PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 3:
Textbook reading: Chapter 1

1. **Continuation of scattering theory discussion**
2. **Center of mass reference fame and its relationship to "laboratory" reference frame**
3. **Analytical evaluation of the differential scattering cross section**

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PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	OPL 103	http://www.wfu.edu/~natalie/f19phy711/

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/26/2019	Chap. 1	Introduction	#1	8/30/2019
2	Wed, 8/28/2019	Chap. 1	Scattering theory	#2	9/02/2019
3	Fri, 8/30/2019	Chap. 1	Scattering theory	#3	9/04/2019
4	Mon, 9/02/2019	Chap. 1	Scattering theory		
5	Wed, 9/04/2019	Chap. 1	Scattering theory		
6	Fri, 9/06/2019	Chap. 2	Non-inertial coordinate systems		
7	Mon, 9/9/2019	Chap. 3	Calculus of Variation		
8	Wed, 9/11/2019	Chap. 3	Calculus of Variation		

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PHY 711 -- Assignment #2

Aug. 28, 2019

Read Chapter 1 in **Fetter & Walecka**.

1. In class, we started the derivation of the differential cross section for the elastic scattering of a beam of particles of mass *m* having an initial velocity $\mathbf{u}_i$ hitting a stationary spherical hard sphere target having mass *M* (uniformly distributed) with mutual radius *R*, scattered at an angle θ in the laboratory frame of reference. Complete the derivation to find the expression for the differential cross section as a function of *R*.

Update: Note that the general solution for arbitrary $x=m/M$ is complicated. Work out the case for $x=0$ in detail. Also consider the case of $x=1$. Extra credit for additional analysis.

PHY 711 -- Assignment #3

Aug. 30, 2019

Read Chapter 1 in **Fetter & Walecka**.

1. Work out the details of the equations relating the laboratory scattering angle to the scattering angle in the center of mass reference frame. Similarly, work out the details of the relationship between the differential scattering cross sections in the laboratory and center of mass frames.

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Scattering theory:

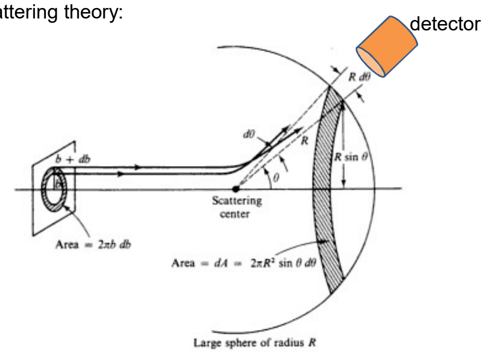

Figure 5.5 The scattering problem and relation of cross section to impact parameter.

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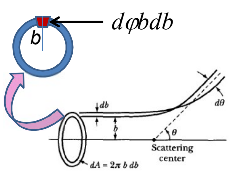
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$= \text{Area of incident beam that is scattered into detector at angle } \theta$



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

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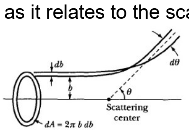
Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



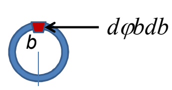


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

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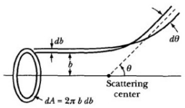
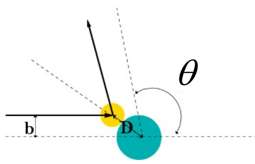
Simple example – collision of hard spheres



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

$$b(\theta) = ?$$



$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

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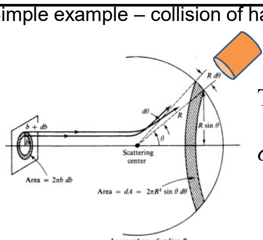
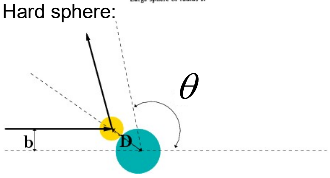
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

Hard sphere:



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

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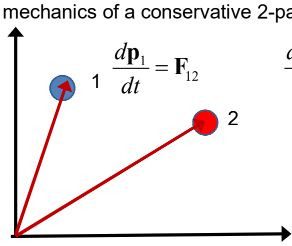
Relationship of scattering cross-section to particle interactions --
Classical mechanics of a conservative 2-particle system.

$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$



$$\mathbf{F}_{12} = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) \quad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

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Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1-\mathbf{r}_2)=V(|\mathbf{r}_1-\mathbf{r}_2|)\equiv V(r)$

Hard sphere: $V(r)=\begin{cases}\infty & r\le a\\ 0 & r>a\end{cases}$

Coulomb or gravitational: $V(r)=\frac{K}{r}$

Lennard-Jones: $V(r)=\frac{A}{r^{12}}-\frac{B}{r^6}$

Scattering theory can help us analyze the interaction potential *V(r)*. First we need to simply the number of variables.

