

**PHY 711 Classical Mechanics and  
Mathematical Methods**  
10-10:50 AM MWF Olin 103

**Plan for Lecture 35**

**Review**

- 1. Comments on presentation and exam**
- 2. Mathematical methods**
- 3. Classical mechanics concepts**

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21	Fri, 10/18/2019	Chap. 7	Contour integrals; Exam due	#15	10/23/2019
22	Mon, 10/21/2019	Chap. 7	More about contour integrals		
23	Wed, 10/23/2019	Chap. 5	Rigid body motion	#16	10/25/2019
24	Fri, 10/25/2019	Chap. 5	Rigid body motion	#17	10/28/2019
25	Mon, 10/28/2019	Chap. 8	Elastic two-dimensional membranes	#18	11/01/2019
26	Wed, 10/30/2019	Chap. 9	Mechanics of 3 dimensional fluids		
27	Fri, 11/01/2019	Chap. 9	Fluid mechanics	#19	11/04/2019
28	Mon, 11/04/2019	Chap. 9	Sound waves		
	Wed, 11/06/2019	No class	NAWH out of town		
29	Fri, 11/08/2019	Chap. 9	Sound waves; Project Topic due	#20	11/11/2019
30	Mon, 11/11/2019	Chap. 9	Non-linear waves and shocks	#21	11/15/2019
31	Wed, 11/13/2019	Chap. 10	Surface waves in water	#22	11/18/2019
32	Fri, 11/15/2019	Chap. 10	Surface waves – non linear effects		
33	Mon, 11/18/2019	Chap. 11	Heat conduction	#23	11/22/2019
34	Wed, 11/20/2019	Chap. 12	Effects of viscosity on fluid dynamics		
	Fri, 11/22/2019		Class cancelled		
35	Mon, 11/25/2019		Review Chap. 1-12		
	Wed, 11/27/2019		Thanksgiving holiday		
	Fri, 11/29/2019		Thanksgiving holiday		
	Mon, 12/2/2019		Presentations I		
	Wed, 12/4/2019		Presentations II		
	Fri, 12/6/2019		Presentations III		

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2

**Comments on the “Computational Project”.**

Presumably you have all completed or nearly completed your project. Nevertheless, just to remind –

The purpose of this assignment is to provide an opportunity for you to study a topic of your choice in greater depth. The general guideline for your choice of project is that it should have something to do with classical mechanics, and there should be some degree of analytical or numerical computation associated with the project. The completed project will include a presentation to the class (~20 min + 5 min for questions). After completing your presentation, please turn in a copy of your presentation slides plus any notes and references.

If you have not already signed up for your presentation time, please do so as soon as possible.

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Comments on presentation schedule:  
**Schedule for Monday, December 2, 2019**

	Presenter	Topic
10:00-10:25		
10:25-10:50		

**Schedule for Wednesday, December 4, 2019**

	Presenter	Topic
10:00-10:25		
10:25-10:50		

**Schedule for Friday, December 6, 2019**

	Presenter	Topic
10:00-10:25		
10:25-10:50		

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Comments on take-home final –  
 Similar to mid-term in form  
**Available:** Fri. Dec. 6, 2019  
**Due before:** Mon. Dec. 16, 2019 before 11 AM

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**Review of mathematical methods**  
 Some useful identities for vectors and vector operators

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

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Vector relations for spherical polar coordinates

$$\nabla\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}$$

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<https://dlmf.nist.gov/>

NIST Digital Library of Mathematical Functions

Project News  
 2018-09-15 DLMF Update, Version 1.0.20  
 2018-06-22 DLMF Update, Version 1.0.19  
 2018-06-22 DLMF Update, Version 1.0.18  
 2018-03-27 DLMF Update, Version 1.0.17  
[More news](#)

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Example – special functions

A 10 Bessel Functions  
 A Bessel and Hankel Functions

10.1 Special Notation 10.3 Graphics

§10.2 Definitions

Contents

§10.200 Bessel's Equation

§10.200 Standard Solutions

§10.200 Numerically Satisfactory Pairs of Solutions

§10.2(i) Bessel's Equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0.$$

This differential equation has a regular singularity at  $z = 0$  with indices  $\pm\nu$ , and an irregular singularity at  $z = \infty$  of rank 1; compare §§2.7(i) and 2.7(ii).

