

Special case: $\tau(x) = 1 = \sigma(x)$ $v(x) = 0$

$$-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{for } 0 \leq x \leq a, \quad \text{with } f_n(0) = f_n(a) = 0$$

$$f_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2$$

Fourier series representation of function $h(x)$ in the interval $0 \leq x \leq a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$A_n = \sqrt{\frac{2}{a}} \int_0^a dx' h(x') \sin\left(\frac{n\pi x'}{a}\right)$$

*Note that if $h(x)$ does not vanish at $x = 0$ and $x = a$, the more general

$$\text{expression applies: } h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) + \sum_{n=0}^{\infty} B_n \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right)$$

(with some restrictions).

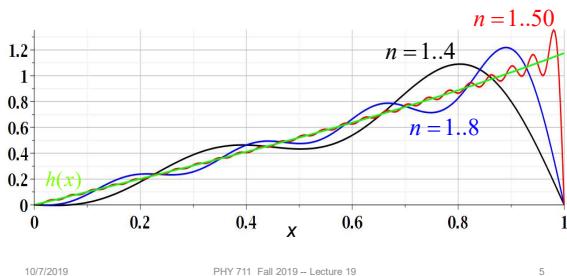
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Example

$$h(x) = \sinh(x) \approx 2\pi \sinh(1) \left(\frac{\sin(\pi x)}{\pi^2 + 1} - \frac{2\sin(2\pi x)}{4\pi^2 + 1} + \dots - (-1)^n n \frac{\sin(n\pi x)}{n^2 \pi^2 + 1} + \dots \right)$$



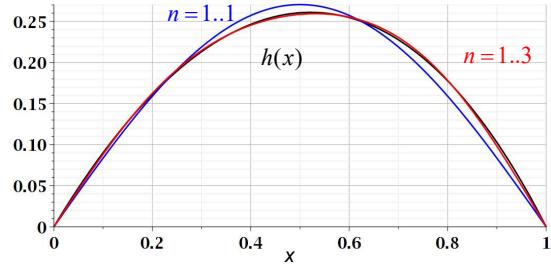
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Example

$$h(x) = (1-x)\sinh(x) \approx 4\pi \left(\frac{\sin(\pi x)(1+\sinh(1))}{(\pi^2+1)^2} + \frac{2\sin(2\pi x)(1-\sinh(1))}{(4\pi^2+1)^2} + \dots + n \frac{\sin(n\pi x)(1-(-1)^n \sinh(1))}{(n^2\pi^2+1)^2} + \dots \right)$$



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Fourier series representation of function $h(x)$ in the interval $0 \leq x \leq a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \text{ with } A_n = \sqrt{\frac{2}{a}} \int_0^a dx' h(x') \sin\left(\frac{n\pi x'}{a}\right)$$

Can show that the series converges provided that $h(x)$ is

piecewise continuous.

Note that this analysis can also apply to time dependent functions. In the remainder of the lecture, we will consider time dependent functions.

$$x \rightarrow t \quad a \rightarrow T \quad 0 \leq t \leq T \quad \frac{n\pi}{a} \rightarrow \frac{n\pi}{T} \equiv \omega_n$$

$$h(t) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{T}} \sin(\omega_n t) \quad A_n = \sqrt{\frac{2}{T}} \int_0^T dt' h(t') \sin(\omega_n t')$$

Note that for this finite time range, Fourier series is discrete in frequency and continuous in time.

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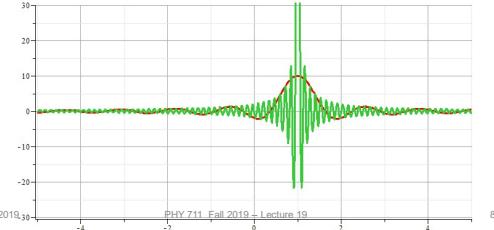
Generalization to infinite range -- Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt \ e^{-i(\omega - \omega_0)t} = 2\pi\delta(\omega - \omega_0)$$

Note that

$$\int_{-T}^T dt e^{-i(\omega - \omega_0)t} = \frac{2 \sin[(\omega - \omega_0)T]}{\omega - \omega_0} \underset{T \rightarrow \infty}{\approx} 2\pi\delta(\omega - \omega_0)$$



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Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Check :

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' f(t') \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

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Properties of Fourier transforms – Parseval's theorem:

$$\int_{-\infty}^{\infty} dt (f(t))^* f(t) = 2\pi \int_{-\infty}^{\infty} d\omega (F(\omega))^* F(\omega)$$

Check:

$$\begin{aligned} \int_{-\infty}^{\infty} dt (f(t))^* f(t) &= \int_{-\infty}^{\infty} dt \left(\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \\ &= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t} \\ &= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') 2\pi \delta(\omega' - \omega) \\ &= 2\pi \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega) \end{aligned}$$

Note that for an infinite time range, the Fourier transform is continuous in both time and frequency.

