

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103
Plan for Lecture 10:**

Continue reading Chapter 3 & 6

- 1. Constants of the motion**
- 2. Conserved quantities**
- 3. Legendre transformations**

9/16/2018

PHY 711 Fall 2019 -- Lecture 10

1

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

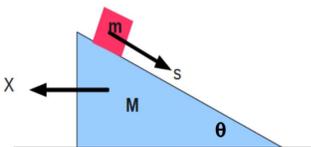
Date	F&W Reading	Topic	Assignment	Due
1 Mon, 8/26/2019	Chap. 1	Introduction	#1	8/30/2019
2 Wed, 8/28/2019	Chap. 1	Scattering theory	#2	9/02/2019
3 Fri, 8/30/2019	Chap. 1	Scattering theory	#3	9/04/2019
4 Mon, 9/02/2019	Chap. 1	Scattering theory	#4	9/06/2019
5 Wed, 9/04/2019	Chap. 2	Non-inertial coordinate systems	#5	9/09/2019
6 Fri, 9/06/2019	Chap. 3	Calculus of Variation	#6	9/11/2019
7 Mon, 9/9/2019	Chap. 3	Calculus of Variation	#7	9/13/2019
8 Wed, 9/11/2019	Chap. 3	Lagrangian Mechanics		
9 Fri, 9/13/2019	Chap. 3	Lagrangian Mechanics	#8	9/16/2019
10 Mon, 9/16/2019	Chap. 3 & 6	Constants of the motion	#9	9/20/2019
11 Wed, 9/18/2019	Chap. 3 & 6	Hamiltonian equations of motion		

9/16/2018

PHY 711 Fall 2019 -- Lecture 10

2

Homework #9



- The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ , m , M , etc.) and solve for the equations of motion, assuming that the system is initially at rest. (Note that X and s represent components of vectors whose directions are related by the angle θ .)

9/16/2018

PHY 711 Fall 2019 -- Lecture 10

3

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = (\text{constant})$$

9/16/2018

PHY 711 Fall 2019 – Lecture 10

4

Examples of constants of the motion:

Example 1: one - dimensional potential :

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} m\dot{x} = 0 \quad \Rightarrow m\dot{x} \equiv p_x \quad (\text{constant})$$

$$\Rightarrow \frac{d}{dt} m\dot{y} = 0 \quad \Rightarrow m\dot{y} \equiv p_y \quad (\text{constant})$$

$$\Rightarrow \frac{d}{dt} m\dot{z} = -\frac{\partial V}{\partial z}$$

9/16/2018

PHY 711 Fall 2019 – Lecture 10

5

Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt} m r^2 \dot{\phi} = 0 \quad \Rightarrow m r^2 \dot{\phi} \equiv p_\phi \quad (\text{constant})$$

$$\Rightarrow \frac{d}{dt} m \dot{r} = m r \dot{\phi}^2 - \frac{\partial V}{\partial r} = \frac{p_\phi^2}{m r^3} - \frac{\partial V}{\partial r}$$

9/16/2018

PHY 711 Fall 2019 – Lecture 10

6

Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Also note that:

$$\begin{aligned} \frac{dL}{dt} &= \sum_\sigma \frac{\partial L}{\partial q_\sigma} \dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \ddot{q}_\sigma + \frac{\partial L}{\partial t} \\ &= \frac{d}{dt} \left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) + \frac{\partial L}{\partial t} \\ &\Rightarrow \frac{d}{dt} \left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t} \end{aligned}$$

9/16/2018 PHY 711 Fall 2019 - Lecture 10 7

Additional constant of the motion:

If $\frac{\partial L}{\partial t} = 0$;

then: $\frac{d}{dt} \left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma = -E \text{ (constant)}$$

Example 1: one-dimensional potential:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2 \right) = 0$$

$$\Rightarrow - \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(z) \right) = -E \text{ (constant)}$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \dot{z}^2 + V(z)$$

9/16/2018 PHY 711 Fall 2019 - Lecture 10 8

Additional constant of the motion -- continued:

If $\frac{\partial L}{\partial t} = 0$;

then: $\frac{d}{dt} \left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma = -E \text{ (constant)}$$

