

PHY 711 - Assignment #7

Sep 8, 2019

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f=f(q, \dot{q}, t)$, where it is assumed that $q=q(t)$ and $\dot{q} \equiv dq/dt$.

a. Evaluate

$$\frac{\partial}{\partial q} \frac{df}{dt} - \frac{d}{dt} \frac{\partial f}{\partial \dot{q}}$$

b. Evaluate

$$\frac{\partial}{\partial \dot{q}} \frac{df}{dt} - \frac{d}{dt} \frac{\partial f}{\partial q}$$

c. Evaluate

$$\frac{df}{dt}$$

d. Now suppose that

$$f(q, \dot{q}, t) = q \dot{q}^2 t^2, \quad \text{where} \quad q(t) = e^{-t/\tau}.$$

Here τ is a constant. Evaluate df/dt using the expression you just derived. Now find the expression for f as an explicit function of t ($f(t)$) and take its time derivative directly to check your previous results.

9/9/2019

PHY 711 Fall 2019 - Lecture 7

4

4

Review : for $f\left(y(x), \frac{dy}{dx}, x\right)$,

a necessary condition to extremize $\int_{x_1}^{x_2} f\left(y(x), \frac{dy}{dx}, x\right) dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0 \quad \leftarrow \text{Euler-Lagrange equation}$$

Note that for $f\left(y(x), \frac{dy}{dx}, x\right)$,

$$\begin{aligned} \frac{df}{dx} &= \left(\frac{\partial f}{\partial y}\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right) \\ &= \left(\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)}\right)\right) \frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right) \frac{d}{dx} \frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right) \end{aligned}$$

$$\Rightarrow \frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x}\right) \quad \leftarrow \text{Alternate Euler-Lagrange equation}$$

9/9/2019

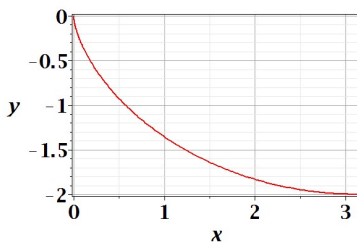
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5

5

Brachistochrone problem: (solved by Newton in 1696)

<http://mathworld.wolfram.com/BrachistochroneProblem.html>



A particle of weight mg travels frictionlessly down a path of shape $y(x)$. What is the shape of the path $y(x)$ that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

9/9/2019

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6

6

$$T = \int_{x_i, y_i}^{x_f, y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx \quad \text{because } \frac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = 0 \quad \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x, y} \right] = 0,$$

$$\frac{d}{dx} \left(\frac{1}{\sqrt{-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right)}} \right) = 0$$

Note that for the original form of Euler-Lagrange equation:

differential equation is more complicated:

$$-\frac{1}{2} \frac{1}{\sqrt{-y^3}} - \frac{d}{dx} \left(\frac{\frac{dy}{dx}}{\sqrt{-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right)}} \right) = 0$$

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7

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{\sqrt{-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right)}} \right) = 0 \quad -y \left(1 + \left(\frac{dy}{dx} \right)^2 \right) = K \equiv 2a$$

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8

$$-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right) = K \equiv 2a$$

$$\frac{dy}{dx} = -\sqrt{\frac{2a}{-y} - 1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y} - 1}} = dx$$

Let $y = -2a \sin^2 \frac{\theta}{2} = a(\cos \theta - 1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y} - 1}} = \frac{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta}{\sqrt{\frac{2a}{2a \sin^2 \frac{\theta}{2}} - 1}} = dx$$

$$x = \int_0^\theta a(1 - \cos \theta') d\theta' = a(\theta - \sin \theta)$$

Parametric equations for Brachistochrone:

