsf{P}].
\end{aligned}$$

Notice that one has

$$\partial(x + x') \in {}^R[\mathsf{E}_{\geqslant 1}\mathsf{G}], \qquad \partial(y + y') \in {}^R[\mathsf{F}_{\geqslant 1}\mathsf{G}], \qquad \partial(z + z') \in {}^R\mathsf{P}.$$

Thus, $\partial(a) = 0$ implies that each one of $x + x'$, $y + y'$ and $z + z'$ is a cycle. Next, we show each is a boundary, by giving details for $x + x'$; the other cases are similar.

Since one has $\mathfrak{p} \cap \mathfrak{q} = 0$ in R, and G_{i-1} is a free R-module, it follows that $\mathfrak{p}(G_{i-1}) \cap \mathfrak{q}(G_{i-1}) = 0$. Therefore, $\partial(x + x') = 0$ implies that x and x' are cycles as well. Let $l(w)$ denote the leftmost letter in the word w. We may express x according to the decomposition

$$
{}^{R}[\mathsf{E}_{\geqslant 2}\mathsf{G}]_i = \bigoplus_{\substack{2\leqslant j\leqslant i\\ w\in \mathsf{G}_{i-j}\\ l(w)\in \mathsf{F}}} {}^{R}[\mathsf{E}_{\mathsf{j}} w].
$$

If w is basis element of degree $i-j$ with $l(w)\in F$ and $2\leqslant j\leqslant i$, then one has $\partial({}^{R}[\mathsf{E}_{\mathsf{j}} w])\subseteq {}^{R}[\mathsf{E}_{\mathsf{j-1}} w]$. Hence, each component of x in the decomposition above is a cycle. For similar reasons the components of x' in ${}^{R}[\mathsf{F}_1\mathsf{E}_{\geqslant 1}\mathsf{G}]$

$$
{}^{R}[\mathsf{F}_1\mathsf{E}_{\geqslant 1}\mathsf{G}] = \bigoplus_{\substack{2\leqslant j\leqslant i\\ w\in \mathsf{G}_{i-j}\\ l(w)\in \mathsf{F}}} {}^{R}[\mathsf{F}_1\mathsf{E}_{\mathsf{j-1}} w]
$$

are cycles. Therefore, it is enough to show that every cycle of the form

$$
\begin{aligned}
x &= \sum_{e\in \mathsf{E}_j} r_e e w \in {}^{R}[\mathsf{E}_{\mathsf{j}} w] \quad \text{or}\\
x' &= \sum_{f\in \mathsf{F}_1}\sum_{e\in \mathsf{E}_{j-1}} r_{fe} f e w \in {}^{R}[\mathsf{F}_1\mathsf{E}_{\mathsf{j-1}} w]
\end{aligned}
$$

where w is a fixed word, and r_e, r_{fe} are in R, is a boundary.

We give details for x, the other case is similar. We first show that $r_e\in\mathfrak{m}$ for each $e\in\mathsf{E}_j$. Indeed, there is a commutative diagram of R-modules

$$
\begin{tikzcd}
\cdots \arrow[r] & {}^{R}[\mathsf{E}_3 w] \arrow[r] \arrow[d, "\widetilde{\sigma}_3"] & {}^{R}[\mathsf{E}_2 w] \arrow[r] \arrow[d, "\widetilde{\sigma}_2"] & {}^{R}[\mathsf{E}_1 w] \arrow[r] \arrow[d, "\widetilde{\sigma}_1"] & Rw \arrow[r] \arrow[d, "\widetilde{\sigma}_0"] & 0\\
\cdots \arrow[r] & E_3 \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & 0
\end{tikzcd}
\tag{1.9.1}
$$

where the vertical maps send $\sum_{e\in\mathsf{E}_j} r_e e w$ to $\sum_{e\in\mathsf{E}_j}\sigma(r_e)e$. The image of x in E is a cycle, and hence a boundary, of E. As E is minimal, the claim follows.

Suppose $t=\partial(f)$ for $f\in\mathsf{F}_1$. Then $tew=\partial(few)$ is a boundary of G. As F is a resolution of k over T, the images of F_1 form a minimal generating set for $\mathfrak{q}$. Hence $\mathfrak{q}ew$ consists entirely of boundaries.

The claims above show it suffices to prove the theorem when the coefficients are in $\mathfrak{p}$. In diagram (1.9.1), we may also define a morphism of complexes $\widetilde{\gamma}:\mathfrak{p}E\to\mathfrak{p}^{R}[\mathsf{E}w]$ by sending $\sum_{e\in\mathsf{E}_j} s_e e$ to $\sum_{e\in\mathsf{E}_j} s_e e w$, viewing $s_e\in\mathfrak{p}\subset R$. Note that $\widetilde{\gamma}$ and $\widetilde{\sigma}|_{\mathfrak{p}^{R}[\mathsf{E}w]}$ are inverses of one another.

Suppose that $x\in\mathfrak{p}^{R}[\mathsf{E}w]$ is a cycle. Then $\widetilde{\sigma}(x)$ is a cycle in E. Hence there exists u so that $\partial^{E}(u)=\widetilde{\sigma}(x)$. One then has

$$
\partial\big(\widetilde{\gamma}(u)\big)=\widetilde{\gamma}\big(\partial^{E}(u)\big)=\widetilde{\gamma}\big(\widetilde{\sigma}(x)\big)=x. \qquad \square
$$

Two special cases of the theorem are used in Section 3.

