

**PHY 711 Classical Mechanics and  
Mathematical Methods  
10-10:50 AM MWF Olin 103**

**Plan for Lecture 30: Chap. 9 of F&W**

**Wave equation for sound beyond the linear approximation**

- 1. Non-linear effects in sound waves**
- 2. Shock wave analysis**

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23	Wed, 10/23/2019	Chap. 5	Rigid body motion	#16	10/25/2019
24	Fri, 10/25/2019	Chap. 5	Rigid body motion	#17	10/28/2019
25	Mon, 10/28/2019	Chap. 8	Elastic two-dimensional membranes	#18	11/01/2019
26	Wed, 10/30/2019	Chap. 9	Mechanics of 3 dimensional fluids		
27	Fri, 11/01/2019	Chap. 9	Fluid mechanics	#19	11/04/2019
28	Mon, 11/04/2019	Chap. 9	Sound waves		
	Wed, 11/06/2019	No class	NAWH out of town		
29	Fri, 11/08/2019	Chap. 9	Sound waves; Project Topic due	#20	11/11/2019
30	Mon, 11/11/2019	Chap. 9	Non-linear waves and shocks	#21	11/15/2019
31	Wed, 11/13/2019				
32	Fri, 11/15/2019				
33	Mon, 11/18/2019				
34	Wed, 11/20/2019				
35	Fri, 11/22/2019				
36	Mon, 11/25/2019				
	Wed, 11/27/2019		Thanksgiving holiday		
	Fri, 11/29/2019		Thanksgiving holiday		
	Mon, 12/2/2019		Presentations I		
	Wed, 12/4/2019		Presentations II		
	Fri, 12/6/2019		Presentations III		

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Effects of nonlinearities in fluid equations  
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Continuity equation :  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume spatial variation confined to  $x$  direction ;  
assume that  $\mathbf{v} = v \hat{\mathbf{x}}$  and  $\mathbf{f}_{\text{applied}} = 0$ .

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

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$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$

$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$

Expressing  $p$  in terms of  $\rho$ :  $p = p(\rho)$

$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x} \equiv c^2(\rho) \frac{\partial \rho}{\partial x}$  where  $\frac{\partial p}{\partial \rho} \equiv c^2(\rho)$

For adiabatic ideal gas:  $\frac{dp}{p} = \gamma \frac{d\rho}{\rho}$   $p = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma$

$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2 \left( \frac{\rho}{\rho_0} \right)^{\gamma-1}$  where  $c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$

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$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$

$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$

Expressing variation of  $v$  in terms of  $v(\rho)$ :

$\frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial t} + v \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$

$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} = 0$

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Some more algebra :

From Euler equation :  $\frac{\partial v}{\partial \rho} \left( \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} \right) + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$

From continuity equation :  $\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = -\rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x}$

Combined equation :  $\frac{\partial v}{\partial \rho} \left( -\rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} \right) + \frac{c^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0$

$\Rightarrow \left( \frac{\partial v}{\partial \rho} \right)^2 = \frac{c^2(\rho)}{\rho^2}$   $\frac{\partial v}{\partial \rho} = \pm \frac{c}{\rho}$

$\Rightarrow \frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$

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Assuming adiabatic process:  $c^2 = c_0^2 \left( \frac{\rho}{\rho_0} \right)^{\gamma-1}$   $c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm \frac{c}{\rho} \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left( \frac{\rho'}{\rho_0} \right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma-1} \left[ \left( \frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - 1 \right]$$

$$\Rightarrow c = c_0 \left( \frac{\rho}{\rho_0} \right)^{(\gamma-1)/2}$$

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**Summary :**

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process:  $c^2 = c_0^2 \left( \frac{\rho}{\rho_0} \right)^{\gamma-1}$   $c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left( \frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} \quad v = \pm \frac{2c_0}{\gamma-1} \left[ \left( \frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - 1 \right]$$

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**Traveling wave solution:**

Assume:  $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self - consistent equations for propagation velocity  $u(\rho)$  using equations

