

**PHY 711 Classical Mechanics and  
Mathematical Methods  
10-10:50 AM MWF Olin 103**

## **Plan for Lecture 18:**

## **Read Chapter 7 & Appendices A-D**

**Generalization of the one dimensional wave equation → various mathematical problems and techniques including:**

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  1. Sturm-Liouville equations
  2. Eigenvalues; orthogonal function expansions
  3. Green's functions methods
  4. Laplace transformation
  5. Contour integration methods

10/04/2019

PHY 711 Fall 2019 -- Lecture 18

1

Date	F&W Reading	Topic	Assignment Due
1 Mon, 8/26/2019	Chap. 1	Introduction	#1 8/30/2019
2 Wed, 8/28/2019	Chap. 1	Scattering theory	#2 9/02/2019
3 Fri, 8/30/2019	Chap. 1	Scattering theory	#3 9/04/2019
4 Mon, 9/02/2019	Chap. 1	Scattering theory	#4 9/06/2019
5 Wed, 9/04/2019	Chap. 2	Non-inertial coordinate systems	#5 9/09/2019
6 Fri, 9/06/2019	Chap. 3	Calculus of Variation	#6 9/11/2019
7 Mon, 9/09/2019	Chap. 3	Calculus of Variation	#7 9/13/2019
8 Wed, 9/11/2019	Chap. 3	Lagrangian Mechanics	
9 Fri, 9/13/2019	Chap. 3	Lagrangian Mechanics	#8 9/16/2019
10 Mon, 9/16/2019	Chap. 3 & 6	Constants of the motion	#9 9/20/2019
11 Wed, 9/18/2019	Chap. 3 & 6	Hamiltonian equations of motion	
12 Fri, 9/20/2019	Chap. 3 & 6	Liouville theorem	#10 9/23/2019
13 Mon, 9/23/2019	Chap. 3 & 6	Canonical transformations	
14 Wed, 9/25/2019	Chap. 4	Small oscillations about equilibrium	#11 9/30/2019
15 Fri, 9/27/2019	Chap. 4	Normal modes of vibration	#12 10/02/2019
16 Mon, 9/30/2019	Chap. 7	Motion of strings	#13 10/04/2019
17 Wed, 10/02/2019	Chap. 7	Sturm-Liouville equations	#14 10/07/2019
18 Fri, 10/04/2019	Chap. 7	Sturm-Liouville equations	
19 Mon, 10/07/2019	Chap. 7	Fourier transform methods	
20 Wed, 10/09/2019	Chap. 1-4,6-7	Review	
Fri, 10/11/2019	No class	Fall break	
Mon, 10/14/2019	No class	Take-home exam	
Wed, 10/16/2019	No class	Take-home exam	
21 Fri, 10/18/2019	Chap. 7	Take-home exam due	

Eigenvalues and eigenfunctions of Sturm-Liouville equations

In the domain  $a \leq x \leq b$ :

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

Alternative boundary conditions; 1.  $f_m(a) = f_m(b) = 0$

$$\text{or } 2. \tau(x) \frac{df_m(x)}{dx} \Big|_a = \tau(x) \frac{df_m(x)}{dx} \Big|_b = 0$$

$$\text{or 3. } f_m(a) = f_m(b) \text{ and } \frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$$

### Properties:

Eigenvalues  $\lambda_n$  are real

Eigenfunctions are orthogonal:  $\int_{-L}^L \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n$ ,

where  $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx$ .

10/04/2019 PHY 711 Fall 2019 -- Lecture 18

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3

### Variation approximation to lowest eigenvalue

In general, there are several techniques to determine the eigenvalues  $\lambda_n$  and eigenfunctions  $f_n(x)$ . When it is not possible to find the "exact" functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle}, \quad S(x) \equiv -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x)$$

where  $\tilde{h}(x)$  is a variable function which satisfies the correct boundary values. The "proof" of this inequality is based on the notion that  $\tilde{h}(x)$  can in principle be expanded in terms of the (unknown) exact eigenfunctions  $f_n(x)$ :

$$\tilde{h}(x) = \sum_n C_n f_n(x), \quad \text{where the coefficients } C_n \text{ can be assumed to be real.}$$

10/4/2019

PHY 711 Fall 2019 – Lecture 18

4

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4

### Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\langle \tilde{h} | S | \tilde{h} \rangle = \int_a^b \tilde{h}(x) S(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\langle \tilde{h} | \sigma | \tilde{h} \rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n,$$

$$\text{Therefore } \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \geq \lambda_0.$$

10/4/2019

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5

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5

### Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \leq \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

$$\text{Example: } -\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{with } f_n(0) = f_n(a) = 0 \\ \text{trial function } f_{\text{trial}}(x) = x(x-a)$$

$$\text{Exact value of } \lambda_0 = \frac{\pi^2}{a^2} = \frac{9.869604404}{a^2}$$

$$\text{Raleigh-Ritz estimate: } \frac{\langle x(x-a) | -\frac{d^2}{dx^2} | x(x-a) \rangle}{\langle x(x-a) | x(x-a) \rangle} = \frac{10}{a^2}$$

10/4/2019

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6

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6



Some details –  
suggested that:  $h(x) \approx \sum_n C_n f_n(x)$ ,

where  $C_n = \frac{1}{N_n} \int_a^b \sigma(x) h(x) f_n(x) dx$ .