Example 1.10. When $M=k$, we can take $\mathsf{P}=\mathsf{E}$. Let D be the resolution given by Theorem 1.8. Since $\mathsf{P}_0=\{1\}$, we can also replace all basis elements of the form $w1$ with a basis element w of the same degree, and set $\partial^{D}(1_R)=0$. Therefore, in low degrees, D has the form

$$\begin{tikzcd}[row sep=tiny]
{}^R\mathsf{E}_3 \arrow[dr] & & & \\
 & {}^R\mathsf{E}_2 \arrow[ddr] & & \\
{}^R[\mathsf{F}_1\mathsf{E}_2] \arrow[ur] & & & \\
 & & {}^R\mathsf{E}_1 \arrow[ddddr] & \\
{}^R[\mathsf{E}_1\mathsf{F}_1\mathsf{E}_1] \arrow[dr] & & & \\
 & {}^R[\mathsf{F}_1\mathsf{E}_1] \arrow[uur] & & \\
{}^R[\mathsf{F}_2\mathsf{E}_1] \arrow[ur] & & & \\
 & & & R \\
{}^R[\mathsf{E}_2\mathsf{F}_1] \arrow[dr] & & & \\
 & {}^R[\mathsf{E}_1\mathsf{F}_1] \arrow[ddr] & & \\
{}^R[\mathsf{F}_1\mathsf{E}_1\mathsf{F}_1] \arrow[ur] & & & \\
 & & {}^R\mathsf{F}_1 \arrow[uuuur] & \\
{}^R[\mathsf{E}_1\mathsf{F}_2] \arrow[dr] & & & \\
 & {}^R\mathsf{F}_2 \arrow[uur] & & \\
{}^R\mathsf{F}_3 \arrow[ur] & & &
\end{tikzcd}$$

Example 1.11. When $M = T$, applying the theorem with the roles of S and T reversed, one has $P_i = 0$ for $i \neq 0$, and $\mathsf{P}_0 = \{1\}$. Let C be the resolution given by Theorem 1.8. Letting w denote the basis element $w1$ as above, we see that C is given by the top half of the diagram in Example 1.10.

2. Cohomology of coproducts of graded connected algebras

In this section, we remind the reader of the construction of coproducts of connected algebras, and collect some facts regarding their depth and Hilbert functions. The results in this section are applied to the Ext algebras of rings as in 1.3 in Section 4.

2.1. If B and C are graded connected k-algebras, the coproduct of B and C in this category is the free product of B and C, denoted $B \sqcup C$, and can be described as follows. A k-basis for $(B \sqcup C)_n$ consists of all elements of the form $x_1 \otimes \cdots \otimes x_p$ with $|x_1| + \cdots + |x_p| = n$, and the factors x_i alternate between elements of the homogeneous bases of B_+ and C_+. Multiplication in $B \sqcup C$ is given by

$$(v \otimes \cdots \otimes x)(y \otimes \cdots \otimes w) = \begin{cases} v \otimes \cdots \otimes xy \otimes \cdots \otimes w & \text{for } x, y \in B \text{ or } x, y \in C; \\ v \otimes \cdots \otimes x \otimes y \otimes \cdots \otimes w & \text{otherwise.} \end{cases}$$

We say that a tensor product of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ has length p, and that it starts in B_i (respectively C_i) if x_1 is in B_i (respectively C_i); the expression that the product ends in B_i (respectively C_i) has a similar meaning. In the sequel, we set $A = B \sqcup C$, and we consider B and C as subalgebras of A. Thus, A is free as a right graded B-module with basis given by those elements in the basis of A described above that end in C.

2.2. For a graded left B-module M, $A \otimes_B M$ is a graded left A-module. A k-basis for $(A \otimes_B M)_n$ consists of all elements of the form $x_1 \otimes \cdots \otimes x_{p-1} \otimes m_p$ such that $|x_1| + \cdots + |x_{p-1}| + |m_p| = n$, $p \geqslant 1$, and the x_i alternate between being elements of the homogeneous bases of B_+ and C_+, with x_{p-1} in C. The action of A on $A \otimes_B M$ is given by

$$(v \otimes \cdots \otimes x)(y \otimes \cdots \otimes w) = \begin{cases} v \otimes \cdots \otimes xy \otimes \cdots \otimes w & \text{for } x, y \in B \text{ or } x, y \in C; \\ v \otimes \cdots \otimes xw & \text{for } x \in B \text{ and } y \in M; \\ v \otimes \cdots \otimes x \otimes y \otimes \cdots \otimes w & \text{otherwise.} \end{cases}$$

We say that a tensor of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ has length p, and starts in B_i (respectively C_i) if x_1 is in B_i (respectively C_i). We identify the span of those tensors of length one with M, and hence M is a left B-submodule of $A \otimes_B M$.

Put $L = A \otimes_B M$. One then sees that every homogeneous element $l \in L$ can be written in the form

$$l = l_B + l_C + l_M \tag{2.2.1}$$

where the terms in l_B and l_C start in B and C, respectively, and l_M is in M.