§10.2(ii) Standard Solutions

Bessel Function of the First Kind

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z^2}{4}\right)^k.$$

The solution of (10.2.1) is an analytic function of  $z \in \mathbb{C}$ , except for a branch point at  $z = 0$  when  $\nu$  is not an integer. The principal branch of  $J_\nu(z)$  corresponds to the principal value of  $\left(\frac{z}{2}\right)^\nu$  (§4.2(ii)) and is analytic in the  $z$ -plane cut along the interval  $(-\infty, 0]$ .

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**Complex numbers**  $i \equiv \sqrt{-1}$   $i^2 = -1$   
 Define  $z = x + iy$   
 $|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$   
**Polar representation**  
 $z = \rho(\cos \phi + i \sin \phi) = \rho e^{i\phi}$

**Functions of complex variables**  
 $f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x, y) + iv(x, y)$

**Derivatives: Cauchy-Riemann equations**

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i \frac{\partial v(z)}{\partial x} \quad \frac{\partial f(z)}{\partial y} = \frac{\partial u(z)}{\partial y} + i \frac{\partial v(z)}{\partial y} = \frac{\partial v(z)}{\partial y} - i \frac{\partial u(z)}{\partial y}$$

Argue that  $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{\partial y} \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$  and  $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

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**Analytic function**

$f(z)$  is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Riemann conditions

➔ A closed integral of an analytic function is zero.

However:

Behavior of  $f(z) = \frac{1}{z^n}$  about the point  $z = 0$ :

For an integer  $n$ , consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

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**Contour integration methods --**

$$\oint_C f(z) dz = 2\pi i \sum_p \text{Res}(f(z_p))$$

The diagram shows a complex plane with a red contour  $C$  that encloses several poles  $z_p$ . One pole is labeled "No contribution" and another is labeled  $2\pi i \text{Res}(f(z_p))$ . A formula  $f(z) \approx \frac{\text{Res}(f(z_p))}{z - z_p}$  is shown near a pole. The axes are labeled  $\Re(z) = x$  and  $\Im(z) = y$ .

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General formula for determining residue:

Suppose that in the neighborhood of  $z_p$ ,  $f(z) \approx \frac{g(z)}{(z-z_p)^m} \equiv \frac{\text{Res}(f(z_p))}{z-z_p}$

Since  $g(z)$  is analytic near  $z_p$ , we can make a Taylor expansion about  $z_p$ :

$$g(z) \approx g(z_p) + (z-z_p) \frac{dg(z_p)}{dz} + \dots + \frac{(z-z_p)^{m-1}}{(m-1)!} \frac{d^{m-1}g(z_p)}{dz^{m-1}} + \dots$$

$$\Rightarrow \text{Res}(f(z_p)) = \lim_{z \rightarrow z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1} \left( (z-z_p)^m f(z) \right)}{dz^{m-1}} \right\}$$

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Fourier transforms --

Definition of Fourier Transform for a function  $f(t)$ :

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' f(t') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' f(t') \delta(t'-t)$$

**Note: The location of the  $2\pi$  factor varies among texts.**

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Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt (f(t))^* f(t) = 2\pi \int_{-\infty}^{\infty} d\omega (F(\omega))^* F(\omega)$$

Check:

$$\int_{-\infty}^{\infty} dt (f(t))^* f(t) = \int_{-\infty}^{\infty} dt \left( \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right)^* \int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t}$$

$$= \int_{-\infty}^{\infty} d\omega' F^*(\omega') \int_{-\infty}^{\infty} d\omega F(\omega) \int_{-\infty}^{\infty} dt e^{i(\omega'-\omega)t}$$

$$= \int_{-\infty}^{\infty} d\omega' F^*(\omega') \int_{-\infty}^{\infty} d\omega F(\omega) 2\pi \delta(\omega'-\omega)$$

$$= 2\pi \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$

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Doubly discrete Fourier Transforms