Example 2: Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) - m\dot{r}^2 - m r^2 \dot{\phi}^2 \right) = 0$$

$$\Rightarrow - \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r) \right) = -E \text{ (constant)}$$

For this case, we also have $m r^2 \dot{\phi} \equiv p_\phi$

$$\Rightarrow E = \frac{p_\phi^2}{2m r^2} + \frac{1}{2} m \dot{r}^2 + V(r)$$

9/16/2018 PHY 711 Fall 2019 - Lecture 10 9

Other examples

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{x}y + y\dot{x})$$

$$\frac{\partial L}{\partial z} = 0 \Rightarrow m\dot{z} = p_z \text{ (constant)}$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{x}y + y\dot{x})$$

$$- \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{2c}B_0(-\dot{x}y + y\dot{x})$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{p_z^2}{2m}$$

9/16/2018

PHY 711 Fall 2019 – Lecture 10

10

Other examples

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c}B_0\dot{x}y$$

$$\frac{\partial L}{\partial z} = 0 \Rightarrow m\dot{z} = p_z \text{ (constant)}$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow m\dot{x} = p_x \text{ (constant)}$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c}B_0\dot{x}y$$

$$- \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}B_0\dot{x}y$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

9/16/2018

PHY 711 Fall 2019 – Lecture 10

11

Lagrangian picture

For independent generalized coordinates $q_{\sigma}(t)$:

$$L = L(\{q_{\sigma}(t)\}, \{\dot{q}_{\sigma}(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} = 0$$

\Rightarrow Second order differential equations for $q_{\sigma}(t)$

Switching variables – Legendre transformation

9/16/2018

PHY 711 Fall 2019 – Lecture 10

12

Mathematical transformations for continuous functions of several variables & Legendre transforms:
 Simple change of variables:
 $z(x, y) \Leftrightarrow x(y, z) ???$

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

But: $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y}$

9/16/2018 PHY 711 Fall 2019 – Lecture 10 13

Simple change of variables -- continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y} \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{(\partial z / \partial x)_y}$$

9/16/2018 PHY 711 Fall 2019 – Lecture 10 14

Simple change of variables -- continued:

Example: $z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$
 $z(x, y) = e^{x^2+y}$
 $x(y, z) = (\ln z - y)^{1/2} \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y} \quad \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{(\partial z / \partial x)_y}$$

$$\frac{1}{2(\ln z - y)^{1/2}} \overset{\checkmark}{=} -\frac{e^{x^2+y}}{2xe^{x^2+y}} \quad \frac{1}{2z(\ln z - y)^{1/2}} \overset{\checkmark}{=} \frac{1}{2xe^{x^2+y}}$$

9/16/2018 PHY 711 Fall 2019 – Lecture 10 15

Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

Let $u \equiv \left(\frac{\partial z}{\partial x}\right)_y$ and $v \equiv \left(\frac{\partial z}{\partial y}\right)_x$

Define new function

$$w(u, y) \Rightarrow dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

For $w = z - ux$, $dw = dz - udx - xdu = \cancel{u}dx + vdy - \cancel{u}dx - xdu$
 $dw = -xdu + vdy$

$$\Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \quad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

9/16/2018 PHY 711 Fall 2019 - Lecture 10 16

For thermodynamic functions:

Internal energy: $U = U(S, V)$
 $dU = TdS - PdV$
 $dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$
 $\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \quad P = -\left(\frac{\partial U}{\partial V}\right)_S$

Enthalpy: $H = H(S, P) = U + PV$
 $dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$
 $\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \quad V = \left(\frac{\partial H}{\partial P}\right)_S$

9/16/2018 PHY 711 Fall 2019 - Lecture 10 17

Name	Potential	Differential Form
Internal energy	$E(S, V, N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E, V, N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S, P, N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T, V, N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T, P, N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T, V, \mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

9/16/2018 PHY 711 Fall 2019 - Lecture 10 18

Lagrangian pictureFor independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

⇒ Second order differential equations for $q_\sigma(t)$ **Switching variables – Legendre transformation**Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

9/16/2018

PHY 711 Fall 2019 – Lecture 10

19

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

9/16/2018

PHY 711 Fall 2019 – Lecture 10

20
