

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 32:

Chapter 10 in F & W: Surface waves

- **Comment on Bessel functions (HW #21)**
- **Summary of linear surface wave solutions**
- **Non-linear contributions and soliton solutions**

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Comment on Bessel functions

<https://dlmf.nist.gov/>

NIST Digital Library of Mathematical Functions

Project News

2019-09-15 [DLMF Update, Version 1.0.24](#)
2019-06-15 [DLMF Update, Version 1.0.23](#)
2019-03-15 [DLMF Update, Version 1.0.22](#)
2018-12-15 [DLMF Update, Version 1.0.21](#)
[More news](#)

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§10.2(i) Bessel's Equation

10.2.1
$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0.$$

→ Note that, in principle, solutions for $+\nu$ and $-\nu$ are related; and for integer ν they are not independent. The “standard” conventions are given as follows.

§10.4 Connection Formulas

Other solutions of (10.2.1) include $J_{-\nu}(z)$, $Y_{-\nu}(z)$, $H_{-\nu}^{(1)}(z)$, and $H_{-\nu}^{(2)}(z)$.

10.4.1

$$\begin{aligned} J_{-n}(z) &= (-1)^n J_n(z), \\ Y_{-n}(z) &= (-1)^n Y_n(z), \\ H_{-n}^{(1)}(z) &= (-1)^n H_n^{(1)}(z), \end{aligned}$$

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This material is covered in Chapter 10 of your textbook using similar notation.

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29	Fri, 11/08/2019	Chap. 9	Sound waves; Project Topic due	#20	11/11/2019
30	Mon, 11/11/2019	Chap. 9	Non-linear waves and shocks	#21	11/15/2019
31	Wed, 11/13/2019	Chap. 10	Surface waves in water	#22	11/18/2019
32	Fri, 11/15/2019	Chap. 10	Surface waves -- non linear effects		
33	Mon, 11/18/2019				
34	Wed, 11/20/2019				
35	Fri, 11/22/2019				
36	Mon, 11/25/2019				
	Wed, 11/27/2019		Thanksgiving holiday		
	Fri, 11/29/2019		Thanksgiving holiday		
	Mon, 12/2/2019		Presentations I		
	Wed, 12/4/2019		Presentations II		
	Fri, 12/6/2019		Presentations III		

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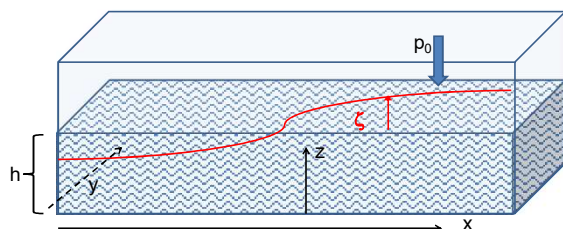
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Consider a container of water with average height h and surface $h+\zeta(x,y,t)$

Atmospheric pressure p_0 is in equilibrium at the surface

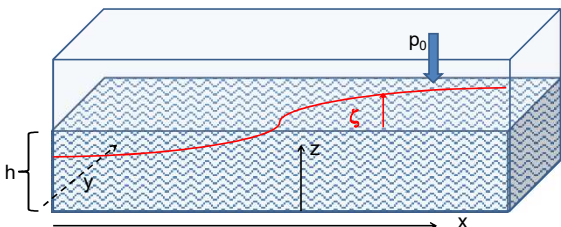


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Euler's equation for incompressible fluid: For irrotational flow -- $\mathbf{v} = -\nabla\Phi$

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} = -\nabla U - \frac{\nabla p}{\rho}$$

Linearized equation: $\nabla \left(-\frac{\partial\Phi}{\partial t} + g(z-h) + \frac{p}{\rho} \right) = 0$

Continuity equation within the fluid: $\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \Rightarrow \nabla \cdot \mathbf{v} = 0$

At surface: $z = h + \zeta \quad -\frac{\partial\Phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

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Keep only linear terms and assume that horizontal variation is only along x :