2.3. Let B and C be graded connected k-algebras, and choose minimal homogeneous sets of generators X of B and Y of C as k-algebras. One then has resolutions of k of the form

$$\cdots \longrightarrow {}^B\mathsf{U} \xrightarrow{\partial_2^B} {}^B\mathsf{X} \xrightarrow{\partial_1^B} B \longrightarrow 0$$
$$\cdots \longrightarrow {}^C\mathsf{V} \xrightarrow{\partial_2^C} {}^C\mathsf{Y} \xrightarrow{\partial_1^C} C \longrightarrow 0$$

where ∂_1^B in 2.3 is given by $\partial_1^B(x) = x$, and one may similarly take $\partial_1^C(y) = y$. One can form a free resolution of k over A, which starts as

$$F = \qquad \cdots \longrightarrow {}^A\mathsf{U} \oplus {}^A\mathsf{V} \xrightarrow{\partial_2} {}^A\mathsf{X} \oplus {}^A\mathsf{Y} \xrightarrow{\partial_1} A \longrightarrow 0, \tag{2.3.1}$$

where the maps are given by the following formulas, see [5, Example 21(c)]:

$$\partial_1 = A \otimes_B \partial_1^B + A \otimes_C \partial_1^C, \qquad \partial_2 = \begin{pmatrix} A \otimes_B \partial_2^B & 0 \\ 0 & A \otimes_C \partial_2^C \end{pmatrix}$$

Proposition 2.4. *Let B and C be graded connected k-algebras with $B_1 \neq 0 \neq C_1$, and let M be a graded left B-module with $M_i = 0$ for $i < 0$ and $M_0 \neq 0$. If either of the conditions*

(i) $C_+C_1 \neq 0 \neq C_2$ *or* $B_+B_1 \neq 0 \neq B_2$;
(ii) $C_+ = C_1$ *and* $\dim_k C_1 \geqslant 2$, *or* $B_+ = B_1$ *and* $\dim_k B_1 \geqslant 2$

are satisfied, then one has

$$\mathrm{Ext}_A^1(k, A \otimes_B M)^{\geqslant 2} \neq 0.$$

Proof. Put $L = A \otimes_B M$. With F as in (2.3.1), standard isomorphisms identify $\mathrm{Homgr}_A(F, L)$ with the complex

$$0 \to L \xrightarrow{\partial_1^*} L^{\mathsf{X}} \oplus L^{\mathsf{Y}} \to L^{\mathsf{U}} \oplus L^{\mathsf{V}} \to \cdots,$$

where ∂_1^* is given by

$$\partial_1^*(l) = \big((xl)_{x \in \mathsf{X}}, (yl)_{y \in \mathsf{Y}}\big).$$

Fix elements β, γ of the same degree and set

$$\phi_{\beta,\gamma} = \big((x\beta)_{x \in \mathsf{X}}, (y\gamma)_{y \in \mathsf{Y}}\big). \tag{2.4.1}$$

One checks directly that this is a cocycle. We first prove that $\operatorname{depth}_A L = 1$ holds when C satisfies $C_+ C_1 \neq 0 \neq C_2$ by choosing β and γ so that $\phi_{\beta,\gamma}$ represents a nonzero cohomology class.

If C satisfies hypothesis (i), then one can choose elements $c, c' \in \mathsf{Y}$ with $cc' \neq 0$ and nonzero elements $b \in B_1$ and $m \in M_0$. Set

$$\beta = cc'm \quad \text{and} \quad \gamma = bcm.$$

Suppose that $\phi = \phi_{\beta,\gamma}$ is a coboundary. Then there exists $l \in L$ so that $\phi = ((xl)_{x\in\mathsf{X}}, (yl)_{y\in\mathsf{Y}})$. Write l in the form (2.2.1) and fix $y \in \mathsf{Y}_a$ for some $a \geqslant 1$. Using the decomposition in (2.2.1), one has

$$ybcm = yl_B + yl_C + yl_M. \tag{2.4.2}$$

Note that $ybcm$ starts in C_a, while yl_B starts in $C_{>a}$. Also, the length of $ybcm$ is 4 and the length of yl_M is 2. Therefore, it follows that $yl_C = yl_M = 0$ for all $y \in \mathsf{Y}$, and 2.2 implies $l_M = 0$. One can similarly deduce $\mathsf{X}l_B = 0$ by considering the equation $xcc'm = xl_B + xl_C$.

Returning to (2.4.2), one now has $ybcm = yl_B$. From 2.2, one concludes $l_B = bcm$. One can similarly deduce that $l_C = cc'm$. For each $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, one then has

$$\begin{aligned} x\beta + y\gamma = \phi(x+y) &= (x+y)(\beta+\gamma) \\ &= x\beta + x\gamma + y\beta + y\gamma. \end{aligned}$$

Therefore, $x\gamma + y\beta = 0$ holds. Arguing as before, we obtain $x\gamma = y\beta = 0$ for each $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, and hence $yc = 0$ for each $y \in \mathsf{Y}$, contradicting our choice of c.

If B satisfies hypotheses (i), choose $b, b' \in \mathsf{X}$ such that $bb' \neq 0$. For any $c \in \mathsf{Y}$, one can argue as above, taking $\beta = cbcm$ and $\gamma = bb'cm$ to deduce that $\phi_{\beta,\gamma}$ is not a coboundary.