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Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1+m_2\mathbf{r}_2=(m_1+m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1+m_2\dot{\mathbf{r}}_2=(m_1+m_2)\dot{\mathbf{R}}_{CM}=(m_1+m_2)\mathbf{V}_{CM}$$

$$E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2+V(\mathbf{r}_1-\mathbf{r}_2)$$

$$=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu|\mathbf{v}_1-\mathbf{v}_2|^2+V(\mathbf{r}_1-\mathbf{r}_2)$$

where: $\mu\equiv\frac{m_1m_2}{m_1+m_2}$

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Classical mechanics of a conservative 2-particle system -- continued

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu|\mathbf{v}_1-\mathbf{v}_2|^2+V(\mathbf{r}_1-\mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1-\mathbf{r}_2)=V(|\mathbf{r}_1-\mathbf{r}_2|)\equiv V(r_{12})$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12}\equiv\mathbf{r}_{12}\times\mu\mathbf{v}_{12}$$

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu v_{12}^2+\frac{L_{12}^2}{2\mu r_{12}^2}+V(r_{12})$$

Simpler notation:

$$E=\frac{1}{2}(m_1+m_2)V_{CM}^2+\frac{1}{2}\mu\dot{r}^2+\frac{\ell^2}{2\mu r^2}+V(r)$$

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Simpler notation:

$$E = \frac{1}{2}(m_1+m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

constants

vary in time

For scattering analysis only need to know trajectory **before** and **after** the collision.

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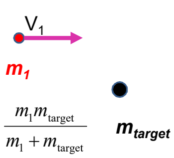
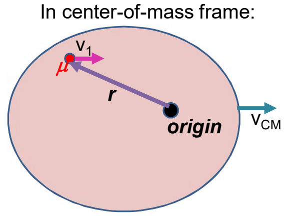
Note: The following analysis will be carried out in the center of mass frame of reference.

In laboratory frame: In center-of-mass frame:



$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = |\mathbf{r} \times \mu \mathbf{v}_1|$$



Also note: We are assuming that the interaction between particle and target *V(r)* conserves energy and angular momentum.

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It is often convenient to analyze the scattering cross section in the center of mass reference frame.

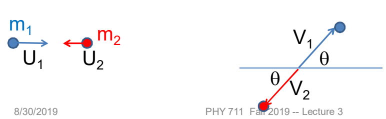
Relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before After

Center of mass reference frame:

Before After



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Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest :

$$\mathbf{V}_{CM} = \frac{m_1}{m_1+m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1+m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1+m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

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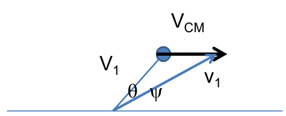
Relationship between center of mass and laboratory frames of reference



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$v_1 \sin\psi = V_1 \sin\theta$$
$$v_1 \cos\psi = V_1 \cos\theta + V_{CM}$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

For elastic scattering

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Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2 + \tfrac{1}{2}m_2U_2^2 + \tfrac{1}{2}\cancel{(m_1+m_2)V_{CM}^2}$$
$$= \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 + \tfrac{1}{2}\cancel{(m_1+m_2)V_{CM}^2}$$

Also note:

$$m_1\mathbf{U}_1 + m_2\mathbf{U}_2 = 0 \qquad m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = 0$$

$$\mathbf{U}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM} \qquad \mathbf{U}_2 = -\mathbf{V}_{CM}$$

$$\Rightarrow |\mathbf{U}_1| = |\mathbf{V}_1| \quad \text{and} \quad |\mathbf{U}_2| = |\mathbf{V}_2| = |\mathbf{V}_{\mathbf{CM}}|$$

Also note that : $m_1|\mathbf{U}_1| = m_2|\mathbf{U}_2|$

So that : $V_{CM}/V_1 = V_{CM}/U_1 = m_1/m_2$

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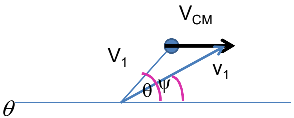
Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$v_1 \sin\psi = V_1 \sin\theta$$
$$v_1 \cos\psi = V_1 \cos\theta + V_{CM}$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\text{Also:} \quad \cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2m_1/m_2 \cos\theta + (m_1/m_2)^2}}$$

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Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}} = \left|\frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi}\right| = \left|\frac{d\cos\theta}{d\cos\psi}\right|$$