For $0 \leq z \leq h + \zeta$: $\nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x, z, t) = 0$

Consider a periodic waveform: $\Phi(x, z, t) = Z(z) \cos(k(x - ct))$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x, 0, t) = 0$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \quad Z(z) = A \cosh(kz)$$

At surface: $z = h + \zeta \quad \frac{\partial\zeta}{\partial t} = v_z(x, h + \zeta, t) = -\frac{\partial\Phi(x, h + \zeta, t)}{\partial z}$

Also: $-\frac{\partial\Phi(x, h + \zeta, t)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

$$\Rightarrow -\frac{\partial^2\Phi(x, h + \zeta, t)}{\partial t^2} + g\frac{\partial\zeta}{\partial t} = -\frac{\partial^2\Phi(x, h + \zeta, t)}{\partial t^2} - g\frac{\partial\Phi(x, h + \zeta, t)}{\partial z} = 0$$

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Velocity potential: $\Phi(x, z, t) = A \cosh(kz) \cos(k(x - ct))$

At surface: $\Phi(x, (h + \zeta), t) = A \cosh(k(h + \zeta)) \cos(k(x - ct))$

$$A \cosh(k(h + \zeta)) \cos(k(x - ct)) \left(k^2 c^2 - gk \frac{\sinh(k(h + \zeta))}{\cosh(k(h + \zeta))} \right) = 0$$

$$\Rightarrow c^2 = \frac{g}{k} \frac{\sinh(k(h + \zeta))}{\cosh(k(h + \zeta))} \approx \frac{g}{k} \tanh(kh)$$

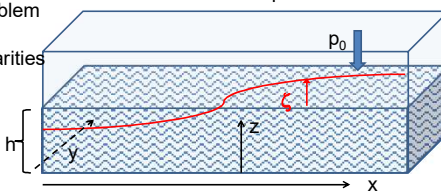
Note that this solution represents a pure plane wave. More likely, there would be a linear combination of wavevectors k . Additionally, your text considers the effects of surface tension. In this lecture, we will focus on the effects of the non-linear effects of Euler and continuity equations.

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Surface waves in an incompressible fluid

General problem including non-linearities



Within fluid: $0 \leq z \leq h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + g(z-h) = \text{constant} \quad \Phi = \Phi(x, y, z, t)$$

$$-\nabla^2 \Phi = 0 \quad \mathbf{v} = \mathbf{v}(x, y, z, t) = -\nabla \Phi(x, y, z, t)$$

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\frac{\partial \Phi(x, y, z, t)}{\partial z} \Big|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h + \zeta, t)$$

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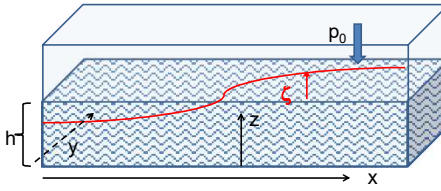
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Further simplifications; assume trivial y -dependence

$$\Phi = \Phi(x, z, t) \quad \zeta = \zeta(x, t)$$

Within fluid: $0 \leq z \leq h + \zeta$

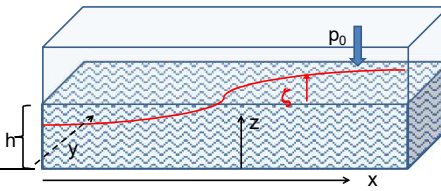
At surface: $v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$



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Non-linear effects in surface waves:



Dominant non-linear effects \Rightarrow soliton solutions

$$\zeta(x, t) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right) \quad \eta_0 = \text{constant}$$

where $c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h} \right)$

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Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x, z, t)$; $\mathbf{v}(x, z, t) = -\nabla\Phi(x, z, t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

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Defining equations for $\Phi(x, z, t)$ and $\zeta(x, t)$

where $0 \leq z \leq h + \zeta(x, t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \Rightarrow \frac{\partial^2 \Phi(x, z, t)}{\partial x^2} + \frac{\partial^2 \Phi(x, z, t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x, z, t)}{\partial t} + \frac{1}{2} \left[\underbrace{\left(\frac{\partial \Phi(x, z, t)}{\partial x} \right)^2}_{v_x^2} + \underbrace{\left(\frac{\partial \Phi(x, z, t)}{\partial z} \right)^2}_{v_z^2} \right] + g(z - h) = 0.$$