For case (ii), assume that $C_+ = C_1$ and $\dim_k C_1 \geqslant 2$, so that we may choose $c, c' \in \mathsf{Y}$. In the notation from (2.4.1), we set $\phi = (0, (y\gamma_y)) \in L^{\mathsf{X}} \oplus L^{\mathsf{Y}}$ where

$$\gamma_y = \begin{cases} bcm, & y = c, \\ bc'm, & y \neq c. \end{cases}$$

One checks directly that ϕ is a cocycle.

If ϕ is a coboundary, then there exists an $l \in L$ so that $\phi = ((xl)_{x\in\mathsf{X}}, (yl)_{y\in\mathsf{Y}})$. Arguing as before, one gets $ybcm = yl_B$. Since the elements of Y are left non-zerodivisors on elements of L starting in B, one has $l_B = bcm$. However, one also gets by a similar argument that $l_B = bc'm$, a contradiction. Therefore, ϕ is not a coboundary. One may give a similar argument if the hypothesis on B in case (ii) is satisfied. $\square$

Definition 2.5. For a graded connected k-algebra B and a graded left B-module M, one defines the depth of M over B by means of the formula

$$\operatorname{depth}_B M = \inf\bigl\{n \in \mathbb{N} \bigm| \operatorname{Ext}^n_A(k, M) \neq 0\bigr\}.$$

The following proposition appears in [5, Example 36(e)2] when B and C are universal enveloping algebras of graded Lie algebras. Our argument is an adaptation of what appears there.

Proposition 2.6. *Let B and C be graded connected k-algebras with $B_1 \neq 0 \neq C_1$, and let M be a graded left B-module with $M_i = 0$ for $i < 0$ and $M_0 \neq 0$. One then has*

$$\operatorname{depth}_A(A \otimes_B M) \geqslant 1.$$

In particular, equality holds if any of the conditions of Proposition 2.4 *are satisfied.*

Proof. We show that for any $b \in B_1 \setminus \{0\}$, $c \in C_1 \setminus \{0\}$, one of bl or cl is nonzero. If $l_M \neq 0$, then one has $cl_M \neq 0$, and hence every term in cl_M starts in C and has length 2. But every term in cl_B has length at least 3, and each term in cl_C starts in $C_{\geqslant 2}$. Therefore, none of the terms in cl_B or cl_C can cancel the terms in cl_M, and hence $cl \neq 0$. One can similarly argue that if one of l_B or l_C is nonzero, then cl_B or bl_C, respectively, is nonzero. $\square$

Lemma 2.7. *If $B \cong k[x]/(x^2)$ and $C \cong k[y]/(y^2)$, then for each A-module L, one has*

$$\operatorname{depth}_A L \leqslant 1.$$

In particular, for each B-module M, one has $\operatorname{depth}_A(A \otimes_B M) = 1$.

Proof. Consider the k-subalgebra D of A generated by $xy + yx$. The algebra D is a central polynomial subalgebra such that A is a free graded left and right D-module with basis $\{1, x, y, xy\}$. In such a situation, the depth of A-modules may be instead computed over D, see [1, Corollary A.7]. Therefore, the depth of A-modules are bounded above by the global dimension of D, which is one. One may now appeal to Proposition 2.6 for the final claim. $\square$

We conclude this section by collecting some numerical data on modules over coproducts of algebras.

2.8. Let M be a graded free k-vector space with $M_i = 0$ for $i \ll 0$ and $\dim_k M_i$ finite for all i. The Hilbert series of M is the formal Laurent series

$$M(t) = \sum_i \dim_k M_i t^i.$$

If $M(t)$ is defined, we say that M has a Hilbert series.

2.9. Let M and N be graded vector spaces with Hilbert series. One then has

$$(M \otimes_k N)(t) = M(t)N(t). \tag{2.9.1}$$

Let $\mathbb{T}_k(M)$ denote the tensor algebra of M over k. If $M_i = 0$ for $i \leqslant 0$, then one also has

$$\mathbb{T}_k(M)(t) = \frac{1}{1 - M(t)}. \tag{2.9.2}$$

The first claim is clear. For the second, note that since $M_i = 0$ for $i \leqslant 0$, $\mathbb{T}_k(M)$ has a Hilbert series. Furthermore, there is an isomorphism of graded vector spaces $\mathbb{T}_k(M) \cong k \oplus (M \otimes_k \mathbb{T}_k(M))$. The desired result follows.

Lemma 2.10. *Let B and C be graded connected k-algebras, M a graded right B-module, and suppose that B, C, and M have Hilbert series. One then has*

$$(A \otimes_B M)(t) = \frac{M(t)C(t)}{B(t) + C(t) - B(t)C(t)}.$$

Proof. The basis and multiplication table of A given in 2.1 shows there is an isomorphism of right B-modules

$$A \cong C \otimes_k \mathbb{T}_k(B_+ \otimes_k C_+) \otimes_k B.$$

Tensoring over B with M on the right gives

$$A \otimes_B M \cong \big(C \otimes_k \mathbb{T}_k(B_+ \otimes_k C_+)\big) \otimes_k M.$$

A computation of Hilbert series using (2.9.1) and (2.9.2) gives the desired equality. □

3. The Yoneda algebra

The notation and assumptions from Theorem 1.8 are in force throughout the section.