Using:

$$\cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2(m_1/m_2)\cos\theta + (m_1/m_2)^2}}$$

$$\left|\frac{d\cos\psi}{d\cos\theta}\right| = \frac{(m_1/m_2)\cos\theta + 1}{\left(1 + 2(m_1/m_2)\cos\theta + (m_1/m_2)^2\right)^{3/2}}$$

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Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

$$\text{where:} \quad \tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

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$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\text{where :}\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

Example: suppose $m_1 = m_2$

$$\text{In this case :}\quad \tan\psi=\frac{\sin\theta}{\cos\theta+1}\quad\Rightarrow\psi=\frac{\theta}{2}$$

$$\text{note that}\quad 0\le\psi\le\frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

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Summary --
Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

For elastic scattering

$$\text{where :}\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

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$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\text{where :}\quad \tan\psi=\frac{\sin\theta}{\cos\theta+m_1/m_2}$$

Example: suppose $m_1 = m_2$

$$\text{In this case :}\quad \tan\psi=\frac{\sin\theta}{\cos\theta+1}\quad\Rightarrow\psi=\frac{\theta}{2}$$

$$\text{note that}\quad 0\le\psi\le\frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

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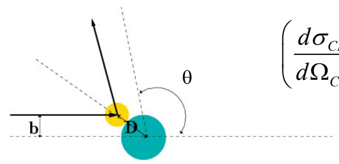
Example of hard spheres



$$\left(\frac{d\sigma_{CM}}{d\Omega_{CM}}\right) = \frac{D^2}{4}$$

Cross section in lab frame when $m_1 = m_2$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + 1} \quad \Rightarrow \psi = \frac{\theta}{2} \quad \Rightarrow \text{note that} \quad 0 \le \psi \le \frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right) \cdot 4\cos\psi = D^2\cos\psi$$

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Hard sphere example – continued

$m_1 = m_2$

Center of mass frame

Lab frame

$$\left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right) = \frac{D^2}{4} \qquad \left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = D^2\cos\psi \quad \psi = \frac{\theta}{2}$$

$$\int \frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}} d\Omega_{CM} = \qquad \int \frac{d\sigma_{lab}(\psi)}{d\Omega_{lab}} d\Omega_{lab} =$$

$$\frac{D^2}{4} 4\pi = \pi D^2 \qquad 2\pi D^2 \int_0^{\pi/2} \cos\psi \, \sin\psi \, d\psi = \pi D^2$$

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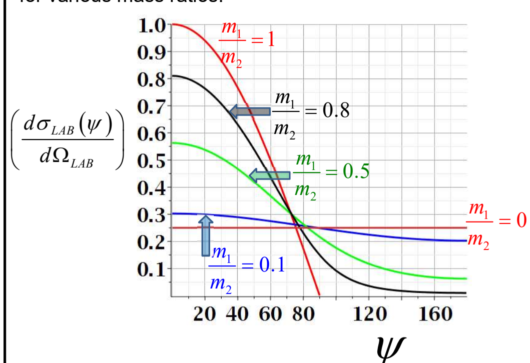
Scattering cross section for hard sphere in lab frame for various mass ratios:



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For visualization, is convenient to make a "parametric" plot of

$$\left(\frac{d\sigma_{LAB}}{d\Omega}(\theta)\right) \text{ vs } \psi(\theta)$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta+1}$$

$$\tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

Maple syntax:

```
> plot( { [psi(theta, 0), sigma(theta, 0), theta = 0.001 ..3.14], [psi(theta, .1), sigma(theta, .1), theta
   = 0.001 ..3.14], [psi(theta, .5), sigma(theta, .5), theta = 0.001 ..3.14], [psi(theta, .8),
   sigma(theta, .8), theta = 0.001 ..3.14], [psi(theta, 1), sigma(theta, 1), theta = 0.001 ..3.14]},
   thickness = 3, font = ['Times','bold', 24], gridlines = true, color = [red, blue, green, black,
   orange])
```


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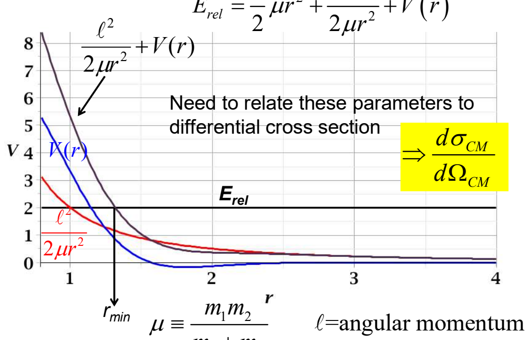
For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

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Focusing on the center of mass frame of reference:

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \le D \\ 0 & r > D \end{cases}$ X

Coulomb or gravitational: $V(r) = \frac{K}{r}$ ←

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$

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