

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 26:

Chap. 8 in F & W: Summary of two-dimensional membrane analysis

Chap. 9 in F & W: Introduction to hydrodynamics

- 1. Motivation for topic**
- 2. Newton's laws for fluids**
- 3. Conservation relations**

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Today!



3 PM in Olin 101

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 Duke University
 Mechanical Engineering and Materials Science

“Atomic Dynamics in Energy and Functional Materials: Scattering Experiments and First-Principles Simulations”

A detailed view of atomic motions in materials is needed to refine microscopic theories of transport and thermodynamics, and to design next-generation energy materials. In particular, the behavior of atomic vibrations (phonons) is key to rationalize numerous functional properties, ranging from ferroelectrics for sonar, to superionics for safer solid batteries, to thermoelectrics for waste-heat harvesting, or metal-insulator transitions for ultrafast transistors. Near phase transitions associated with phonon instabilities, one needs to properly account for the effect of strong anharmonicity, which disrupts the quasharmonic phonon gas model through large phonon-phonon coupling terms. Large phonon amplitudes can also amplify the electron-phonon interaction and lead to renormalization of a material's electronic structure. These interactions, while often neglected in textbooks and traditional studies, could open the door to further tuning of materials properties for improved functionality.

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13	Mon, 9/23/2019	Chap. 3 & 6	Canonical transformations		
14	Wed, 9/25/2019	Chap. 4	Small oscillations about equilibrium	#11	9/30/2019
15	Fri, 9/27/2019	Chap. 4	Normal modes of vibration	#12	10/02/2019
16	Mon, 9/30/2019	Chap. 7	Motion of strings	#13	10/04/2019
17	Wed, 10/02/2019	Chap. 7	Sturm-Liouville equations	#14	10/07/2019
18	Fri, 10/04/2019	Chap. 7	Sturm-Liouville equations		
19	Mon, 10/07/2019	Chap. 7	Fourier transform methods		
20	Wed, 10/09/2019	Chap. 1-4,6-7	Review		
	Fri, 10/11/2019	No class	Fall break		
	Mon, 10/14/2019	No class	Take-home exam		
	Wed, 10/16/2019	No class	Take-home exam		
21	Fri, 10/18/2019	Chap. 7	Contour integrals; Exam due	#15	10/23/2019
22	Mon, 10/21/2019	Chap. 7	More about contour integrals		
23	Wed, 10/23/2019	Chap. 5	Rigid body motion	#16	10/25/2019
24	Fri, 10/25/2019	Chap. 5	Rigid body motion	#17	10/28/2019
25	Mon, 10/28/2019	Chap. 8	Elastic two-dimensional membranes	#18	11/01/2019
26	Wed, 10/30/2019	Chap. 9	Mechanics of 3 dimensional fluids		
27	Fri, 11/01/2019				
28	Mon, 11/04/2019				
	Wed, 11/06/2019	No class	NAWH out of town		
29	Fri, 11/08/2019				

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Review --

Two - dimensional wave equation :

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions :

$$u(x, y, t) = \Re(e^{-i\omega t} \rho(x, y))$$

$$(\nabla^2 + k^2) \rho(x, y) = 0 \quad \text{where } k = \frac{\omega}{c}$$

Consider a square boundary:



Free boundary conditions:

$$\frac{\partial \rho(0, y)}{\partial x} = \frac{\partial \rho(a, y)}{\partial x} = \frac{\partial \rho(x, 0)}{\partial y} = \frac{\partial \rho(x, a)}{\partial y} = 0$$

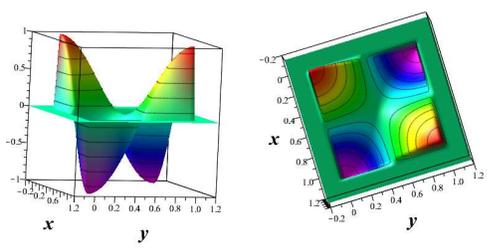
$$\Rightarrow \rho_{mn}(x, y) = A \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \omega_{mn} = ck_{mn}$$

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For $n = m = 1$:

$$\rho_{11}(x, y) = A \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right)$$


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Hydrodynamic analysis

Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

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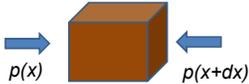
Newton's equations for fluids
 Use Euler formulation; following "particles" of fluid

Variables: Density $\rho(x,y,z,t)$
 Pressure $p(x,y,z,t)$
 Velocity $\mathbf{v}(x,y,z,t)$

$m\mathbf{a} = \mathbf{F}$
 $m \rightarrow \rho dV$
 $\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$
 $\mathbf{F} \rightarrow \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$

