

**PHY 711 Classical Mechanics and  
Mathematical Methods  
10-10:50 AM MWF Olin 103**

**Plan for Lecture 31:**

**Chapter 10 in F & W: Surface waves**

**1. Water waves in a channel**

**2. Wave-like solutions; wave speed**

11/13/2019

PHY 711 Fall 2019 – Lecture 31

1

1

21	Fri, 10/18/2019	Chap. 7	Contour integrals; Exam due	#15	10/23/2019
22	Mon, 10/21/2019	Chap. 7	More about contour integrals		
23	Wed, 10/23/2019	Chap. 5	Rigid body motion	#16	10/25/2019
24	Fri, 10/25/2019	Chap. 5	Rigid body motion	#17	10/28/2019
25	Mon, 10/28/2019	Chap. 8	Elastic two-dimensional membranes	#18	11/01/2019
26	Wed, 10/30/2019	Chap. 9	Mechanics of 3 dimensional fluids		
27	Fri, 11/01/2019	Chap. 9	Fluid mechanics	#19	11/04/2019
28	Mon, 11/04/2019	Chap. 9	Sound waves		
	Wed, 11/06/2019	No class	NAWH out of town		
29	Fri, 11/08/2019	Chap. 9	Sound waves; Project Topic due	#20	11/11/2019
30	Mon, 11/11/2019	Chap. 9	Non-linear waves and shocks	#21	11/15/2019
31	Wed, 11/13/2019	Chap. 10	Surface waves in water	#22	11/18/2019
32	Fri, 11/15/2019				
33	Mon, 11/18/2019				
34	Wed, 11/20/2019				
35	Fri, 11/22/2019				
36	Mon, 11/25/2019				
	Wed, 11/27/2019		Thanksgiving holiday		
	Fri, 11/29/2019		Thanksgiving holiday		
	Mon, 12/2/2019		Presentations I		
	Wed, 12/4/2019		Presentations II		
	Fri, 12/6/2019		Presentations III		

2

Wake Forest College & Graduate School of Arts and Sciences

**WFU Physics**

WFU Physics | People | Events and News | Undergraduate | Graduate | Research | Resources

**Events**

Colloquium: "From Enzyme-Catalyzed Living Polymerization Reactions to Microfluidic Shear Wave Resonator Sensors" – Wednesday Nov. 13, 2019 at 3 PM  
 Professor Stefan Zauscher Department of Mechanical Engineering and Materials Science  
 Duke University George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, Nov. 13, 2019, at 00 PM There will be ...

Colloquium: "On the Design and Tailoring Molecular Co-adsorption in Nanoporous Materials" – Wednesday, Nov. 20, 2019 at 3 PM  
 Kai Tan, PhD Research Scientist The University of Texas at Dallas Department of Materials Science and Engineering George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, Nov. 20, 2019, at ...

11/13/2019 PHY 711 Fall 2019 – Lecture 31 3

3

## Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

11/13/2019

PHY 711 Fall 2019 – Lecture 31

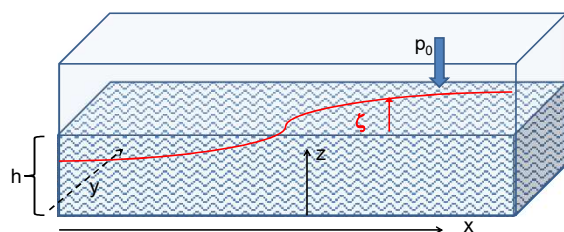
4

4

Consider a container of water with average height  $h$  and surface  $h + \zeta(x, y, t)$ ; ( $h \leftrightarrow z_0$  on some of the slides)

Atmospheric pressure is in equilibrium with the surface of water

$$p_0 = \rho g(h + \zeta) \quad \text{Here } \rho \text{ represents density of water}$$



11/13/2019

PHY 711 Fall 2019 – Lecture 31

5

5

Euler's equation for incompressible fluid :

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

$$\text{Assume that } v_z \ll v_x, v_y \quad \Rightarrow -g - \frac{1}{\rho} \frac{\partial p}{\partial z} \approx 0$$

$$\Rightarrow p(x, y, z, t) = p_0 + \rho g(\zeta(x, y, t) + h - z) \quad \text{within the water}$$