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Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x, 0, t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$\begin{aligned} v_z(x, z, t) \Big|_{z=h+\zeta} &= \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla \zeta + \frac{\partial \zeta}{\partial t} \\ &= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t} \\ \Rightarrow -\frac{\partial \Phi(x, z, t)}{\partial z} + \frac{\partial \Phi(x, z, t)}{\partial x} \frac{\partial \zeta(x, t)}{\partial x} - \frac{\partial \zeta(x, t)}{\partial t} \Big|_{z=h+\zeta} &= 0 \end{aligned}$$

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Analysis assuming water height z is small relative to variations in the direction of wave motion (x)

Taylor's expansion about $z = 0$:

$$\Phi(x, z, t) \approx \Phi(x, 0, t) + z \frac{\partial \Phi}{\partial z}(x, 0, t) + \frac{z^2}{2} \frac{\partial^2 \Phi}{\partial z^2}(x, 0, t) + \frac{z^3}{3!} \frac{\partial^3 \Phi}{\partial z^3}(x, 0, t) + \frac{z^4}{4!} \frac{\partial^4 \Phi}{\partial z^4}(x, 0, t) \dots$$

Note that the zero vertical velocity at the bottom ensures that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x, 0, t)$ vanish from the

Taylor expansion. In addition, the Laplace equation allows us to convert all even derivatives with respect to z to derivatives with respect to x .

$$\rightarrow \frac{\partial^2 \Phi(x, z, t)}{\partial x^2} + \frac{\partial^2 \Phi(x, z, t)}{\partial z^2} = 0$$

$$\text{Modified Taylor's expansion: } \Phi(x, z, t) \approx \Phi(x, 0, t) - \frac{z^2}{2} \frac{\partial^2 \Phi}{\partial x^2}(x, 0, t) + \frac{z^4}{4!} \frac{\partial^4 \Phi}{\partial x^4}(x, 0, t) \dots$$

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Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x, t)$

$$-\frac{\partial \Phi(x, h, t)}{\partial t} + g\zeta(x, t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x, t)$

$$-\frac{\partial \Phi(x, z, t)}{\partial z} - \frac{\partial \zeta(x, t)}{\partial t} \bigg|_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x, h, t)}{\partial z} \approx h \frac{\partial^2 \Phi(x, 0, t)}{\partial x^2} = -\frac{\partial \zeta(x, t)}{\partial t} \quad -\frac{\partial \Phi(x, h, t)}{\partial t} \approx -\frac{\partial \Phi(x, 0, t)}{\partial t} = -g\zeta(x, t)$$

$$\text{Decoupled equations: } \frac{\partial^2 \Phi(x, 0, t)}{\partial t^2} = gh \frac{\partial^2 \Phi(x, 0, t)}{\partial x^2}.$$

→ linear wave equation with $c^2 = gh$

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Analysis of non-linear equations --

Bernoulli equation evaluated at surface:

$$-\frac{\partial \Phi(x, z, t)}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial \Phi(x, z, t)}{\partial x} \right)^2 + \left(\frac{\partial \Phi(x, z, t)}{\partial z} \right)^2 \right] \bigg|_{z=h+\zeta} + g\zeta(x, t) = 0.$$

Consistency of surface velocity

$$-\frac{\partial \Phi(x, z, t)}{\partial z} + \frac{\partial \Phi(x, z, t)}{\partial x} \frac{\partial \zeta(x, t)}{\partial x} - \frac{\partial \zeta(x, t)}{\partial t} \bigg|_{z=h+\zeta} = 0$$

