

PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 5:
Reading: Chapter 2 of Fetter & Walecka

1. Recap of scattering theory from Chapter 1
2. Physics described in accelerated coordinate frames
 - a. Linear acceleration
 - b. Angular acceleration
 - c. Foucault pendulum

9/4/2019 PHY 711 Fall 2019 – Lecture 5 1

1

Course schedule
 (Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment Due
1 Mon, 8/26/2019	Chap. 1	Introduction	#1
2 Wed, 8/28/2019	Chap. 1	Scattering theory	#2
3 Fri, 8/30/2019	Chap. 1	Scattering theory	#3
4 Mon, 9/2/2019	Chap. 1	Scattering theory	#4
5 Wed, 9/4/2019	Chap. 2	Non-inertial coordinate systems	#5
6 Fri, 9/6/2019	Chap. 3	Calculus of Variation	
7 Mon, 9/9/2019	Chap. 3	Calculus of Variation	
8 Wed, 9/11/2019	Chap. 3	Calculus of Variation	

9/4/2019 PHY 711 Fall 2019 – Lecture 5 2

2

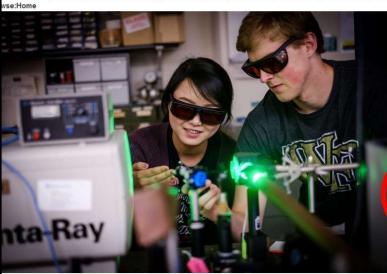
Wednesday colloquium at 3 PM followed by refreshments at 4 PM

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Events

Physics Picnic — Monday Sept. 2, 2019 at 5:30 PM
 The Society for Physics Students (SPS) is hosting a cookout outside of Polo Hall on Monday Sept. 2, 2019 starting at 5:30 PM. All Physics students (undergraduate majors, minors, and graduate students) are invited.

Colloquium: "Solar-Maze WFOF for Energy Efficient Hydrocarbon Separation" – Wednesday Sept. 4, 2019 at 3 PM
 Professor Jing Li (Department of Chemistry and Chemical Biology, Rutgers University, Piscataway, NJ USA) George P. Williams, Jr., Lecture Hall, (Cm 101) Wednesday Sept. 4, 2019, at 3:00 PM. There will ...

News

9/4/2019 PHY 711 Fall 2019 – Lecture 5 3

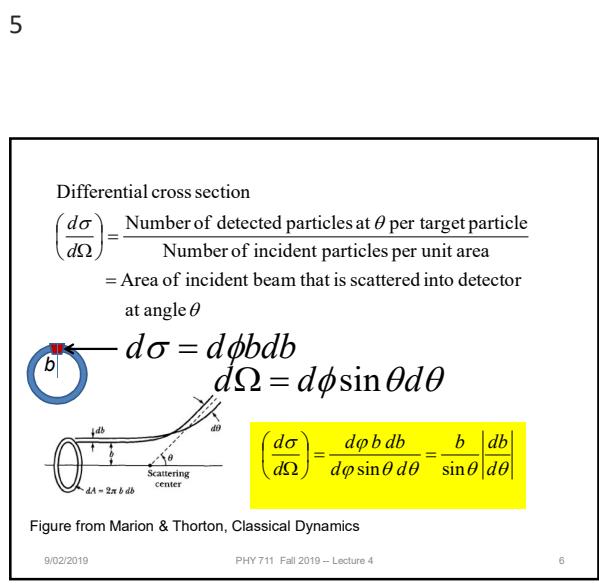
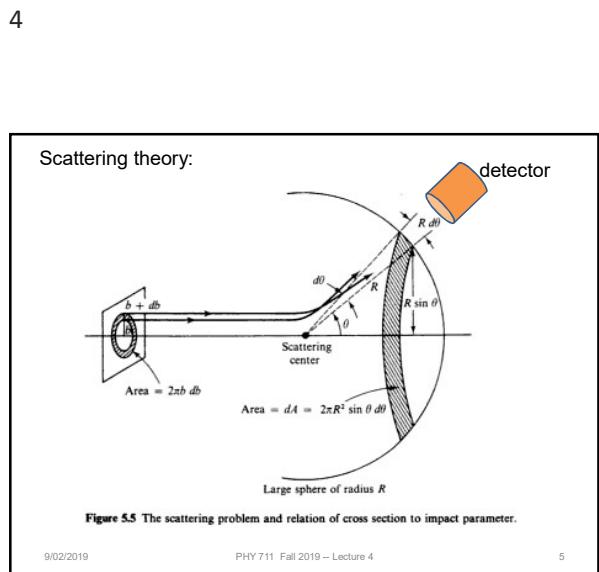
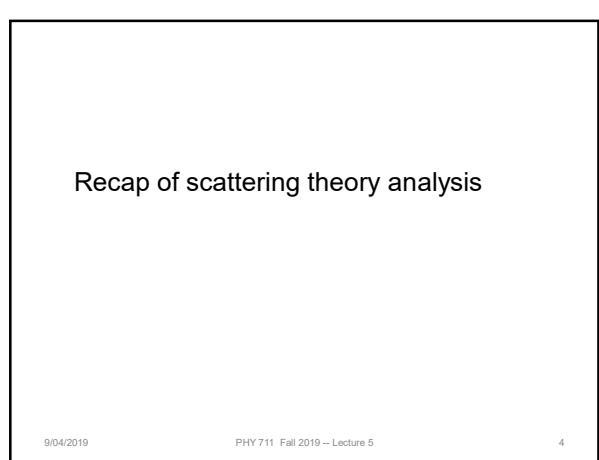
3

