

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 6:

Start reading Chapter 3.17 –

- 1. Introduction to the calculus of variations**
- 2. Example problems**

9/6/2019

PHY 711 Fall 2019 – Lecture 6

1

1

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment	Due
1 Mon, 8/26/2019	Chap. 1	Introduction	#1	8/30/2019
2 Wed, 8/28/2019	Chap. 1	Scattering theory	#2	9/02/2019
3 Fri, 8/30/2019	Chap. 1	Scattering theory	#3	9/04/2019
4 Mon, 9/02/2019	Chap. 1	Scattering theory	#4	9/06/2019
5 Wed, 9/04/2019	Chap. 2	Non-inertial coordinate systems	#5	9/09/2019
6 Fri, 9/06/2019	Chap. 3	Calculus of Variation	#6	9/11/2019
7 Mon, 9/9/2019	Chap. 3	Calculus of Variation		
8 Wed, 9/11/2019	Chap. 3	Calculus of Variation		

9/6/2019

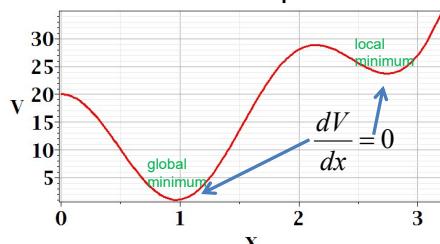
PHY 711 Fall 2019 – Lecture 6

2

2

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

Minimization of a simple function



9/6/2019

PHY 711 Fall 2019 – Lecture 6

3

3

Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx} = 0$

The graph shows a function $V(x)$ plotted against x . The x-axis ranges from 0 to 3, and the y-axis ranges from 0 to 30. The curve starts at $(0, 20)$, decreases to a local minimum near $x=0.7$ (labeled 'global minimum'), increases to a local maximum near $x=1.8$, decreases again to a global minimum near $x=2.2$ (labeled 'local minimum'), and finally increases towards $(3, 30)$. A blue arrow points to the derivative term $\frac{dV}{dx}$ with the label $\frac{dV}{dx} = 0$.

x	V(x)
0.0	20.0
0.5	18.0
0.7	10.0
1.0	12.0
1.5	22.0
1.8	25.0
2.0	22.0
2.2	10.0
2.5	12.0
3.0	30.0

4

Functional minimization

Consider a family of functions $y(x)$, with fixed end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and a function $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example :

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

5

Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Sample functions :

$$y_1(x) = \sqrt{x} \quad L = \int_0^1 \sqrt{1 + \frac{1}{4x}} dx = 1.4789$$

$$y_2(x) = x \quad L = \int_0^1 \sqrt{1 + 1x^2} dx = \sqrt{2} = 1.4142$$

$$y_3(x) = x^2 \quad L = \int_0^1 \sqrt{1 + 4x^2} dx = 1.4789$$

6

6

Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

$$\text{where } L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$$

Necessary condition : $\delta L = 0$

At any x , let $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally :

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx.$$

9/6/2019

PHY 711 Fall 2019 – Lecture 6

7

7

After some derivations, we find

$$\begin{aligned} \delta L &= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx \\ &= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f \\ &\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f \end{aligned}$$

9/6/2019

PHY 711 Fall 2019 – Lecture 6

8

8

“Some” derivations --
Consider term

$$\int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] dx:$$

If $y(x)$ is a well-defined function, then $\delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} \delta y$

$$\int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \delta \left(\frac{dy}{dx} \right) \right] dx = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \frac{d}{dx} \delta y \right] dx$$

$$= \int_{x_i}^{x_f} \left[\frac{d}{dx} \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \delta y \right] - \frac{d}{dx} \left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \delta y \right] dx$$

9/6/2019

PHY 711 Fall 2019 – Lecture 6

9

9

"Some" derivations (continued)--

$$\begin{aligned} & \int_x^{x_f} \left[\frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] - \frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \left[\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \\ &= 0 \quad - \int_{x_i}^{x_f} \left[\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx \end{aligned}$$

Euler-Lagrange equation:

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x,y} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

9/6/2019

PHY 711 Fall 2019 – Lecture 6

10

Example : End points -- $y(0) = 0; y(1) = 1$

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \Rightarrow f \left(\left\{ y(x), \frac{dy}{dx} \right\}, x \right) = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \\ &\left(\frac{\partial f}{\partial y} \right)_{x,y} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \\ &\Rightarrow - \frac{d}{dx} \left(\frac{dy}{dx} \Big/ \sqrt{1 + (dy/dx)^2} \right) = 0 \end{aligned}$$

Solution:

$$\begin{aligned} \left(\frac{dy}{dx} \Big/ \sqrt{1 + (dy/dx)^2} \right) &= K \quad \frac{dy}{dx} = K \equiv \frac{K}{\sqrt{1 - K^2}} \\ \Rightarrow y(x) &= K'x + C \quad y(x) = x \end{aligned}$$

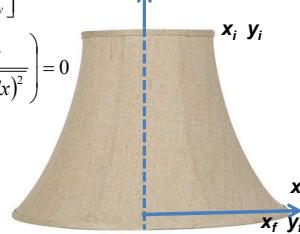
9/6/2019

PHY 711 Fall 2019 – Lecture 6

11

Example : Lamp shade shape $y(x)$

$$\begin{aligned} A &= 2\pi \int_{x_i}^{x_f} x \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \Rightarrow f \left(\left\{ y(x), \frac{dy}{dx} \right\}, x \right) = x \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \\ &\left(\frac{\partial f}{\partial y} \right)_{x,y} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \\ &\Rightarrow - \frac{d}{dx} \left(\frac{x dy}{dx} \Big/ \sqrt{1 + (dy/dx)^2} \right) = 0 \end{aligned}$$