Doubly periodic functions

$$\omega \rightarrow \frac{2\pi\nu}{T} \qquad t \rightarrow \frac{\mu T}{2N+1}$$

$$\tilde{f}_\mu = \frac{1}{2N+1} \sum_{\nu=-N}^N \tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu = \sum_{\mu=-N}^N \tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

→Fast Fourier Transforms (FFT)

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Mechanics topics

- Scattering theory
- Lagrangian mechanics
- Hamiltonian mechanics
- Liouville theorem
- Rigid body motion
- Normal modes of oscillation about equilibrium
- Wave motion
- Fluid mechanics (ideal or including viscosity; linear and nonlinear)
- Heat conduction

Note: The following review slides are necessarily brief. Please refer to the original lecture slides for details. Please email: [natalie@wfu.edu](mailto:natalie@wfu.edu) with any corrections/suggestions

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Scattering theory

**Note:** The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.

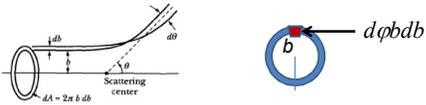


Figure from Marion & Thornton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi b db}{d\phi \sin\theta d\theta} = \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in  $\phi$

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**Lagrangian mechanics**  
 Given the Lagrangian function:  $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$ ,  
 The physical trajectories of the generalized coordinates  $\{q_\sigma(t)\}$   
 Are those which minimize the action:  $S = \int L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$   
 Euler-Lagrange equations:  

$$\sum_\sigma \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$
  
 For the case that there both mechanical and  
 electromagnetic contributions in terms of electric and magnetic fields:  

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$
  

$$L = T - U_{\text{mech}} - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

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**Recipe for constructing the Hamiltonian and analyzing the equations of motion**

1. Construct Lagrangian function :  $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$
2. Compute generalized momenta :  $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression :  $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function :  $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$
5. Analyze canonical equations of motion :  

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

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**Liouville's Theorem (1838)**

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space :  $D = D(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$   

$$\frac{dD}{dt} = \sum_\sigma \left( \frac{\partial D}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$
  
 According to Liouville's theorem :  $\frac{dD}{dt} = 0$

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Rigid body motion

Moment of inertia tensor :

$$\bar{\mathbf{I}} \equiv \sum_p m_p (\mathbf{1}r_p^2 - \mathbf{r}_p \mathbf{r}_p) \quad (\text{dyad notation})$$

In a reference frame attached to the object, there are 3 moments of inertia and 3 distinct principal axes

Representation of rotational kinetic energy:

$$T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$= \frac{1}{2} I_1 [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma]^2$$

$$+ \frac{1}{2} I_2 [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma]^2$$

$$+ \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2$$

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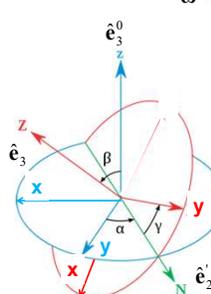
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Euler's transformation between body fixed and inertial reference frames

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2^1 + \dot{\gamma} \hat{\mathbf{e}}_3^2$$


$$\tilde{\boldsymbol{\omega}} = [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma] \hat{\mathbf{e}}_1$$

$$+ [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma] \hat{\mathbf{e}}_2$$

$$+ [\dot{\alpha} \cos \beta + \dot{\gamma}] \hat{\mathbf{e}}_3$$

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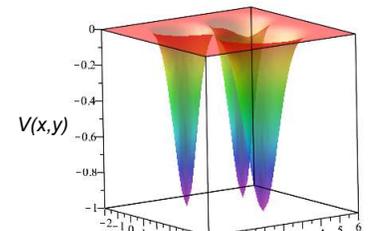
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Normal modes of vibration -- potential in 2 and more dimensions

$$V(x, y) \approx V(x_{eq}, y_{eq}) + \frac{1}{2} (x - x_{eq})^2 \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_{eq}, y_{eq}}$$

$$+ \frac{1}{2} (y - y_{eq})^2 \left. \frac{\partial^2 V}{\partial y^2} \right|_{x_{eq}, y_{eq}} + (x - x_{eq})(y - y_{eq}) \left. \frac{\partial^2 V}{\partial x \partial y} \right|_{x_{eq}, y_{eq}}$$