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$$F_{\text{pressure}}|_x = (-p(x+dx, y, z) + p(x, y, z))dydz$$

$$= \frac{(-p(x+dx, y, z) + p(x, y, z))}{dx} dx dy dz$$

$$= -\frac{\partial p}{\partial x} dV$$

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Newton's equations for fluids -- continued

$m\mathbf{a} = \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{\text{applied}} \rho dV - (\nabla p) dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

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Detailed analysis of acceleration term :

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that :

$$\frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z = \nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

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Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\rho \left(\nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

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Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{\text{applied}} = -\nabla U$ conservative applied force

3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial(-\nabla \Phi)}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

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Bernoulli's integral of Euler's equation for irrotational and incompressible fluid

$$\nabla \left(\frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = C(t)$$

where $\mathbf{v} = -\nabla \Phi(\mathbf{r}, t) = -\nabla(\Phi(\mathbf{r}, t) + C'(t))$

$$\Rightarrow \frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = 0 \quad \text{Bernoulli's theorem}$$

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Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = 0$$

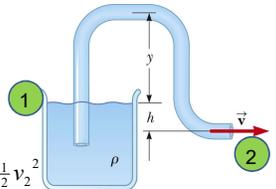
Modified form; assuming $\frac{\partial \Phi}{\partial t} = 0$

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 = \text{constant}$$

$p_1 = p_2 = p_{atm}$

$U_1 - U_2 = gh$

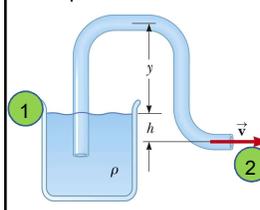
$v_1 \approx 0$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$


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Examples of Bernoulli's theorem -- continued



$p_1 = p_2 = p_{atm}$

$U_1 - U_2 = gh$

$v_1 \approx 0$

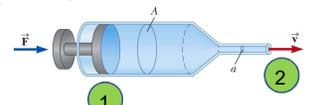
$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$

$$v_2 \approx \sqrt{2gh}$$

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Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 = \text{constant}$$


$$p_1 = \frac{F}{A} + p_{atm} \quad p_2 = p_{atm}$$

$$U_1 = U_2$$

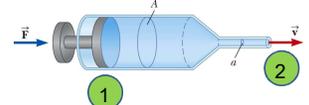
$$v_1 A = v_2 a \quad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$

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Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2}v^2 = \text{constant}$$


$$\frac{2F}{A} = v_2^2 \left(1 - \left(\frac{a}{A} \right)^2 \right)$$

$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A} \right)^2}}$$

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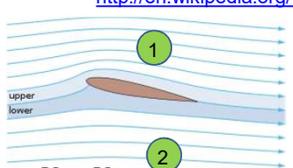
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Examples of Bernoulli's theorem -- continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2}v_2^2$$

$$p_2 - p_1 = \frac{1}{2}\rho(v_1^2 - v_2^2)$$

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Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$

$$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \quad \text{alternative form}$$

of continuity equation

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Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

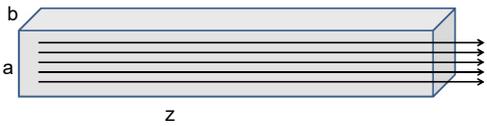
Irrotational flow: $\nabla \times \mathbf{v} = 0 \quad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

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Example – uniform flow



z

$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution:

$$\Phi = -v_0 z$$

$$\mathbf{v} = -\nabla \Phi = v_0 \hat{\mathbf{z}}$$

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Example – flow around a long cylinder (oriented in the Y direction)

$\nabla^2 \Phi = 0$

$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$

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Laplace equation in cylindrical coordinates
 (r, θ , defined in x - z plane; y representing cylinder axis)

$\nabla^2 \Phi = 0 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$

In our case, there is no motion in the y dimension

$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$

From boundary condition : $v_z(r \rightarrow \infty) = v_0$

$\frac{\partial \Phi}{\partial z}(r \rightarrow \infty) = -v_0 \Rightarrow \Phi(r \rightarrow \infty, \theta) = -v_0 r \cos \theta$

Note that : $\frac{\partial^2 \cos \theta}{\partial \theta^2} = -\cos \theta$

Guess form : $\Phi(r, \theta) = f(r) \cos \theta$

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Necessary equation for radial function

$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$

$f(r) = Ar + \frac{B}{r}$ where A, B are constants

Boundary condition on cylinder surface :

$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$

$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$

$\Rightarrow B = Aa^2$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

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$$\Phi(r, \theta) = -v_0 \left(r + \frac{a^2}{r} \right) \cos \theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0 \left(1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = v_0 \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2 \Phi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2}$$

to be continued ...

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