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Use of Fourier transforms to solve wave equation

Wave equation: $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

Suppose $u(x, t) = e^{-i\omega t} \tilde{F}(x, \omega)$ where $\tilde{F}(x, \omega)$ satisfies the equation:

$$\frac{\partial^2 \tilde{F}(x, \omega)}{\partial x^2} = -\frac{\omega^2}{c^2} \tilde{F}(x, \omega) \equiv -k^2 \tilde{F}(x, \omega) \quad \text{More generally: } u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(x, \omega) e^{-i\omega t}$$

Further assume that fixed boundary conditions apply: $0 \leq x \leq L$

with $\tilde{F}(0, \omega) = 0$ and $\tilde{F}(L, \omega) = 0$

For $n = 1, 2, 3, \dots$

$$\begin{aligned} \tilde{F}_n(x, \omega) &= \sin\left(\frac{n\pi x}{L}\right) & k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c} \\ u(x, t) &= e^{-i\omega_n t} \sin(k_n x) = e^{-i\omega_n t} \frac{(e^{ik_n x} - e^{-ik_n x})}{2i} = \frac{(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)})}{2i} \end{aligned}$$

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Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition: Suppose $u(x, t) = \sum_n A_n e^{-i\omega_n t} \tilde{F}_n(x, \omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x, \omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \tilde{F}_n(x, \omega_n) \equiv -k_n^2 \tilde{F}_n(x, \omega_n)$$

$$\text{For } \tilde{F}_n(x, \omega) = \sin\left(\frac{n\pi x}{L}\right) \quad k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$\begin{aligned} \Rightarrow u(x, t) &= \sum_n A_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{A_n}{2i} e^{-i\omega_n t} (e^{ik_n x} - e^{-ik_n x}) \\ &= \sum_n \frac{A_n}{2i} (e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}) \equiv f(x-ct) + g(x+ct) \end{aligned}$$

Note that at this point, we do not know the coefficients A_n ; however, it clear that the solutions are consistent with D'Alembert's analysis of the wave equation.

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Now consider the Fourier transform for a time periodic function:

Suppose $f(t+nT) = f(t)$ for any integer n

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_0^T dt f(t) e^{i\omega(t+nT)} \right)$$

Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{v=-\infty}^{\infty} \delta(\omega - v\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

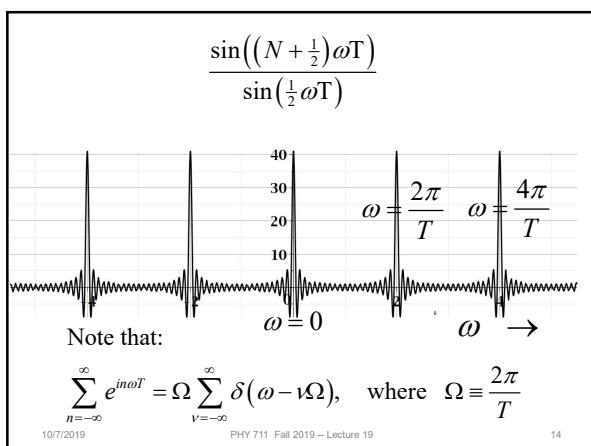
Details:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N e^{in\omega T} = \lim_{N \rightarrow \infty} \frac{\sin((N + \frac{1}{2})\omega T)}{\sin(\frac{1}{2}\omega T)}$$