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Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

Potential contribution for spring 13:

$$V_{13} = \frac{1}{2}k(|\ell_{13} + \mathbf{u}_3 - \mathbf{u}_1| - |\ell_{13}|)^2$$

$$\approx \frac{1}{2}k\left(\frac{\ell_{13} \cdot (\mathbf{u}_3 - \mathbf{u}_1)}{|\ell_{13}|}\right)^2$$

$$\approx \frac{1}{2}k\left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1})\right)^2$$

$$\ell_{13} = |\ell_{13}|\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right)$$

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Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

Potential contributions:  $V = V_{12} + V_{13} + V_{23}$

$$\approx \frac{1}{2}k\left(\frac{\ell_{12} \cdot (\mathbf{u}_2 - \mathbf{u}_1)}{|\ell_{12}|}\right)^2 + \frac{1}{2}k\left(\frac{\ell_{13} \cdot (\mathbf{u}_3 - \mathbf{u}_1)}{|\ell_{13}|}\right)^2$$

$$+ \frac{1}{2}k\left(\frac{\ell_{23} \cdot (\mathbf{u}_3 - \mathbf{u}_2)}{|\ell_{23}|}\right)^2$$

$$\approx \frac{1}{2}k(u_{x2} - u_{x1})^2$$

$$+ \frac{1}{2}k\left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1})\right)^2$$

$$+ \frac{1}{2}k\left(\frac{1}{2}(u_{x2} - u_{x3}) - \frac{\sqrt{3}}{2}(u_{y2} - u_{y3})\right)^2$$

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Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

$$\frac{k}{m} \begin{bmatrix} \frac{5}{4} & -1 & -\frac{1}{4} & \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} \\ -1 & \frac{5}{4} & -\frac{1}{4} & 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 \\ \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} & \frac{3}{4} & 0 & -\frac{3}{4} \\ 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & \frac{3}{4} & -\frac{3}{4} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & -\frac{3}{4} & -\frac{3}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix} = \omega^2 \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix}$$

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Discrete particle interactions → continuous media →  
 The wave equation

Initial value solutions  $\mu(x,t)$  to the wave equation;  
 attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}(\varphi(x-ct) + \varphi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

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Mechanical motion of fluids

Newton's equations for fluids  
 Use Euler formulation; following "particles" of fluid

Variables: Density  $\rho(x,y,z,t)$   
 Pressure  $p(x,y,z,t)$   
 Velocity  $\mathbf{v}(x,y,z,t)$

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v} + \frac{1}{\rho} \left( \zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \mathbf{v})$$

Continuity condition  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

} Viscosity contributions

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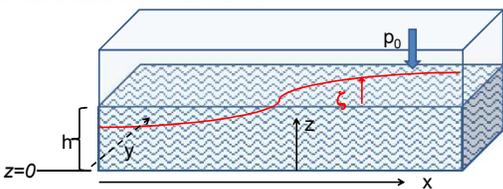
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Fluid mechanics of incompressible fluid plus surface  
 Non-linear effects in surface waves:



Dominant non-linear effects ⇒ soliton solutions

$$\zeta(x,t) = \eta_0 \operatorname{sech}^2 \left( \sqrt{\frac{3\eta_0}{h}} \frac{x-ct}{2h} \right) \quad \eta_0 = \text{constant}$$

where  $c = \sqrt{\frac{gh}{1 + \frac{\eta_0}{2h}}} \approx \sqrt{gh} \left( 1 + \frac{\eta_0}{2h} \right)$

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