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} = -g \frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y} = -g \frac{\partial \zeta}{\partial y}$$

11/13/2019

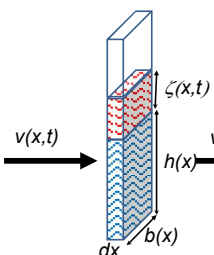
PHY 711 Fall 2019 – Lecture 31

6

6

Consider a surface  $\zeta(x,t)$  wave moving in the  $x$ -direction in a channel of width  $b(x)$  and height  $h(x)$ :

Continuity condition in integral form:



$$\frac{d}{dt} \int_V \rho dV + \int_A \rho \mathbf{v} \cdot d\mathbf{A} = 0$$

$$b(x) \left( h(x) + \zeta(x,t) \right) dx$$

Evaluating continuity condition:

$$b(x) \frac{\partial \zeta}{\partial t} = - \frac{\partial}{\partial x} (h(x) b(x) v(x,t))$$

11/13/2019 PHY 711 Fall 2019 – Lecture 31 7

7

---

---

---

---

---

---

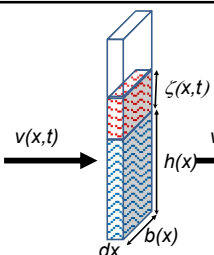
---

---

From continuity condition:

$$b(x) \frac{\partial \zeta}{\partial t} = - \frac{\partial}{\partial x} (h(x) b(x) v(x,t))$$

Example (Problem 10.3):

$$b(x) = b_0 \quad h(x) = \kappa x$$


$$b_0 \frac{\partial \zeta}{\partial t} = - \frac{\partial}{\partial x} ((\kappa x) b_0 v(x,t))$$

From Newton-Euler equation:

$$\frac{\partial \zeta}{\partial t} = -\kappa \left( v + x \frac{\partial v}{\partial x} \right) \quad \frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g \frac{\partial \zeta}{\partial x}$$

11/13/2019 PHY 711 Fall 2019 – Lecture 31 8

8

---

---

---

---

---

---

---

---

Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa \left( v + x \frac{\partial v}{\partial x} \right) \Rightarrow \frac{\partial^2 \zeta}{\partial t^2} = -\kappa \left( \frac{\partial v}{\partial t} + x \frac{\partial^2 v}{\partial x \partial t} \right)$$

$$\frac{\partial v}{\partial t} = -g \frac{\partial \zeta}{\partial x} \Rightarrow \frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left( \frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = C J_0 \left( \frac{2\omega}{\sqrt{\kappa g}} \sqrt{x} \right) \cos(\omega t)$$

Note that  $J_0(u)$  satisfies the equation:  $\left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + 1 \right) J_0(u) = 0$

Therefore, for  $u = \frac{2\omega}{\sqrt{\kappa g}} \sqrt{x}$

$$\left( x \frac{d^2}{dx^2} + \frac{d}{dx} \right) J_0(u) = \frac{\omega^2}{\kappa g} \left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} \right) J_0(u) = -\frac{\omega^2}{\kappa g} J_0(u)$$

11/13/2019 PHY 711 Fall 2019 – Lecture 31 9

9

---

---

---

---

---

---

---

---

Example continued

$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left( \frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

$$\Rightarrow \zeta(x, t) = C J_0 \left( \frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

Check:

$$-\omega^2 C J_0 \left( \frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t) = \kappa g \left( \frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} \right) C J_0 \left( \frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

10

10

---

---

---

---

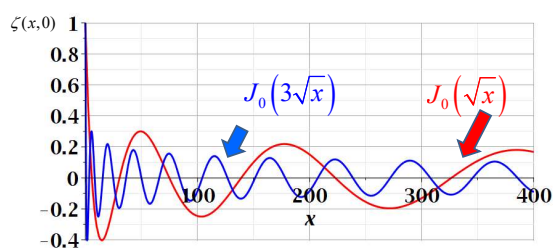
---

---

---

---

$$\zeta(x, t) = C J_0 \left( \frac{2\omega}{\sqrt{\kappa g}} \sqrt{x} \right) \cos(\omega t)$$