Representation of velocity potential from Taylor's expansion:

$$\Phi(x, z, t) \approx \Phi(x, 0, t) - \frac{z^2}{2} \frac{\partial^2 \Phi}{\partial x^2}(x, 0, t) + \frac{z^4}{4!} \frac{\partial^4 \Phi}{\partial x^4}(x, 0, t) \dots$$

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Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2} \frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left((h+\zeta) \frac{\partial^2 \phi}{\partial x^2} \right)^2 \right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2} \frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\Rightarrow \frac{\partial}{\partial x} \left((h+\zeta(x,t)) \frac{\partial \phi}{\partial x} \right) - \frac{h^3}{3!} \frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

These equations represent non-linear coupling of $\phi(x,t)$ and $\zeta(x,t)$.

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Coupled equations: $-\frac{\partial \phi}{\partial t} + \frac{h^2}{2} \frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + g\zeta = 0.$

$$\frac{\partial}{\partial x} \left((h+\zeta(x,t)) \frac{\partial \phi}{\partial x} \right) - \frac{h^3}{3!} \frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

Traveling wave solutions with new notation:

$$u \equiv x - ct \quad \phi(x,t) \equiv \chi(u) \quad \text{and} \quad \zeta(x,t) \equiv \eta(u)$$

Note that the wave "speed" c will be consistently determined

$$c \frac{d\chi(u)}{du} - \frac{ch^2}{2} \frac{d^3 \chi(u)}{du^3} + \frac{1}{2} \left(\frac{d\chi(u)}{du} \right)^2 + g\eta(u) = 0.$$

$$\frac{d}{du} \left((h+\eta(u)) \frac{d\chi(u)}{du} \right) - \frac{h^3}{6} \frac{d^4 \chi(u)}{du^4} + c \frac{d\eta(u)}{du} = 0.$$

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Integrating and re-arranging coupled equations

$$c \frac{d\chi(u)}{du} - \frac{ch^2}{2} \frac{d^3 \chi(u)}{du^3} + \frac{1}{2} \left(\frac{d\chi(u)}{du} \right)^2 + g\eta(u) = 0.$$

$$\chi' = -\frac{g}{c} \eta + \frac{h^2}{2} \chi''' - \frac{1}{2c} (\chi')^2 \approx -\frac{g}{c} \eta - \frac{h^2 g}{2c} \eta'' - \frac{g^2}{2c^3} \eta^2$$

$$\frac{d}{du} \left((h+\eta(u)) \frac{d\chi(u)}{du} \right) - \frac{h^3}{6} \frac{d^4 \chi(u)}{du^4} + c \frac{d\eta(u)}{du} = 0.$$

$$\Rightarrow (h+\eta) \frac{d\chi(u)}{du} - \frac{h^3}{6} \frac{d^3 \chi(u)}{du^3} + c\eta(u) = 0$$

Now we can express $\frac{d\chi(u)}{du} = \chi'$ in terms of η :

$$\chi' \approx -\frac{g}{c} \eta - \frac{h^2 g}{2c} \eta'' - \frac{g^2}{2c^3} \eta^2$$

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Integrating and re-arranging coupled equations – continued --
Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h + \eta) \left(-\frac{g}{c} \eta - \frac{h^2 g}{2c} \eta'' - \frac{g^2}{2c^3} \eta^2 \right) + \frac{h^3 g}{6c} \eta''' + c\eta = 0$$

$$\Rightarrow \left(1 - \frac{gh}{c^2} \right) \eta - \frac{gh^3}{3c^2} \eta'' - \frac{g}{c^2} \left(1 + \frac{gh}{2c^2} \right) \eta^2 = 0$$

$$\Rightarrow \left(1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3} \eta''(u) - \frac{3}{2h} [\eta(u)]^2 = 0.$$

Note: $c^2 = gh + \dots$

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Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of η

$$\Rightarrow \left(1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3} \eta''(u) - \frac{3}{2h} [\eta(u)]^2 = 0.$$

Soliton solution

$$\zeta(x, t) = \eta(x - ct) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right)$$

$$c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h} \right) \quad \text{where } \eta_0 \text{ is a constant}$$