Recap of scattering theory analysis

9/04/2019

PHY 711 Fall 2019 – Lecture 5

4



Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$E = E_{CM} + E_{rel}$$

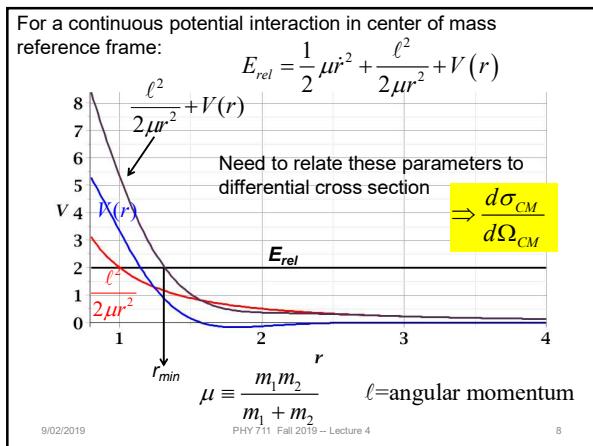
Focus of analysis

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

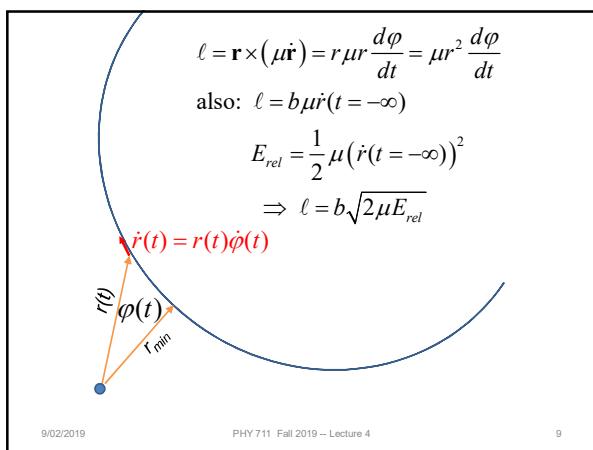
$$= \frac{1}{2}\mu\dot{r}^2 + V_{eff}^\ell(r)$$

8/30/2019 PHY 711 Fall 2019 – Lecture 3 7

7

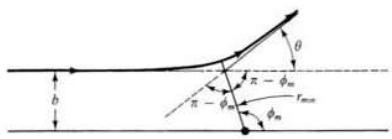


8



9

Relationship between ϕ_{\max} and θ :



$$2(\pi - \varphi_{\max}) + \theta = \pi$$

$$\Rightarrow \varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

9/2/2019

PHY 711 Fall 2019 – Lecture 4

10

10

$$\varphi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E_{rel}}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E_{rel}}}} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E_{rel}}}} \right)$$

9/2/2019

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11

11

Recap of equations for scattering cross section in the center of mass frame of reference