11/13/2019

PHY 711 Fall 2019 – Lecture 31

11

11

---

---

---

---

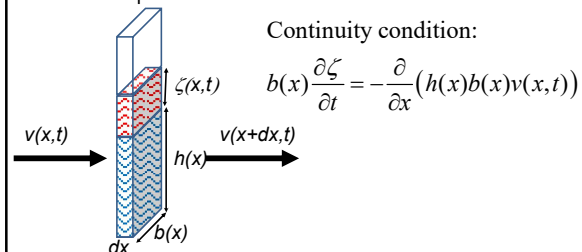
---

---

---

---

Another example:



Special case, where  $b$  and  $h$  are constant --

For constant  $b$  and  $h$ :

$$\frac{\partial \zeta}{\partial t} = -h \frac{\partial}{\partial x} (v(x, t))$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

12

12

---

---

---

---

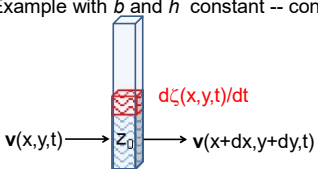
---

---

---

---

Example with  $b$  and  $h$  constant -- continued



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h \nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations:  $\frac{\partial \mathbf{v}}{\partial t} = -g \nabla \zeta$

Equation for surface function:  $\frac{\partial^2 \zeta}{\partial t^2} - gh \nabla^2 \zeta = 0$

11/13/2019 PHY 711 Fall 2019 -- Lecture 31 13

13

For uniform channel:

Surface wave equation:

$$\frac{\partial^2 \zeta}{\partial t^2} - c^2 \nabla^2 \zeta = 0 \quad c^2 = gh$$

More complete analysis finds:

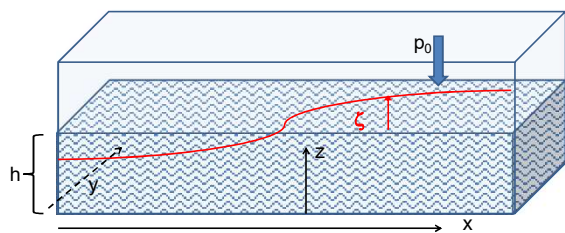
$$c^2 = \frac{g}{k} \tanh(kh) \quad \text{where } k = \frac{2\pi}{\lambda}$$

11/13/2019 PHY 711 Fall 2019 -- Lecture 31 14

14

More details: -- recall setup --

Consider a container of water with average height  $h$  and surface  $h + \zeta(x,y,t)$



11/13/2019 PHY 711 Fall 2019 -- Lecture 31 15

15

Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} v^2 \right) + \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla U - \frac{\nabla p}{\rho}$$

Assume that  $\nabla \times \mathbf{v} = 0$  (irrotational flow)  $\Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla \left( -\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + U + \frac{p}{\rho} \right) = 0$$

$$\Rightarrow -\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

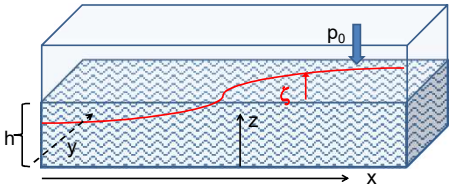
$$\nabla \cdot \mathbf{v} = -\nabla^2 \Phi = 0$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

16

16



Within fluid:  $0 \leq z \leq h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + g(z - h) = \text{constant} \quad (\text{We have absorbed } p_0 \text{ in "constant"})$$

$$-\nabla^2 \Phi = 0$$

At surface:  $z = h + \zeta$  with  $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h + \zeta, t)$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

17

17

Full equations:

Within fluid:  $0 \leq z \leq h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + g(z - h) = \text{constant} \quad (\text{We have absorbed } p_0 \text{ in "constant"})$$

$$-\nabla^2 \Phi = 0$$

At surface:  $z = h + \zeta$  with  $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h + \zeta, t)$$

Linearized equations:

$$\text{For } 0 \leq z \leq h + \zeta: \quad -\frac{\partial \Phi}{\partial t} + g(z - h) = 0 \quad -\nabla^2 \Phi = 0$$

$$\text{At surface: } z = h + \zeta \quad \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} = v_z(x, y, h + \zeta, t)$$

$$-\frac{\partial \Phi(x, y, h + \zeta, t)}{\partial t} + g\zeta = 0$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