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Steps to solution

$$\left(1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3} \eta''(u) - \frac{3}{2h} [\eta(u)]^2 = 0.$$

$$\text{Let } 1 - \frac{hg}{c^2} \equiv \frac{\eta_0}{h} \Rightarrow \frac{\eta_0}{h} \eta(u) - \frac{h^2}{3} \eta''(u) - \frac{3}{2h} [\eta(u)]^2 = 0.$$

$$\text{Multiply equation by } \eta'(u) \Rightarrow \frac{d}{du} \left(\frac{\eta_0}{2h} \eta^2(u) - \frac{h^2}{6} \eta'^2(u) - \frac{1}{2h} \eta^3(u) \right) = 0$$

Integrate wrt u and assume solution vanishes for $u \rightarrow \infty$

$$\frac{\eta_0}{2h} \eta^2(u) - \frac{h^2}{6} \eta'^2(u) - \frac{1}{2h} \eta^3(u) = 0$$

$$\eta'^2(u) = \frac{3}{h^2} \eta^2(u) (\eta_0 - \eta(u))$$

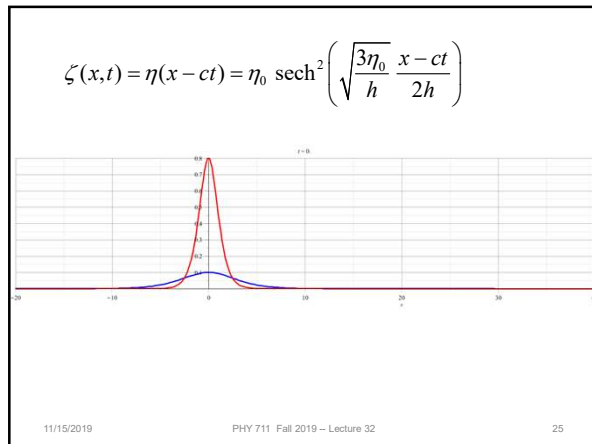
$$\frac{d\eta}{\eta(\eta_0 - \eta)^{1/2}} = \sqrt{\frac{3}{h^2}} du \Rightarrow \eta(u) = \frac{\eta_0}{\cosh^2 \left(\sqrt{\frac{3\eta_0}{4h^3}} u \right)}$$

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Relationship to “standard” form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \quad \bar{x} = \sqrt{\frac{3}{2h}} \frac{x}{h}, \quad \text{and} \quad \bar{t} = \sqrt{\frac{3}{2h}} \frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta \frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x}, \bar{t}) = \frac{\beta}{2} \operatorname{sech}^2 \left[\frac{\sqrt{\beta}}{2} (\bar{x} - \beta \bar{t}) \right].$$

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More details

Modified surface amplitude equation in terms of η :

$$\left(1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3} \eta''(u) - \frac{3}{2h} [\eta(u)]^2 = 0.$$

Some identities: $\frac{\eta_0}{h} = 1 - \frac{gh}{c^2}$; $\frac{\partial \eta}{\partial t} = -c \frac{d\eta}{du}$; $\frac{\partial \eta}{\partial x} = \frac{d\eta}{du}$.

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h} \eta' - \frac{h^2}{3} \eta''' - \frac{3}{h} \eta \eta' = 0.$$

Expression in terms of x and t :

$$-\frac{\eta_0}{ch} \frac{\partial \eta}{\partial t} - \frac{h^2}{3} \frac{\partial^3 \eta}{\partial x^3} - \frac{3}{h} \eta \frac{\partial \eta}{\partial x} = 0.$$

Expression in terms of \bar{x} and \bar{t} :

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta \frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

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Summary

Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x-ct}{2h} \right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h} \right) \quad \text{where } \eta_0 \text{ is a constant}$$

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Photo of canal soliton <http://www.ma.hw.ac.uk/solitons/>



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John Scott Russell and the solitary wave



Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves" (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

https://www.macs.hw.ac.uk/~chris/scott_russell.html

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

([Cet passage en français](#))

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.

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