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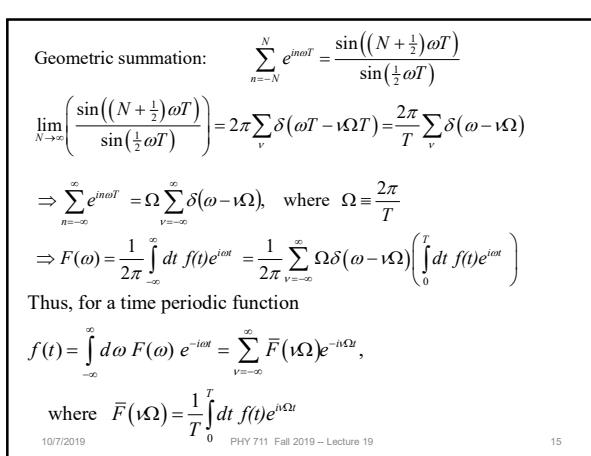
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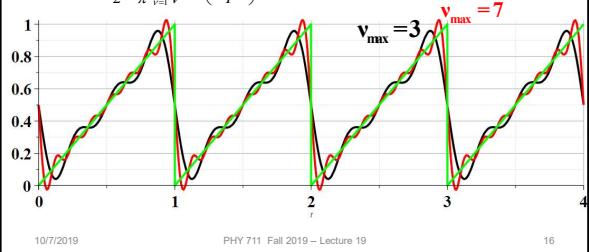
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Example:

Suppose: $f(t) = \frac{t-nT}{T}$ for $nT \leq t \leq (n+1)T$; $n = \dots -3, -2, -1, 0, 1, 2, 3 \dots$

$$\bar{F}(\nu\Omega) = \frac{1}{T} \int_0^T t e^{i \frac{\nu^2 \pi t}{T}} dt = \bar{F}^*(-\nu\Omega) = \frac{-i}{2\pi\nu} \quad \text{for } \nu = 1, 2, 3, \dots \quad \bar{F}(0) = \frac{1}{2}$$

$$f(t) = \frac{1}{2} - \frac{2}{\pi} \sum_{v=1}^{\infty} \frac{1}{v} \sin\left(\frac{2\pi vt}{T}\right)$$



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Summary –

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Find discrete frequencies ω for functions $f(t)$ over finite time domain or for functions $f(t)$ which are periodic: $f(t)=f(t+nT)$

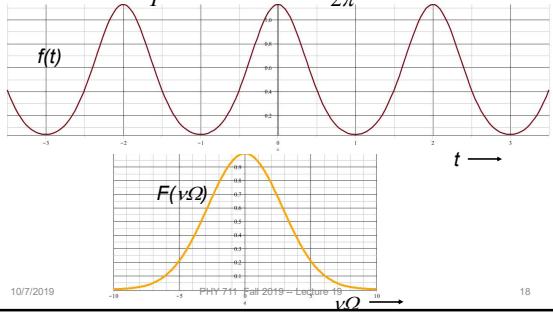
→ Numerically, there is an advantage of tabulating double discrete Fourier transforms (discrete in ω and in t).

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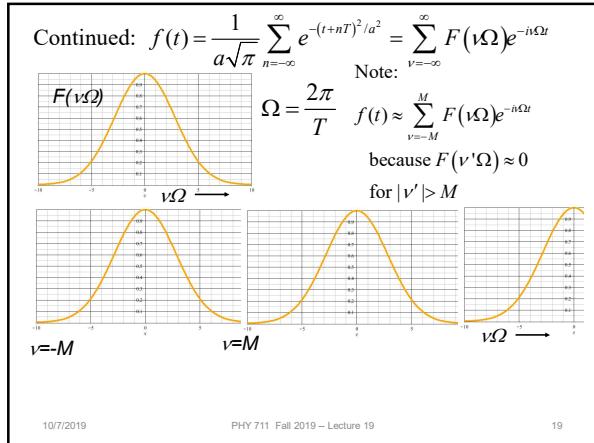
Example:

$$\text{Suppose: } f(t) = \frac{1}{a\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{v=-\infty}^{\infty} F(v\Omega)e^{-iv\Omega t}$$

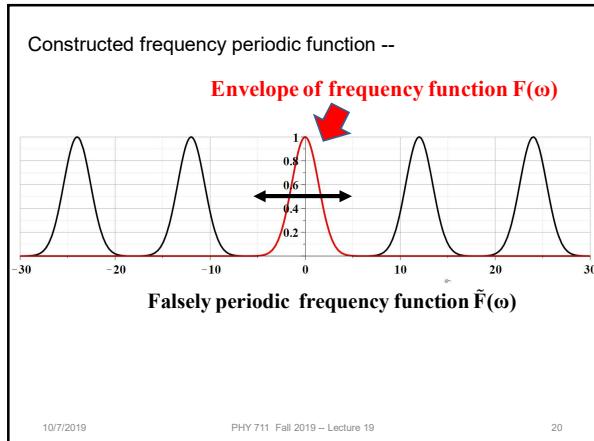
where $\Omega \equiv \frac{2\pi}{T}$ and $F(\nu\Omega) = \frac{1}{2\pi} e^{-a^2 \nu^2 \Omega^2 / 4}$



