PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 31:

Chapter 10 in F & W: Surface waves

1. **Water waves in a channel**
2. **Wave-like solutions; wave speed**

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21	Fri, 10/18/2019	Chap. 7	Contour integrals; Exam due	#15	10/23/2019
22	Mon, 10/21/2019	Chap. 7	More about contour integrals		
23	Wed, 10/23/2019	Chap. 5	Rigid body motion	#16	10/25/2019
24	Fri, 10/25/2019	Chap. 5	Rigid body motion	#17	10/28/2019
25	Mon, 10/28/2019	Chap. 8	Elastic two-dimensional membranes	#18	11/01/2019
26	Wed, 10/30/2019	Chap. 9	Mechanics of 3 dimensional fluids		
27	Fri, 11/01/2019	Chap. 9	Fluid mechanics	#19	11/04/2019
28	Mon, 11/04/2019	Chap. 9	Sound waves		
	Wed, 11/06/2019	No class	NAWH out of town		
29	Fri, 11/08/2019	Chap. 9	Sound waves; Project Topic due	#20	11/11/2019
30	Mon, 11/11/2019	Chap. 9	Non-linear waves and shocks	#21	11/15/2019
31	Wed, 11/13/2019	Chap. 10	Surface waves in water	#22	11/18/2019
32	Fri, 11/15/2019				
33	Mon, 11/18/2019				
34	Wed, 11/20/2019				
35	Fri, 11/22/2019				
36	Mon, 11/25/2019				
	Wed, 11/27/2019		Thanksgiving holiday		
	Fri, 11/29/2019		Thanksgiving holiday		
	Mon, 12/2/2019		Presentations I		
	Wed, 12/4/2019		Presentations II		
	Fri, 12/6/2019		Presentations III		

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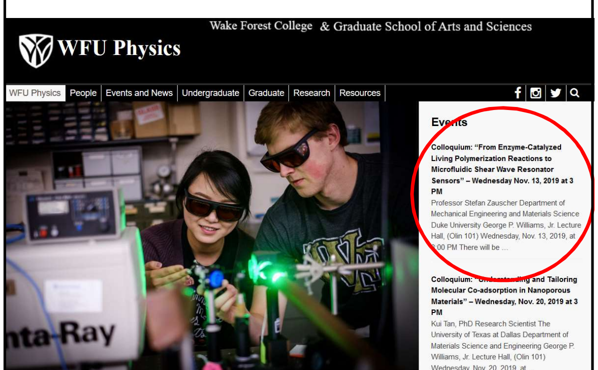

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Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

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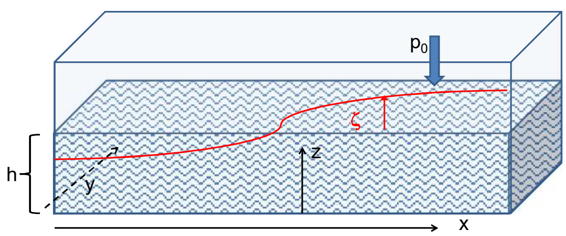
Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ z_0 on some of the slides)

Atmospheric pressure is in equilibrium with the surface of water

$p_0 = \rho g(h+\zeta)$ Here ρ represents density of water



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Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

$\Rightarrow p(x,y,z,t) = p_0 + \rho g(\zeta(x,y,t) + h - z)$ within the water

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$

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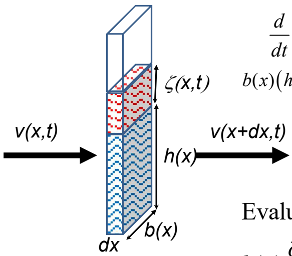
Consider a surface $\zeta(x,t)$ wave moving in the x-direction in a channel of width $b(x)$ and height $h(x)$:

Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

$b(x)\left(h(x)+\zeta(x,t)\right)dx$ $\quad$ $b(x)\left(h(x)+\zeta(x,t)\right)\hat{\mathbf{x}}$



Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$

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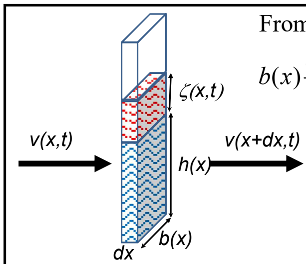

From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

$$b_0\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left((\kappa x)b_0 v(x,t)\right)$$

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$

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Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t}\right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1\right)J_0(u) = 0$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)J_0(u) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du}\right)J_0(u) = -\frac{\omega^2}{\kappa g}J_0(u)$$

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Example continued

$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left(\frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

$$\Rightarrow \zeta(x,t) = C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

Check:

$$-\omega^2 C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t) = \kappa g \left(\frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} \right) C J_0 \left(\frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

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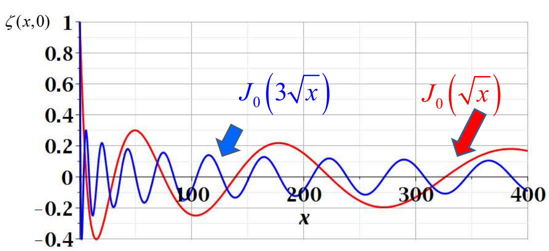
$$\zeta(x,t) = C J_0 \left(\frac{2\omega}{\sqrt{\kappa g}} \sqrt{x} \right) \cos(\omega t)$$



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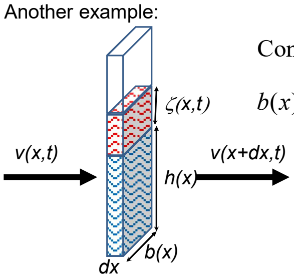
Another example:



Continuity condition:

$$b(x) \frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x} \big(h(x) b(x) v(x,t) \big)$$

Special case, where *b* and *h* are constant --

For constant *b* and *h*:

$$\frac{\partial \zeta}{\partial t} = -h \frac{\partial}{\partial x} \big(v(x,t) \big)$$

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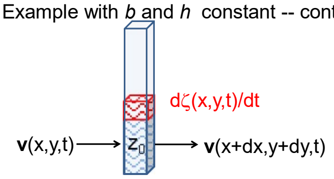
Example with *b* and *h* constant -- continued



Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations: $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla\zeta$

Equation for surface function: $\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2\zeta = 0$

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For uniform channel:

Surface wave equation:

$$\frac{\partial^2 \zeta}{\partial t^2} - c^2\nabla^2\zeta = 0 \qquad c^2 = gh$$

More complete analysis finds:

$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

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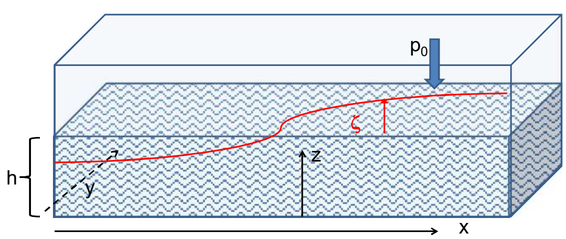
More details: -- recall setup --

Consider a container of water with average height h and surface h+ζ(x,y,t)



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Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$

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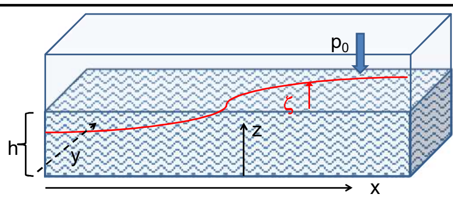

Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g\left(z-h\right) = \text{constant}$$

(We have absorbed p_0 in "constant")

$$-\nabla^2\Phi = 0$$

At surface: $z = h+\zeta$ with $\zeta = \zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}\left(x,y,h+\zeta,t\right)$$

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Full equations:

Within fluid: $0 \le z \le h+\zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g\left(z-h\right) = \text{constant}$$

(We have absorbed p_0 in "constant")

$$-\nabla^2\Phi = 0$$

At surface: $z = h+\zeta$ with $\zeta = \zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}\left(x,y,h+\zeta,t\right)$$

Linearized equations:

For $0 \le z \le h+\zeta$: $\quad -\frac{\partial\Phi}{\partial t} + g(z-h) = 0 \qquad -\nabla^2\Phi = 0$

At surface: $z = h+\zeta \quad \frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} = v_z(x,y,h+\zeta,t)$

$$-\frac{\partial\Phi(x,y,h+\zeta,t)}{\partial t} + g\zeta = 0$$

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For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For} \quad 0 \le z \le h+\zeta: \qquad \nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t)=0$$

Consider and periodic waveform: $\Phi(x,z,t)=Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left(\frac{d^2}{dz^2}-k^2\right)Z(z)=0$$

Boundary condition at bottom of tank: $v_z(x,0,t)=0$

$$\Rightarrow \frac{dZ}{dz}(0)=0 \qquad Z(z)=A\cosh(kz)$$

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For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface}: \quad z=h+\zeta \qquad \frac{\partial\zeta}{\partial t}=v_z(x,h+\zeta,t)=-\frac{\partial\Phi(x,h+\zeta,t)}{\partial z}$$

$$-\frac{\partial\Phi(x,h+\zeta,t)}{\partial t}+g\zeta=0$$

$$-\frac{\partial^2\Phi(x,h+\zeta,t)}{\partial t^2}+g\frac{\partial\zeta}{\partial t}=-\frac{\partial^2\Phi(x,h+\zeta,t)}{\partial t^2}-g\frac{\partial\Phi(x,h+\zeta,t)}{\partial z}=0$$

$$\text{For} \quad \Phi(x,(h+\zeta),t)=A\cosh(k(h+\zeta))\cos\big(k(x-ct)\big)$$

$$A\cosh(k(h+\zeta))\cos\big(k(x-ct)\big)\left(k^2c^2-gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right)=0$$

$$\Rightarrow c^2=\frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

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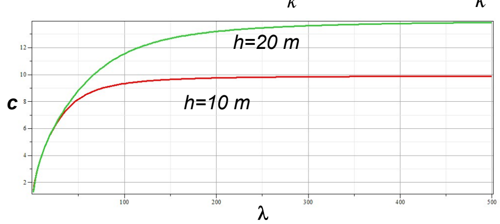
For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2=\frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}=\frac{g}{k}\tanh(k(h+\zeta))$$

$$\text{Assuming } \zeta \ll h: \qquad c^2=\frac{g}{k}\tanh(kh) \qquad \lambda=\frac{2\pi}{k}$$



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For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h$, $\quad c^2 \approx gh$

(solutions are consistent with previous analysis)

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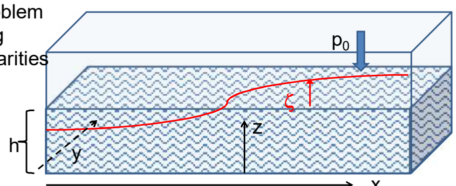
General problem including non-linearities



Within fluid : $\quad 0 \le z \le h+\zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2\Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

At surface : $\quad z = h+\zeta \qquad$ with $\zeta = \zeta(x,y,t)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x,y,h+\zeta,t)$$

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