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E_{rel}}}} \right)$$

where r_{\min} is found from

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E_{rel}} = 0$$

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12

12

Physics in accelerated reference frames

9/04/2019

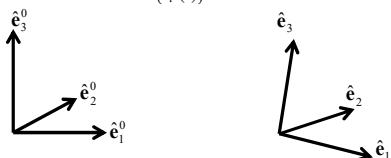
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13

13

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\{\hat{e}_i^0\}$
- For some problems, it is convenient to transform the equations into a non-inertial coordinate system $\{\hat{e}_i(t)\}$



9/04/2019

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14

14

Comparison of analysis in “inertial frame” versus “non-inertial frame”

Denote by \hat{e}_i^0 a fixed coordinate system

Denote by \hat{e}_i a moving coordinate system

$$\mathbf{V} = \sum_{i=1}^3 V_i^0 \hat{e}_i^0 = \sum_{i=1}^3 V_i \hat{e}_i$$

$$\left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \sum_{i=1}^3 \frac{dV_i^0}{dt} \hat{e}_i^0 = \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

$$\text{Define : } \left(\frac{d\mathbf{V}}{dt} \right)_{body} \equiv \sum_{i=1}^3 \frac{dV_i}{dt} \hat{e}_i$$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{e}_i}{dt}$$

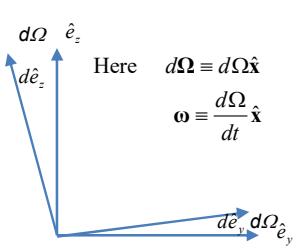
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15

15

Properties of the frame motion (rotation only):



Here $d\Omega \equiv d\Omega \hat{x}$

$$\omega \equiv \frac{d\Omega}{dt} \hat{x}$$

$$\begin{aligned} d\hat{e}_y &= d\Omega \hat{e}_z \\ d\hat{e}_z &= -d\Omega \hat{e}_y \\ \Rightarrow d\hat{\mathbf{e}} &= d\Omega \times \hat{\mathbf{e}} \\ \frac{d\hat{\mathbf{e}}}{dt} &= \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \\ \boxed{\frac{d\hat{\mathbf{e}}}{dt} = \omega \times \hat{\mathbf{e}}} \end{aligned}$$

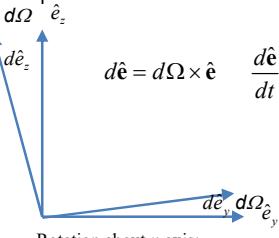
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16

16

Properties of the frame motion (rotation only):



Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \quad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) - 1 & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) - 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

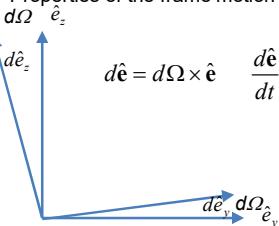
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17

17

Properties of the frame motion (rotation only):



Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} = -d\Omega e_z \hat{e}_y + d\Omega e_y \hat{e}_z = d\Omega \hat{x} \times \hat{\mathbf{e}}$$

9/4/2019

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18

18

Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \sum_{i=1}^3 V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt} \right)_{inertial} = \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration (rotation only):

$$\left(\frac{d}{dt} \frac{d\mathbf{V}}{dt} \right)_{inertial} = \left[\left(\frac{d}{dt} \right)_{body} + \boldsymbol{\omega} \times \right] \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2} \right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2} \right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt} \right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

9/4/2019

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19

19

Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration $\left(\frac{d^2\mathbf{a}}{dt^2} \right)_{inertial}$

Newton's laws; Let \mathbf{r} denote the position of particle of mass m :

$$m \left(\frac{d^2\mathbf{r}}{dt^2} \right)_{inertial} = \mathbf{F}_{ext}$$

$$m \left(\frac{d^2\mathbf{r}}{dt^2} \right)_{body} = \mathbf{F}_{ext} - m \left(\frac{d^2\mathbf{a}}{dt^2} \right)_{inertial} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{body} - m \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