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Thus, for any periodic function: $f(t) = \sum_{v=-\infty}^{\infty} F(v\Omega)e^{-iv\Omega t}$

Now suppose that the transformed function is bounded;

$$|F(v\Omega)| \leq \varepsilon \quad \text{for } |v| \geq N$$

Define a periodic transform function

$$\tilde{F}(\nu\Omega + \sigma W) \equiv \tilde{F}(\nu\Omega) \text{ for } \sigma = \dots -3, -2, -1, 0, 1, 2, 3 \dots \text{ where } W \equiv ((2N+1)\Omega)$$

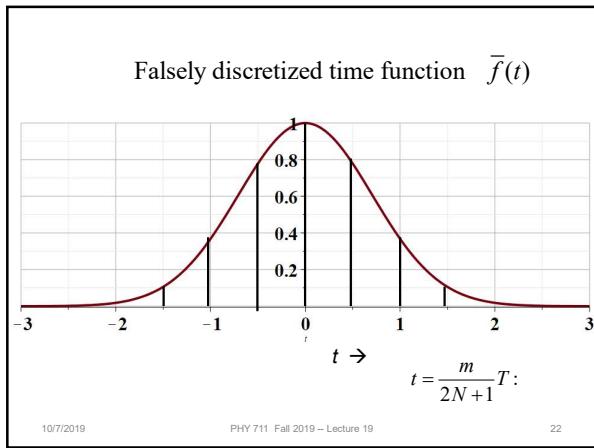
Recall that: $\sum_{n=-\infty}^{\infty} e^{j n \omega T} = \Omega \sum_{v=-\infty}^{\infty} \delta(\omega - v\Omega)$, where $\Omega \equiv \frac{2\pi}{T}$

$$f(t) = \sum_{\nu=-\infty}^{\infty} \tilde{F}(\nu\Omega)e^{-i\nu\Omega t} = \frac{2\pi}{(2N+1)\Omega} \sum_{\nu=-N}^N \tilde{F}(\nu\Omega)e^{-i\nu\Omega t} \sum_{\mu} \delta\left(t - \frac{\mu T}{2N+1}\right)$$

$$\text{For } t = \frac{m}{2N+1}T : \Rightarrow f\left(\frac{\nu}{2N+1}\right) = \sum_{v=-M}^M F(v\Omega)e^{-i2\pi v m/(2N+1)}$$

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Doubly periodic functions

$$t \rightarrow \frac{\mu T}{2N+1}$$

$$\tilde{f}_\mu = \frac{1}{2N+1} \sum_{v=-N}^N \tilde{F}_v e^{-i2\pi v \mu / (2N+1)}$$

$$\tilde{F}_v = \sum_{\mu=-N}^N \tilde{f}_\mu e^{i2\pi v\mu/(2N+1)}$$

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More convenient notation

$$2N+1 \rightarrow M$$

$$\tilde{f}_\mu = \frac{1}{M} \sum_{\nu=0}^{M-1} \tilde{F}_\nu e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_\nu = \sum_{\mu=0}^M \tilde{f}_\mu e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \dots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \dots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \dots$$

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Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \dots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \dots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \dots$$

However, $W^M = (e^{i2\pi/M})^M = 1$

$$\text{and } W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965)

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<http://www.fftw.org/>



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Introduction

FFTW is a C subroutine library for computing the discrete Fourier transform (DFT) in one or more dimensions, of arbitrary input size, and of both real and complex data (as well as of even/odd data, i.e. the discrete cosine/sine transforms or DCT/DST). We believe that FFTW, which is [free software](#), should become the [FFT](#) library of choice for most applications.

The latest official release of FFTW is version **3.3.8**, available from [our download page](#). Version 3.3 introduced support for the AVX x86 extensions, a distributed-memory implementation on top of MPI, and a Fortran 2003 API. Version 3.3.1 introduced support for the ARM Neon extensions. See the [release notes](#) for more information.

The FFTW package was developed at [MIT](#) by [Matteo Frigo](#) and [Steven G. Johnson](#).

Our [benchmarks](#), performed on a variety of platforms, show that FFTW's performance is typically superior to that of other publicly available FFT software, and is even competitive with vendor-tuned codes. In contrast to vendor-tuned codes, however, FFTW's performance is *portable*: the same program will perform well on most architectures without modification. Hence the name, "FFTW," which stands for the somewhat whimsical title of "**F**astest **Fourier T**ransform in the **W**est."

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