September 7, 2002

PHY 113 – "Derivation" of projectile motion equations

In class, we have stressed the notion that it is important to understand the relationships between the basic equations of physics and the relations that can be derived from them. While these dreaded derivations are not generally a favorite lecture topic, they are very important for understanding the meaning and the proper use of the equations. These notes show an example of a simple derivation for projectile motion. While this derivation is not as formal nor as rigorous as a mathematical "proof", it is adequate for these purposes.

The basic equations for our starting point, are the equations which define the relationships between position, velocity, and acceleration. For this purpose, we will assume that we can describe these quantities in the $x-y$ plane:

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}, \tag{1}$$

$$\mathbf{v}(t) = v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}}, \tag{2}$$

$$\mathbf{a}(t) = a_x(t)\hat{\mathbf{x}} + a_y(t)\hat{\mathbf{y}}. \tag{3}$$

The velocity is the rate of change of position:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}, \tag{4}$$

and the acceleration is the rate of change of velocity:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}. \tag{5}$$

Additional relationships can be obtained by taking the antiderivatives. The equations (1-5) are general definitions.

Now, let us consider a special case – where the acceleration is constant in space and time and is given by

$$\mathbf{a}(t) = -g\hat{\mathbf{y}}, \tag{6}$$

with $g = 9.8\text{m/s}^2$, representing the acceleration of gravity in the $-\hat{\mathbf{y}}$ direction near the surface of the earth. By evaluating the relationships (1-5) for this case, we find:

$$v_x(t) = v_{xi} \qquad v_y(t) = v_{yi} - gt \tag{7}$$

and

$$x(t) = x_i + v_{xi}t \qquad y(t) = y_i + v_{yi}t - \frac{1}{2}gt^2, \tag{8}$$

where x_i, y_i, v_{xi}, v_{yi} denote initial ($t = 0$) positions and velocities.

The equations (6-8) describes how an object moves in a plane. If we know the constants x_i, y_i, v_{xi}, v_{yi}, we can determine its position and velocity at any time $t \geq 0$. Now our task is to use these relationships (1-8) to find the direct relationship between the x and y. This will enable us to spatially trace out the path of the object as it moves on its trajectory. We can do this by manipulating Eq. (8) with the following steps:

1. Solve the first equation for t:

$$x(t) = x_i + v_{xi}t \quad \Rightarrow \quad t = \frac{x(t) - x_i}{v_{xi}}. \tag{9}$$

2. Replace t in the second equation with the expression above:

$$y(t) = y_i + v_{yi}\left(\frac{x(t) - x_i}{v_{xi}}\right) - \frac{1}{2}g\left(\frac{x(t) - x_i}{v_{xi}}\right)^2. \tag{10}$$

This result, shows that for every horizontal position of the object $x(t)$, the vertical position $y(t)$ traces out a parabolic path.