Coriolis
force

Centrifugal
force

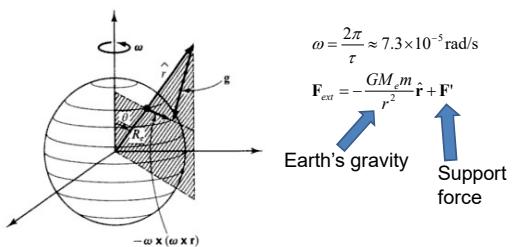
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20

20

Motion on the surface of the Earth:



Main contributions:

$$m \left(\frac{d^2\mathbf{r}}{dt^2} \right)_{earth} = -\frac{GM_m}{r^2} \hat{\mathbf{r}} + F' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{earth} - m \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$$

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21

21

Non-inertial effects on effective gravitational "constant"

$$m \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{\text{earth}} = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}' - 2m\omega \times \left(\frac{d\mathbf{r}}{dt} \right)_{\text{earth}} - m \frac{d\omega}{dt} \times \mathbf{r} - m\omega \times \omega \times \mathbf{r}$$

For $\left(\frac{d\mathbf{r}}{dt} \right)_{\text{earth}} = 0$ and $\left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{\text{earth}} = 0$,

$$0 = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}' - m\omega \times \omega \times \mathbf{r}$$

$\mathbf{F}' = -mg$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2} \hat{\mathbf{r}} - \omega \times \omega \times \mathbf{r} \Big|_{r=R_e}$$

$$= \left(-\frac{GM_e}{R_e^2} + \omega^2 R_e \sin^2 \theta \right) \hat{\mathbf{r}} + \sin \theta \cos \theta \omega^2 R_e \hat{\theta}$$

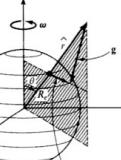
↑
0.03 m/s²

9.80 m/s²

9/4/2019

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22

 $-\omega \times (\omega \times \mathbf{r})$

22

Foucault pendulum http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm



The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

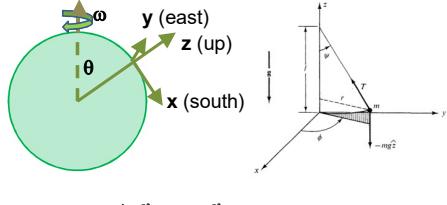
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23

Equation of motion on Earth's surface

$$m \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{\text{surface}} = -\frac{GM_e m}{r^2} \hat{\mathbf{r}} + \mathbf{F}' - 2m\omega \times \left(\frac{d\mathbf{r}}{dt} \right)_{\text{surface}} - m \frac{d\omega}{dt} \times \mathbf{r} - m\omega \times \omega \times \mathbf{r}$$



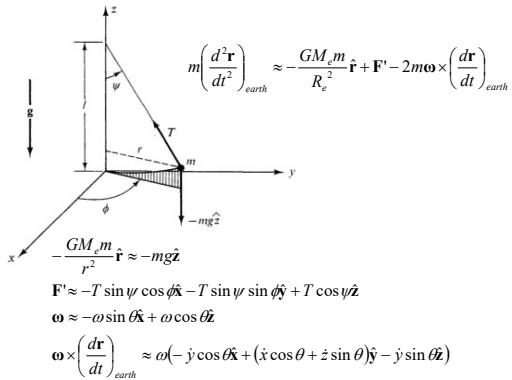
$$\omega \approx -\omega \sin \theta \hat{x} + \omega \cos \theta \hat{z}$$

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24

Foucault pendulum continued – keeping leading terms:



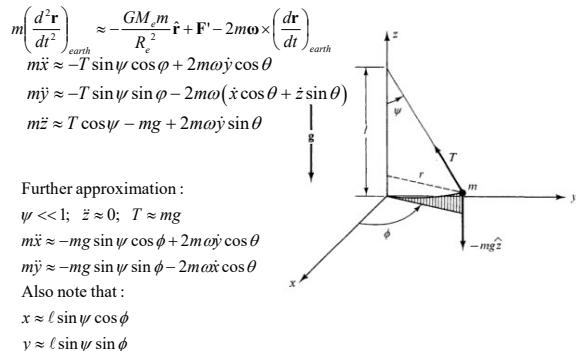
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25

25

Foucault pendulum continued – keeping leading terms:



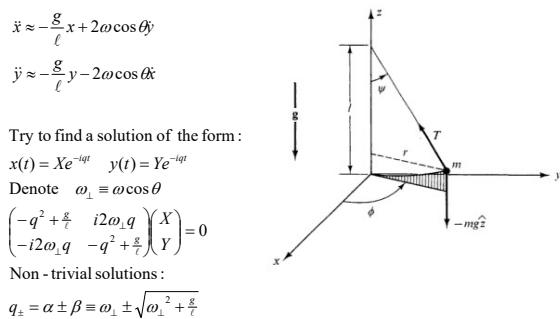
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26

26

Foucault pendulum continued – coupled equations:



9/4/2019

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27

Foucault pendulum continued – coupled equations:

Solution continued :

$$x(t) = X e^{-i\alpha t} \quad y(t) = Y e^{-i\beta t}$$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_{\perp}q \\ -i2\omega_{\perp}q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non-trivial solutions :

$$q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}^2 + \frac{g}{\ell}}$$

$$\text{Amplitude relationship : } X = iY$$

General solution with complex amplitudes C and D :

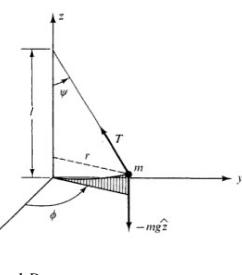
$$x(t) = \text{Re}\{Ce^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\}$$

$$y(t) = \text{Re}\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\}$$

9/4/2019

PHY 711 Fall 2019 – Lecture 5

28



28

General solution with complex amplitudes C and D :

$$x(t) = \text{Re}\{Ce^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\}$$

$$y(t) = \text{Re}\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\}$$

$$q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}^2 + \frac{g}{\ell}} \approx \omega_{\perp} \pm \sqrt{\frac{g}{\ell}}$$

$$\text{since } \omega_{\perp} \approx 7 \times 10^{-5} \cos \theta \text{ rad/s} \ll \sqrt{\frac{g}{\ell}}$$

$$\text{Suppose: } x(0) = X_0 \quad y(0) = 0$$

$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \cos(\omega_{\perp}t) \quad \text{Note that } \omega = \frac{2\pi}{24 \cdot 3600 \text{ s}} = 7 \times 10^{-5} \text{ rad/sec}$$

$$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \sin(\omega_{\perp}t)$$

9/4/2019

29

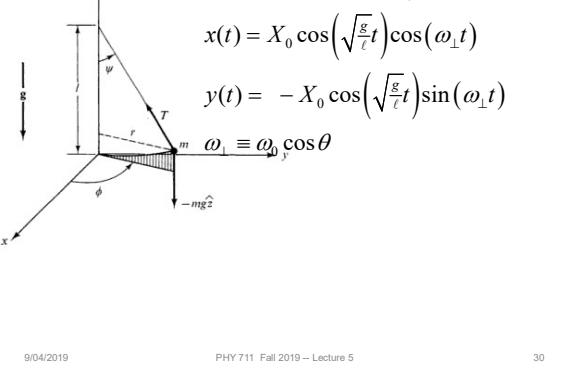
29

Summary of approximate solution for Foucault pendulum:

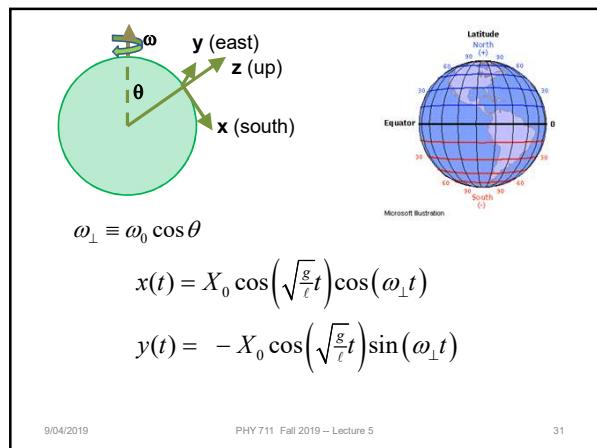
$$\text{Suppose: } x(0) = X_0 \quad y(0) = 0$$

$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \cos(\omega_{\perp}t)$$

$$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \sin(\omega_{\perp}t)$$



30



9/4/2019

PHY 711 Fall 2019 – Lecture 5

31

31