3.1. For a ring A and an A-module L, we say L has a Poincaré series if there exists a free resolution of L such that each free module is finitely generated. In this case, one then defines the Poincaré series of L over A to be the formal power series

$$P_L^A(t) := \sum_i \dim_k \operatorname{Ext}_A^i(L,k)t^i.$$

For example, if S and M are as in 1.3(1) with S noetherian and M finitely generated, M has a Poincaré series.

In addition to the setup given in Theorem 1.8, we assume for the remainder of the section that M and k have Poincaré series over S, and that k has a Poincaré series over T.

Theorem 3.2. *One has an equality of formal power series*

$$\frac{1}{P_M^R(t)} = \frac{P_k^S(t)}{P_M^S(t)}\left(\frac{1}{P_k^S(t)} + \frac{1}{P_k^T(t)} - 1\right).$$

Proof. One may describe the basis G in Theorem 1.8 as a basis of the k-vector space ${}^k\mathsf{F} \otimes_k \mathbb{T}_k({}^k\mathsf{E}_{\geqslant 1} \otimes_k {}^k\mathsf{F}_{\geqslant 1}) \otimes_k {}^k\mathsf{P}$. Therefore, (2.9.1) and (2.9.2) give

$$G(t) = \frac{P(t)F(t)}{1 - (E(t)-1)(F(t)-1)}.$$

Since G is a minimal resolution of M over R, the formula above gives the desired equality. □

The equality of Poincaré series given in Theorem 3.2 was first obtained in [6] for $M = k$ and in [3, Theorem 1] in general.

3.3. Let $\mathcal{R} = \operatorname{Ext}_R(k,k)$, $\mathcal{S} = \operatorname{Ext}_S(k,k)$ and $\mathcal{T} = \operatorname{Ext}_T(k,k)$ denote the Ext algebras of R, S, and T, respectively. The functor $\operatorname{Ext}_-(k,k)$ applied to the diagram of homomorphisms of rings

$$\begin{tikzcd} R \arrow[r, "\tau"] \arrow[d, "\sigma"'] & T \arrow[d, "\pi_T"] \\ S \arrow[r, "\pi_S"'] & k \end{tikzcd} \quad \text{induces a diagram of graded algebras} \quad \begin{tikzcd} \mathcal{R} & \mathcal{T} \arrow[l, "\tau^*"'] \\ \mathcal{S} \arrow[u, "\sigma^*"] & k \arrow[l, "\pi_S^*"] \arrow[u, "\pi_T^*"'] \end{tikzcd}$$

and hence defines a unique homomorphism of graded k-algebras

$$\phi : \mathcal{S} \sqcup \mathcal{T} \to \mathcal{R}.$$

Theorem 3.4. *The homomorphism of graded k-algebras ϕ is bijective.*

For an S-module M, we let $\mathcal{M}_S$ be the graded left $\mathcal{S}$-module $\operatorname{Ext}_S(M,k)$, and let $\mathcal{M}_R$ be the graded left $\mathcal{R}$-module $\operatorname{Ext}_R(M,k)$. The homomorphism $\sigma : R \to S$ also induces a homomorphism of graded left $\mathcal{S}$-modules $\theta^* = \operatorname{Ext}_\sigma(M,k) : \mathcal{M}_S \to \mathcal{M}_R$. Since θ^* is left σ^*-equivariant, the formula $\xi \otimes \mu \mapsto \xi \cdot \theta^*(\mu)$ defines a homomorphism

$$\lambda : \mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S \to \mathcal{M}_R.$$

Theorem 3.5. *The homomorphism of graded left $\mathcal{R}$-modules λ is bijective.*

In order to prove Theorems 3.4 and 3.5, we set up notation and describe the multiplication tables for the action of $\mathcal{R}$ on $\mathcal{M}_R$.

Notation 3.6. In the notation of Construction 1.7, one has isomorphisms

$$\mathcal{M}_R \cong \operatorname{Hom}_R(G,k) \cong \operatorname{Hom}_R(G/\mathfrak{m}G,k) \cong \operatorname{Hom}_k(G/\mathfrak{m}G,k)$$

of k-vector spaces. Let $\{\xi_w^R \mid w \in \mathsf{G}\} \subseteq \operatorname{Hom}_R(G,k)$ be the graded basis dual to the image of G in $G/\mathfrak{m}(G)$. Also, let $\{\xi_e^S \mid e \in \mathsf{E}\} \subseteq \operatorname{Hom}_R(G,k)$ be the graded basis dual to the basis given by the image of E in $E/\mathfrak{p}E$. We abuse language and say that ξ_w^R starts with (respectively ends in) E if the first (respectively last) letter of w is in E. Also, we say that ξ_w^R has length n if w has length n.

Our first lemma concerns the image of words of length one.

Lemma 3.7. *For $e \in \mathsf{E}$, $f \in \mathsf{F}$, $p \in \mathsf{P}$, and $m \in M$, one has*

$$\sigma^*(\xi_e^S) = \xi_e^R, \qquad \tau^*(\xi_f^T) = \xi_f^R, \quad \textit{and} \quad \theta^*(\xi_p^S) = \xi_p^R.$$

Proof. Set

$$D' = {}^R(\mathsf{D} \setminus \mathsf{E}) + \mathfrak{q}E \subseteq D.$$

The definition of ∂^D shows that D' is a subcomplex of D. Also, since $R/\mathfrak{q} \cong S$ one has $D/D' = E$ as complexes of R-modules. Let $\epsilon^R : D \to k$ and $\epsilon^S : E \to k$ be the augmentation maps, and let ψ be the canonical surjection $\psi : D \to D/D' = E$. One then has $\epsilon^R = \epsilon^S \psi$, and hence $\sigma^*(\xi_e^S) := \xi_e^S \psi = \xi_e^R$, as desired. The other cases are similar. $\square$

Next we provide a partial multiplication table for the left action of $\mathcal{R}$ on $\mathcal{M}_R$.