9/6/2019

PHY 711 Fall 2019 – Lecture 6

12

$$\begin{aligned}
 -\frac{d}{dx} \left(\frac{x dy/dx}{\sqrt{1+(dy/dx)^2}} \right) &= 0 \\
 \frac{x dy/dx}{\sqrt{1+(dy/dx)^2}} &= K_1 \\
 \frac{dy}{dx} &= -\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2 - 1}} \\
 \Rightarrow y(x) &= K_2 - K_1 \ln \left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)
 \end{aligned}$$

9/6/2019

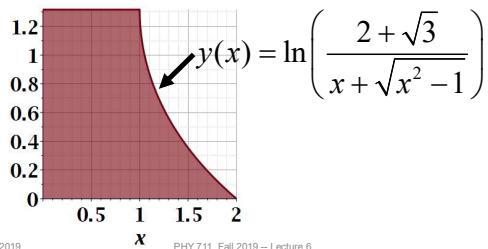
PHY 711 Fall 2019 – Lecture 6

13

13

General form of solution --

$$y(x) = K_2 - K_1 \ln \left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

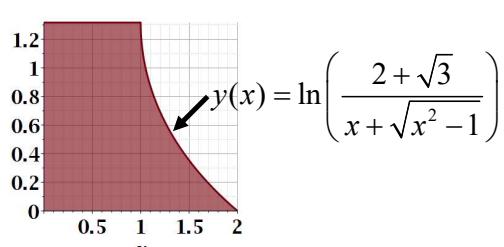
Suppose $K_1 = 1$ and $K_2 = 2 + \sqrt{3}$ 

9/6/2019

PHY 711 Fall 2019 – Lecture 6

14

14



$$A = 2\pi \int_1^2 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 15.02014144$$

(according to Maple)

9/6/2019

PHY 711 Fall 2019 – Lecture 6

15

Another example:

(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0) = 0$ and $y(1) = 1$

that minimize the integral:

$$I = \int_0^1 \left(\left(\frac{dy}{dx} \right)^2 - ay^2 \right) dx \quad \text{for constant } a > 0$$

Euler - Lagrange equation :

$$\frac{d^2y}{dx^2} + ay = 0$$

$$\Rightarrow y(x) = \frac{\sin(\sqrt{a}x)}{\sin(\sqrt{a})}$$

9/6/2019

PHY 711 Fall 2019 – Lecture 6

16

16

Review : for $f\left(y(x), \frac{dy}{dx}\right), x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}\right) dx$:

$$\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right)_{x, y} \right] = 0 \quad \text{Euler-Lagrange equation}$$

Note that for $f\left(y(x), \frac{dy}{dx}\right), x\right)$,

$$\begin{aligned} \frac{df}{dx} &= \left(\frac{\partial f}{\partial y} \right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x} \right) \\ &= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right) \right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (\frac{dy}{dx})} \right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x} \right) \end{aligned}$$

$$\Rightarrow \frac{d}{dx} \left(f - \frac{\partial f}{\partial (\frac{dy}{dx})} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right) \quad \text{Alternate Euler-Lagrange equation}$$

9/6/2019

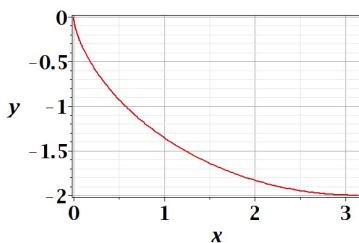
PHY 711 Fall 2019 – Lecture 6

17

17

Brachistochrone problem: (solved by Newton in 1696)

<http://mathworld.wolfram.com/BrachistochroneProblem.html>



A particle of weight mg travels frictionlessly down a path of shape $y(x)$. What is the shape of the path $y(x)$ that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

9/6/2019

PHY 711 Fall 2019 – Lecture 6

18

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-2gy}} dx \quad \text{because } \frac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$

Note that for the original form of Euler-Lagrange equation:

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = 0$$

$$\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] = 0,$$

differential equation is more complicated:

$$\frac{d}{dx} \left(\frac{1}{\sqrt{-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right)}} \right) = 0$$

$$-\frac{1}{2} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx} \left(\frac{\frac{dy}{dx}}{\sqrt{-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right)}} \right) = 0$$

19

PHY 711 Fall 2019 – Lecture 6

$$\begin{aligned} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) &= \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}} \\ \frac{d}{dx}\left(f - \frac{\partial f}{\partial(y/x)} \frac{dy}{dx}\right) &= \left(\frac{\partial f}{\partial x}\right) \\ \Rightarrow \frac{d}{dx} \left(\frac{1}{\sqrt{-y \left(1 + \left(\frac{dy}{dx}\right)^2\right)}} \right) &= 0 \quad -y \left(1 + \left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a \end{aligned}$$

20

PHY 711 Fall 2019 – Lecture 6

$$\begin{aligned}
 -y\left(1+\left(\frac{dy}{dx}\right)^2\right) &= K \equiv 2a & \text{Let } y = -2a \sin^2 \frac{\theta}{2} = a(\cos \theta - 1) \\
 \frac{dy}{dx} &= \sqrt{\frac{2a}{y} - 1} & -\frac{dy}{\sqrt{\frac{2a}{y} - 1}} &= \frac{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta}{\sqrt{\frac{2a}{2a \sin^2 \frac{\theta}{2}}} - 1} = dx \\
 -\frac{dy}{\sqrt{\frac{2a}{y} - 1}} &= dx & x &= \int_0^\theta a(1 - \cos \theta') d\theta' = a(\theta - \sin \theta)
 \end{aligned}$$

Parametric equations for Brachistochrone:

$$x = a(\theta - \sin \theta)$$

9/6/2019

PHY 711 Fall 2019 – Lecture 6