Minimize:  $\chi(\{C_n\}) = \int_a^b dx \sigma(x) \left( h(x) - \sum_n C_n f_n(x) \right)^2$

Necessary condition for minimum:

$$\frac{d\chi}{dC_n} = 0 \quad \int_a^b dx 2\sigma(x) \left( h(x) - \sum_m C_m f_m(x) \right) f_n(x) = 0$$

Note that:  $\int_a^b \sigma(x) f_m(x) f_n(x) dx = N_n \delta_{mn}$

$$\Rightarrow C_n = \frac{1}{N_n} \int_a^b \sigma(x) h(x) f_n(x) dx$$

10/04/2019

PHY 711 Fall 2019 – Lecture 18

10

10

**Green's function solution methods**

Suppose that we can find a Green's function defined as follows:

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Among other things, this is useful for solving  
inhomogeneous equations of the type:

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \psi(x) = F(x)$$

where  $F(x)$ ,  $\tau(x)$ ,  $v(x)$ ,  $\lambda$ , and  $\sigma(x)$  are known,  
and  $\psi(x)$  is to be determined according to:

$$\psi(x) = \int_a^b dx' G(x, x') F(x')$$

10/04/2019

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11

11

**Green's function solution methods**

Suppose that we can find a Green's function defined as follows:

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Completeness of eigenfunctions:

Recall:  $\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$

In terms of eigenfunctions:

$$\begin{aligned} & \left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} \\ & \Rightarrow G_\lambda(x, x') = \sum_n \frac{f_n(x) f_n(x') / N_n}{\lambda_n - \lambda} \end{aligned}$$

10/04/2019

PHY 711 Fall 2019 – Lecture 18

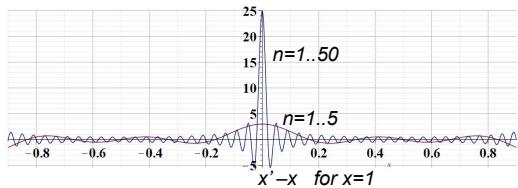
12

12



$$\text{Example: } \frac{2}{L} \sum_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) = \delta(x - x')$$

For  $L=2$



10/04/2019

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16

Green's function:

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Green's function for the example:

$$\left( -\frac{d^2}{dx^2} - \lambda \right) G_\lambda(x, x') = \delta(x - x')$$

$$G(x, x') = \sum_n \frac{f_n(x)f_n(x') / N_n}{\lambda_n - \lambda} = \frac{2}{L} \sum_n \frac{\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1}$$

10/04/2019

PHY 711 Fall 2019 – Lecture 18

17

Using Green's function to solve inhomogeneous equation:

$$\left( -\frac{d^2}{dx^2} - 1 \right) \varphi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

$$\varphi(x) = \varphi_0(x) + \int_0^L G(x, x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$\varphi(x) = \varphi_0(x) + \frac{2}{L} \sum_n \left[ \frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1} \int_0^L \sin\left(\frac{n\pi x'}{L}\right) F_0 \sin\left(\frac{\pi x'}{L}\right) dx' \right]$$

$$\varphi(x) = \varphi_0(x) + \sum_n \left[ \frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1} \delta_{nl} \right] = \varphi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1} \sin\left(\frac{\pi x}{L}\right)$$

10/04/2019

PHY 711 Fall 2019 -- Lecture 18

18

18



Digression on properties of the Wronskian --

Two homogeneous solutions

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) g_i(x) = 0 \quad \text{for } i = a, b$$

$$\text{Let: } W = \tau(x) \left( g_a(x) \frac{d}{dx} g_b(x) - g_b(x) \frac{d}{dx} g_a(x) \right)$$

$$\begin{aligned} \text{Consider: } \frac{dW}{dx} &= \frac{d}{dx} \left( \tau(x) \left( g_a(x) \frac{d}{dx} g_b(x) - g_b(x) \frac{d}{dx} g_a(x) \right) \right) \\ &= 0 \quad \Rightarrow W = \text{constant} \end{aligned}$$