Lemma 3.8. *Let w be a word $\mathsf{D} \cup \mathsf{G}$, x be a letter in $\mathsf{E} \cup \mathsf{F}$, and let $m \in M$. Let $l(w)$ denote the first letter of w and let $r(w)$ denote the last letter of w. Then the left action of $\mathcal{R}$ on $\mathcal{M}_R$ satisfies*

$$\xi_x^R \cdot \xi_w^R = \begin{cases} \xi_{fw}^R & \text{if } l(w) \in \mathsf{E} \cup \mathsf{P} \text{ and } x = f \in \mathsf{F}, \\ \xi_{ew}^R & \text{if } l(w) \in \mathsf{F} \text{ and } x = e \in \mathsf{E}. \end{cases}$$

Proof. Let $w = ew' \in \mathsf{G}_i$, with $e \in \mathsf{E}_j$. We define a chain map $\psi_w \in \mathrm{Hom}_R(G, D)_{-i}$ such that $\epsilon^R \psi_w = \xi_w^R$ as follows. Set

$$\mathsf{G}^w = \{ v \in \mathsf{G} \mid v \notin (\mathsf{G}w \cup \mathsf{G}(\mathsf{E}_{\geqslant j+1})w') \},$$

where $\mathsf{G}w$ denotes elements of G that end in w (including w), and $\mathsf{G}(\mathsf{E}_{\geqslant j+1})w'$ denotes elements of G ending in a letter of $\mathsf{E}_{\geqslant j+1}$, followed by w'. Let ${}^R[\mathsf{G}^w]$ be the free R-module generated by G^w.

The definition of ∂^G shows ${}^R[\mathsf{G}^w]$ is a subcomplex of G. As graded R-modules, $G/{}^R[\mathsf{G}^w]$ is isomorphic to ${}^R[\mathsf{G}w] \oplus {}^R[\mathsf{G}(\mathsf{E}_{\geqslant j+1})w']$. For $v'w \in \mathsf{G}w$, set $\alpha(v'w) = v'$, and extend α by R-linearity to a homomorphism $\alpha: {}^R[\mathsf{G}w] \to D$. One then has

$$\alpha(\partial(v'w)) = \alpha(\partial(v')w) = \partial(v') = \partial(\alpha(v'w)).$$

Let B be the subcomplex of $G/{}^R[\mathsf{G}^w]$ spanned by $\mathsf{G}(\mathsf{E}_{\geqslant j+1})w' \cup \{w\}$, and let C be the resolution of T as an R-module given in Example 1.11. As C is acyclic, and $\mathsf{G}(\mathsf{E}_{\geqslant j+1})w' \cup \{w\}$ is a basis of B, there exists a chain map $\beta: B \to C$ of degree $-i$ satisfying $\beta(w) = 1 \in C_0$.

Now one can extend α to all of ${}^R[\mathsf{G}w] \oplus {}^R[\mathsf{G}(\mathsf{E}_{\geqslant j+1})w']$ by defining it on B to be the composition $B \xrightarrow{\beta} C \hookrightarrow D$. Note that the words in the image of B under α end in letters from E. Let ψ_w denote the composition $G \twoheadrightarrow G/{}^R[\mathsf{G}^w] \xrightarrow{\alpha} D$. One then has $\epsilon^R \psi_w = \xi_w^R$, and hence $\xi_f^R \cdot \xi_w^R = \xi_f^R \psi_w$ for each element f of F.

By construction, one has $\xi_f^R \psi_w(fw) = 1$. We show that $\xi_f^R \psi_w(v)$ is zero for all other basis elements v. If $v \in \mathsf{G}^w$, then $\psi_w(v) = 0$. If $v \in \mathsf{G}w$, then write $v = v'w$. Then

$$\xi_f^R \psi_w(v) = \xi_f^R(v') = \begin{cases} 1 & \text{if } v' = f, \\ 0 & \text{otherwise.} \end{cases}$$

If $v \in \mathsf{G}(\mathsf{E}_{\geqslant j+1})w'$, then $\psi_w(v)$ is in the span of words ending in E. Hence $\xi_f^R \psi_w(v) = 0$. The other cases of the left action are similar, and often easier. $\square$

Proof of Theorem 3.4. Under the hypothesis of the section, the k-algebras $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ are degree-wise finite. Theorem 3.2 together with Lemma 2.10 show that $\mathcal{R}(t) = (\mathcal{S} \sqcup \mathcal{T})(t)$. As ϕ is a homogeneous k-linear map, it suffices to prove that it is surjective. Lemma 3.8 shows that

$$\{ \xi_e^R \mid e \in \mathsf{E} \} \cup \{ \xi_f^R \mid f \in \mathsf{F} \}$$

generates $\mathcal{R}$ as a k-algebra, and Lemma 3.7 shows that these generators are in the image of ϕ. $\square$