From previous derivations:  $\frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$

Apparently:  $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0 \left( \frac{\gamma+1}{\gamma-1} \left( \frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

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Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c) \frac{\partial \rho}{\partial x} = 0$$

$$\text{Assume: } \rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0 \left( \frac{\gamma+1}{\gamma-1} \left( \frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

Solution in linear approximation:

$$u = v + c \approx v_0 + c_0 = c_0 \left( \frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1} \right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f(x - c_0 t)$$

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Traveling wave solution -- full non-linear case:

Visualization for particular waveform:  $\rho = \rho_0 + f(x - u(\rho)t)$

$$\text{Assume: } f(w) \equiv \rho_0 s(w)$$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0 \left( \frac{\gamma+1}{\gamma-1} \left( \frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

$$u = c_0 \left( \frac{\gamma+1}{\gamma-1} (1 + s(x - ut))^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

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Visualization continued:

$$u = c_0 \left( \frac{\gamma+1}{\gamma-1} (1 + s(x - ut))^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

Plot  $s(x - ut)$  for fixed  $t$ , as a function of  $x$ :

Let  $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w, t)$$

$$u(w) = c_0 \left( \frac{\gamma+1}{\gamma-1} (1 + s(w))^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

Parametric equations:

plot  $s(w)$  vs  $x(w, t)$  for range of  $w$  at each  $t$

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**Summary**

$$\frac{\partial \rho}{\partial t} + u(\rho) \frac{\partial \rho}{\partial x} = 0$$

Solution:  $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0(1 + s(x - u(\rho)t))$

For linear case:  $u(\rho) = c_0$

For non-linear case:  $u(\rho) = c_0 \left( \frac{\gamma+1}{\gamma-1} (1 + s(x - ut))^{\frac{(\gamma-1)}{2}} - \frac{2}{\gamma-1} \right)$

Plot  $s(x - ut)$  for fixed  $t$ , as a function of  $x$ :

Let  $w = x - ut \Rightarrow x = w + ut = w + u(w)t \equiv x(w, t)$

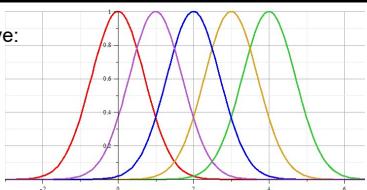
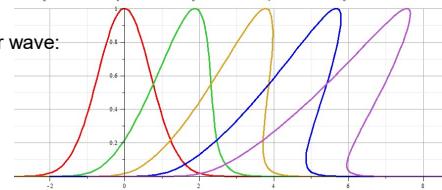
$$u(w) = c_0 \left( \frac{\gamma+1}{\gamma-1} (1 + s(w))^{\frac{(\gamma-1)}{2}} - \frac{2}{\gamma-1} \right)$$

Parametric equations: plot  $s(w)$  vs  $x(w, t)$  for range of  $w$

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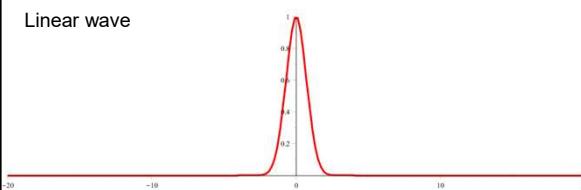
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**Linear wave:****Non-linear wave:**

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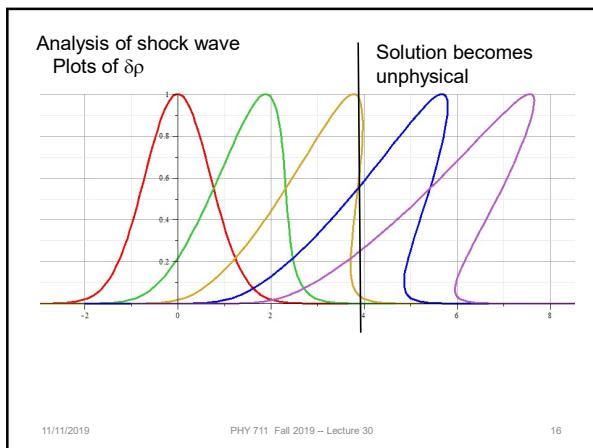
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**Linear wave****Non-linear wave**