18

18

For simplicity, keep only linear terms and assume that horizontal variation is only along  $x$ :

$$\text{For } 0 \leq z \leq h + \zeta: \quad \nabla^2 \Phi = \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x, z, t) = 0$$

Consider a periodic waveform:  $\Phi(x, z, t) = Z(z) \cos(k(x - ct))$

$$\Rightarrow \left( \frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank:  $v_z(x, 0, t) = 0$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \quad Z(z) = A \cosh(kz)$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

19

19

For simplicity, keep only linear terms and assume that horizontal variation is only along  $x$  – continued:

$$\text{At surface: } z = h + \zeta \quad \frac{\partial \zeta}{\partial t} = v_z(x, h + \zeta, t) = -\frac{\partial \Phi(x, h + \zeta, t)}{\partial z} - \frac{\partial \Phi(x, h + \zeta, t)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi(x, h + \zeta, t)}{\partial t^2} + g \frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi(x, h + \zeta, t)}{\partial t^2} - g \frac{\partial \Phi(x, h + \zeta, t)}{\partial z} = 0$$

For  $\Phi(x, (h + \zeta), t) = A \cosh(k(h + \zeta)) \cos(k(x - ct))$

$$A \cosh(k(h + \zeta)) \cos(k(x - ct)) \left( k^2 c^2 - gk \frac{\sinh(k(h + \zeta))}{\cosh(k(h + \zeta))} \right) = 0$$

$$\Rightarrow c^2 = \frac{g \sinh(k(h + \zeta))}{k \cosh(k(h + \zeta))}$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

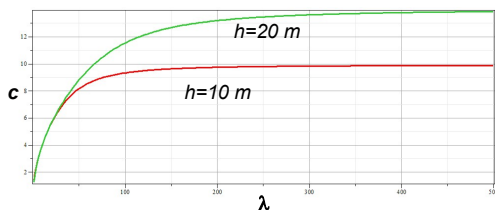
20

20

For simplicity, keep only linear terms and assume that horizontal variation is only along  $x$  – continued:

$$c^2 = \frac{g \sinh(k(h + \zeta))}{k \cosh(k(h + \zeta))} = \frac{g}{k} \tanh(k(h + \zeta))$$

$$\text{Assuming } \zeta \ll h: \quad c^2 = \frac{g}{k} \tanh(kh) \quad \lambda = \frac{2\pi}{k}$$



11/13/2019

PHY 711 Fall 2019 – Lecture 31

21

21

For simplicity, keep only linear terms and assume that horizontal variation is only along  $x$  – continued:

$$c^2 \approx \frac{g}{k} \tanh(kh) \quad \text{For } \lambda \gg h, \quad c^2 \approx gh$$

$$\Phi(x, z, t) = A \cosh(kz) \cos(k(x - ct))$$

$$\zeta(x, t) = \frac{1}{g} \frac{\partial \Phi(x, h + \zeta, t)}{\partial t} \approx \frac{kc}{g} A \cosh(kh) \sin(k(x - ct))$$

Note that for  $\lambda \gg h$ ,  $c^2 \approx gh$   
(solutions are consistent with previous analysis)

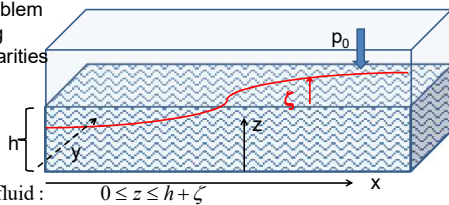
11/13/2019

PHY 711 Fall 2019 – Lecture 31

22

22

General problem  
including  
non-linearities



Within fluid:  $0 \leq z \leq h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + g(z - h) = \text{constant} \quad (\text{We have absorbed } p_0 \text{ in our constant.})$$

$$-\nabla^2 \Phi = 0$$

$$\text{At surface: } z = h + \zeta \quad \text{with } \zeta = \zeta(x, y, t)$$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h + \zeta, t)$$

11/13/2019

PHY 711 Fall 2019 – Lecture 31

23

23