Proof of Theorem 3.5. Under the hypothesis of the section, the graded $\mathcal{R}$-modules $\mathcal{M}_R$ and $\mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S$ are degree-wise finite. Lemma 2.10 and Theorem 3.2 show that the Hilbert series of $\mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S$ and $\mathcal{M}_R$ are equal, so it is enough to prove that λ is surjective. By Lemma 3.8, $\mathcal{M}_R$ is generated as a left $\mathcal{R}$-module by $\{\xi_p^R \mid p \in \mathsf{P}\}$. By Lemma 3.7, $\lambda(1 \otimes \xi_p^S) = \xi_p^R$, and hence λ is surjective. $\square$

The functor $\mathrm{Ext}_R(-, k)$ has a property similar to the one given in Theorem 3.4. We use below the non-standard notation of $\mu^* = \lambda(1 \otimes \mathrm{Ext}_S(\mu, k))$ and similarly define ν^*.

Theorem 3.9. *Let M be an S-module, N be a T-module, and V be a k-vector space for which there exists exist surjective π_S and π_T-equivariant homomorphisms $M \xrightarrow{\mu} V \xleftarrow{\nu} N$ with $\ker\mu = \mathfrak{p}M$ and $\ker\nu = \mathfrak{q}N$. The exact sequence of R-modules*

$$0 \to M \times_V N \xrightarrow{\iota} M \times N \xrightarrow{\mu-\nu} V \to 0$$

induces an exact sequence of graded left $\mathcal{R}$-modules

$$0 \to \mathcal{R} \otimes_k V^* \xrightarrow{(\mu^*,-\nu^*)} \mathcal{M}_R \times \mathcal{N}_R \xrightarrow{\iota^*} \mathcal{L} \to 0$$

where $\mathcal{L} = \operatorname{Ext}_R(M \times_V N, k)$, and $V^* = \operatorname{Hom}_k(V, k)$. In particular, one has

$$P^R_{M\times_V N}(t) = P^R_M(t) + P^R_N(t) - (\operatorname{rank}_k V)P^R_k(t).$$

Proof. The sequence of R-modules defining $M \times_V N$ induces an exact sequence of graded $\mathcal{R}$-modules

$$\Sigma^{-1}\mathcal{L} \to \mathcal{R} \otimes_k V^* \xrightarrow{(\mu^*,-\nu^*)} \mathcal{M}_R \times \mathcal{N}_R \xrightarrow{\iota^*} \mathcal{L} \to \Sigma\big(\mathcal{R} \otimes_k V^*\big).$$

Thus, we need to show that $(\mu^*, -\nu^*)$ is injective. Set $n = \operatorname{rank}_k V$, and let F be a free S-module of rank n. One then has S-linear maps $F \to M \xrightarrow{\mu} V$ that induce homomorphisms of graded left $\mathcal{S}$-modules $\operatorname{Ext}_S(V, k) \to \mathcal{M}_S \to \operatorname{Ext}_S(F, k)$. Tensoring with $\mathcal{R}$ over $\mathcal{S}$ on the left and using Theorem 3.5, one obtains the top row of the diagram:

$$\begin{tikzcd}
\mathcal{R} \otimes_{\mathcal{S}} \operatorname{Ext}_S(V,k) \arrow[r, "\mu^*"] & \mathcal{M}_R \arrow[r] & \operatorname{Ext}_R(F,k) \\
\mathcal{R} \otimes_k V^* \arrow[u, "\cong"] \arrow[rr] & & \big(\mathcal{R}/\mathcal{R}\mathcal{S}^+\big)^n \arrow[u, "\cong"']
\end{tikzcd}$$

Note that $\mathcal{R} \otimes_k V^* \cong \mathcal{R}^n$. The kernel of the bottom arrow is clearly $(\mathcal{R}\mathcal{S}^+)^n$, and therefore the kernel of μ^* is contained in the image of $(\mathcal{R}\mathcal{S}^+)^n$ under the left vertical isomorphism. Similarly, the kernel of ν^* is contained in the image of $\mathcal{R}\mathcal{T}^+$. Hence one has

$$\operatorname{Ker}\big(\mu^*, -\nu^*\big) = \operatorname{Ker}\mu^* \cap \operatorname{Ker}\nu^* \subseteq \mathcal{R}\mathcal{S}^+ \cap \mathcal{R}\mathcal{T}^+ = 0. \qquad \square$$

3.10. In the situation of 1.3(2), the maps λ and ϕ from Theorems 3.4 and 3.5 are homomorphisms of bigraded algebras and modules, respectively. We therefore have the following proposition.

Recall that a graded module M over a graded connected algebra A is said to be *Koszul* when $\operatorname{Ext}^i_A(M,k)^j = 0$ if $j \neq i$. A graded connected algebra A is Koszul if k is Koszul as an A-module. The equivalence of the first two conditions was proved in [2, Theorem 1.c].

Proposition 3.11. *The following conditions are equivalent.*

(i) *The algebra R is Koszul.*
(ii) *The algebras S and T are Koszul.*
(iii) *There exists an S-module M that is Koszul as an R-module.*

4. Depth of cohomology modules

The notation and conventions from 3.3 are still in force.

In this section, we compute the depth of the cohomology module of an R-module, where R is the fiber product of commutative noetherian local rings $(S, \mathfrak{p}, k)$ and $(T, \mathfrak{q}, k)$. For uses of this invariant, see [4] or [1].