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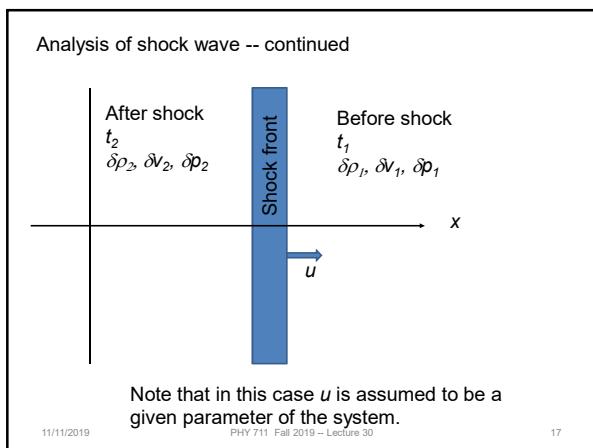
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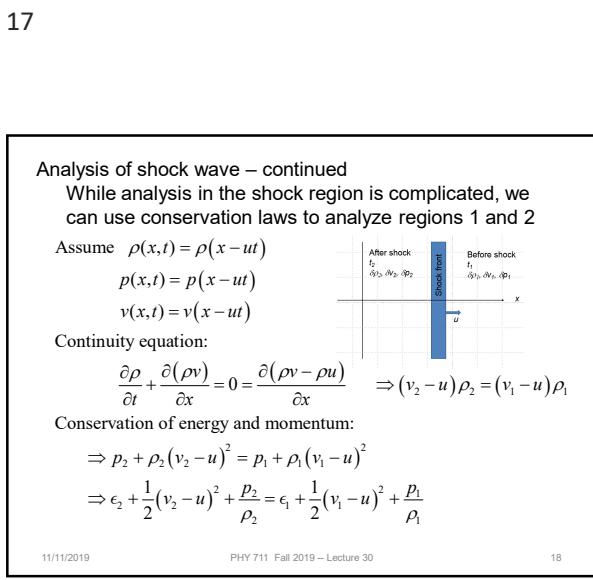
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### Analysis of shock wave – continued

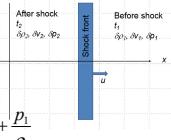
While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Summary of equations

$$\Rightarrow (v_2 - u) \rho_2 = (v_1 - u) \rho_1$$

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2} (v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2} (v_1 - u)^2 + \frac{p_1}{\rho_1}$$



Assume that within each regions (1 & 2), the ideal gas equations apply

$$\epsilon_i + \frac{p_i}{\rho_i} = \frac{\gamma}{\gamma-1} \frac{p_i}{\rho_i}$$

$$\epsilon_2 + \frac{p_2}{\rho_2} = \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2}$$

$$\text{It follows that } \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} + \frac{1}{2} (v_2 - u)^2 = \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} + \frac{1}{2} (v_1 - u)^2$$

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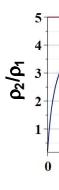
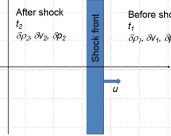
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### Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1} \frac{p_2}{p_1} + 1}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$



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### Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

$$\text{Internal energy density: } \epsilon = \frac{p}{(\gamma-1)\rho} = C_v T$$

$$\text{First law of thermo: } d\epsilon = Tds - pd\left(\frac{1}{\rho}\right)$$

$$ds = \frac{1}{T} \left( d\left(\frac{p}{(\gamma-1)\rho}\right) - pd\left(\frac{1}{\rho}\right) \right) = C_v d\ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = C_v \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = C_v \ln\left(\frac{p_2}{p_1} \left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \quad 0 < s_2 - s_1 < C_v \left( \ln\left(\frac{p_2}{p_1}\right) - \gamma \ln\left(\frac{\gamma+1}{\gamma-1}\right) \right)$$

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