10/04/2019

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22

22

For  $\epsilon \rightarrow 0$ :

$$\begin{aligned} \int_{x'-\epsilon}^{x'+\epsilon} dx \left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_A(x, x') &= \int_{x'-\epsilon}^{x'+\epsilon} dx \delta(x - x') \\ \int_{x'-\epsilon}^{x+\epsilon} dx \left( -\frac{d}{dx} \tau(x) \frac{d}{dx} \right) \frac{1}{W} g_a(x_<) g_b(x_>) &= 1 \\ -\frac{\tau(x)}{W} \left( \frac{d}{dx} g_a(x_<) g_b(x_>) \right) \Big|_{x'-\epsilon}^{x'+\epsilon} &= \frac{\tau(x')}{W} \left( g_a(x') \frac{d}{dx} g_b(x') - g_b(x') \frac{d}{dx} g_a(x') \right) \\ \Rightarrow W &= \tau(x') \left( g_a(x') \frac{d}{dx} g_b(x') - g_b(x') \frac{d}{dx} g_a(x') \right) \end{aligned}$$

Note --  $W$  (Wronskian) is constant, since  $\frac{dW}{dx'} = 0$ .

$\Rightarrow$  Useful Green's function construction in one dimension:

$$G_\lambda(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

10/04/2019

PHY 711 Fall 2019 -- Lecture 18

23

$$\left( -\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = F(x)$$

Green's function solution:

$$\begin{aligned} \varphi_\lambda(x) &= \varphi_{\lambda 0}(x) + \int_{x_l}^{x_u} G_\lambda(x, x') F(x') dx' \\ &= \varphi_{\lambda 0}(x) + \frac{g_b(x)}{W} \int_{x_l}^x g_a(x') F(x') dx' + \frac{g_a(x)}{W} \int_x^{x_u} g_b(x') F(x') dx' \end{aligned}$$

10/04/2019

PHY 711 Fall 2019 -- Lecture 18

24

24

### Example --

$$\frac{d^2}{dx^2} \Phi(x) = -\rho(x) / \epsilon_0 \quad \text{electrostatic potential for charge density } \rho(x)$$

Homogeneous equation:

$$\frac{d^2}{dx^2}g_{a,b}(x)=0$$

Let  $g_a(x) = x$        $g_b(x) = 1$

Wronskian:

$$W = g_a(x) \frac{dg_b(x)}{dx} - g_b(x) \frac{dg_a(x)}{dx} = -1$$

Green's function:

$$G(x, x') = -x_<$$

$$\Phi(x) = \Phi_0(x) + \frac{1}{\epsilon_0} \int_{-\infty}^x dx' x' \rho(x') + \frac{x}{\epsilon_0} \int_x^{\infty} dx' \rho(x')$$

10/04/2019

PHY 711 Fall 2019 -- Lecture 18

25

25

### Example -- continued

$$\frac{d^2}{dx^2} \Phi(x) = -\rho(x) / \epsilon_0 \quad \text{electrostatic potential for charge density } \rho(x)$$

$$\Phi(x) = \Phi_0(x) + \frac{1}{\epsilon_0} \int_{-\infty}^x dx' x' \rho(x') + \frac{x}{\epsilon_0} \int_x^{\infty} dx' \rho(x')$$

$$\text{Suppose } \rho(x) = \begin{cases} \rho_0 x / a & -a \leq x \leq a \\ 0 & x \geq a \end{cases}$$

$$\Phi(x) = \Phi_0(x) + \begin{cases} 0 & x \leq -a \\ \frac{\rho_0}{\epsilon_0 a} \left( \frac{a^3}{3} + \frac{x a^2}{2} - \frac{x^3}{6} \right) & -a \leq x \leq a \\ \frac{2}{3\epsilon_0} \rho_0 a^2 & x \geq a \end{cases}$$

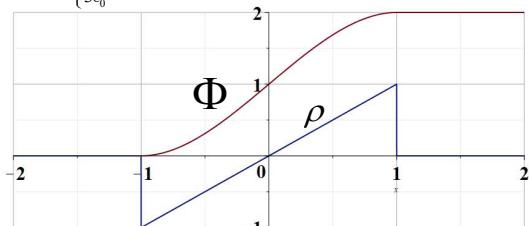
10/04/2019

PHY 711 Fall 2019 – Lecture 18

26

26

$$\Phi(x) = \begin{cases} 0 & x \leq -a \\ \frac{\rho_0}{\epsilon_0 a} \left( \frac{a^3}{3} + \frac{x a^2}{2} - \frac{x^3}{6} \right) & -a \leq x \leq a \\ \frac{2}{3\epsilon_0} \rho_0 a^2 & x \geq a \end{cases}$$



10/04/20