Theorem 4.1. *Let $(S, \mathfrak{p}, k)$ and $(T, \mathfrak{q}, k)$ be commutative noetherian local rings, set $R = S \times_k T$, and let L be a nonzero finitely generated R-module. If $\mathfrak{p} \neq 0$ and $\mathfrak{q} \neq 0$, then one has*

$$\operatorname{depth}_{\mathcal{R}} \mathcal{L} \leqslant 1.$$

Equality holds if L is an S-module or a T-module.

In order to prove Theorem 4.1, we use an observation of Dress and Krämer in [3, Remark 3]. Recall that the syzygy $\Omega_1^R L$ of an R-module L is the kernel of a (graded) free cover $F \to L$; it is defined uniquely up to isomorphism; for $n \geqslant 2$ one sets $\Omega_n^R L = \Omega_1^R \Omega_{n-1}^R L$. Every finitely generated R-module L has a free cover $\varphi : Q \to L$ such that $\ker \varphi \subseteq \mathfrak{m} Q$.

Proposition 4.2. *Let L be a left R-module. Then $\Omega_2^R(L) \cong M \oplus N$ where M is an S-module and N is a T-module.*

Proof. Recall that the maximal ideal of R is $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$. Let $\varphi : Q \to Q'$ be a minimal free presentation of L over R. One then has

$$\begin{aligned} \Omega_2^R(L) = \operatorname{Ker} \varphi &= \operatorname{Ker} \varphi \cap \mathfrak{m} A \\ &= \operatorname{Ker} \varphi \cap (\mathfrak{p} Q \oplus \mathfrak{q} Q) \\ &= (\operatorname{Ker} \varphi \cap \mathfrak{p} Q) \oplus (\operatorname{Ker} \varphi \cap \mathfrak{q} Q). \end{aligned}$$

To see the last equality, suppose that (x_1, x_2) in $\mathfrak{p} Q \oplus \mathfrak{q} Q$ satisfies $\varphi((x_1, x_2)) = 0$. Note that $\varphi((x_1, 0))$ is in $\varphi(\mathfrak{p} Q) \subseteq \mathfrak{p} Q'$ and $\varphi((0, x_2))$ is in $\varphi(\mathfrak{q} Q) \subseteq \mathfrak{q} Q'$. Also, $\mathfrak{p} Q' \cap \mathfrak{q} Q' = 0$, hence $\varphi((x_1, 0)) = 0 = \varphi((0, x_2))$.

Take $M = \operatorname{Ker} \varphi \cap \mathfrak{p} Q$ and $N = \operatorname{Ker} \varphi \cap \mathfrak{q} Q$ to get the desired result. □

By putting together Theorem 3.5 and Proposition 4.2, we obtain a nearly complete description of the cohomology of R-modules.

Corollary 4.3. *Set $\mathcal{L} = \operatorname{Ext}_R(L, k)$, and let $\mathcal{N}_T = \operatorname{Ext}_T(N, k)$. There is then an exact sequence of graded left $\mathcal{R}$-modules*

$$0 \to \left(\Sigma^{-2} \mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S\right) \oplus \left(\Sigma^{-2} \mathcal{R} \otimes_{\mathcal{T}} \mathcal{N}_T\right) \to \mathcal{L} \to \mathcal{L}/\mathcal{L}^{\geqslant 2} \to 0.$$

We can now prove Theorem 4.1.

Proof of Theorem 4.1. If $\operatorname{pd}_R L$ is finite, then $\operatorname{rank}_k \mathcal{L}$ is finite, hence $\operatorname{depth}_{\mathcal{R}} \mathcal{L} = 0$. If $\operatorname{pd}_R L = \infty$, then $\Omega_R^2(L) \neq 0$, and by Proposition 4.2, we have $\Omega_R^2(L) = M \oplus N$ for some S-module M and some T-module N, with M or N nonzero.

Set $\mathcal{X} := \operatorname{Ext}_R(M \oplus N, k)$. Corollary 4.3 provides an exact sequence

$$\operatorname{Hom}_{\mathcal{R}}\left(k, \mathcal{L}/\mathcal{L}^{\geqslant 2}\right) \xrightarrow{\vartheta} \operatorname{Ext}^1_{\mathcal{R}}\left(k, \Sigma^{-2} \mathcal{X}\right) \xrightarrow{\psi} \operatorname{Ext}^1_{\mathcal{R}}(k, \mathcal{L}).$$

If $\mathcal{S} \ncong k[x]/(x^2)$ or $\mathcal{T} \ncong k[y]/(y^2)$, then Proposition 2.4 shows that the module $\mathrm{Ext}^1_{\mathcal{R}}(k, \Sigma^{-2}\mathcal{X})^{\geqslant 2}$ is nonzero. However, $\mathrm{Hom}_{\mathcal{R}}(k, \mathcal{L}/\mathcal{L}^{\geqslant 2})$ is nonzero only in internal degree zero and one. Therefore, ϑ is not surjective, which provides the desired inequality. If L is an S or T-module, then Proposition 2.6 gives equality.

Otherwise, one has $\mathcal{R} \cong k[x]/(x^2) \sqcup k[y]/(y^2)$, and both the inequality and equality follow from Lemma 2.7. $\square$

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