$$x = a(\theta - \sin \theta)$$

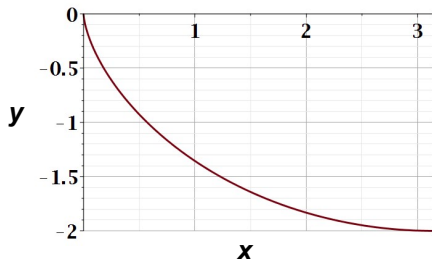
$$y = a(\cos \theta - 1)$$

9/9/2019 PHY 711 Fall 2019 -- Lecture 7 9

9

Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



9/9/2019

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10

10

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points

$y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

9/9/2019

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11

11

Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

9/9/2019

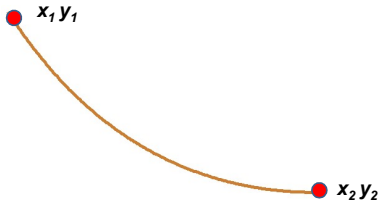
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12

12

Another example optimization problem:

Determine the shape $y(x)$ of a rope of length L and mass density ρ hanging between two points



9/9/2019

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13

13

Potential energy of hanging rope :

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Length of rope :

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Define a composite function to minimize :

$$W \equiv E + \lambda L$$

Lagrange multiplier

9/9/2019

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14

14

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$f\left(y, \frac{dy}{dx}\right) = (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

$$\Rightarrow (\rho g y + \lambda) \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

9/9/2019

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15

15

$$(\rho g y + \lambda) \left(\sqrt{1 + \left(\frac{dy}{dx} \right)^2} - \frac{\left(\frac{dy}{dx} \right)^2}{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}} \right) = K$$

$$(\rho g y + \lambda) \left(\frac{1}{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}} \right) = K$$

$$y(x) = -\frac{1}{\rho g} \left(\lambda + K \cosh \left(\frac{x-a}{K / \rho g} \right) \right)$$

9/9/2019 PHY 711 Fall 2019 – Lecture 7 16

16

$$y(x) = -\frac{1}{\rho g} \left(\lambda + K \cosh \left(\frac{x-a}{K / \rho g} \right) \right)$$

Integration constants : K, a, λ

Constraints : $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = L$$

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17

Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f \left(\left\{ y(x), \frac{dy}{dx} \right\}, x \right) dx \quad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y} \right) - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right) \right] = 0$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

9/9/2019 PHY 711 Fall 2019 – Lecture 7 18

18

Application to particle dynamics

$x \rightarrow t$ (time)

$y \rightarrow q$ (generalized coordinate)

$f \rightarrow L$ (Lagrangian)

$I \rightarrow A$ or S (action)

Denote: $\dot{q} \equiv \frac{dq}{dt}$

$$A = \int_{t_1}^{t_2} L(\{q, \dot{q}\}; t) dt$$

9/9/2019

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19

19

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$A = \int_{t_1}^{t_2} L(\{q, \dot{q}\}; t) dt \equiv \int_{t_1}^{t_2} L\left(\left\{y, \frac{dy}{dt}\right\}; t\right) dt$$

Simple example: vertical trajectory of particle of mass m subject to constant downward acceleration $a = -g$.

Newton's formulation: $m \frac{d^2 y}{dt^2} = -mg$

Resultant trajectory: $y(t) = y_i + v_i t - \frac{1}{2} g t^2$

Lagrangian for this case:

$$L = \frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy$$

9/9/2019

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20

20

<http://www-history.mcs.st-and.ac.uk/Biographies/Hamilton.html>

Sir William Rowan Hamilton

Wednesday, September 11th, 2013



Tribute to Sir William Hamilton

Hello and welcome! This page is dedicated to the life and work of Sir William Rowan Hamilton.

William Rowan Hamilton was Ireland's greatest scientist. He was an mathematician, physicist, and astronomer and made important works in optics, dynamics, and algebra.

His contribution in dynamics plays a important role in the later developed quantum mechanics. His name was perpetuated in one of the fundamental concepts in quantum mechanics, called "Hamiltonian".

The Discovery of Quaternions is probably is his most familiar invention today.

2005 was the Hamilton Year, celebrating his 200th birthday. The year was dedicated to celebrate Irish Science. 2005 was called the Einstein year also, reminding of three great papers of the year 1905. So UNESCO designated 2005 to the World Year of Physics

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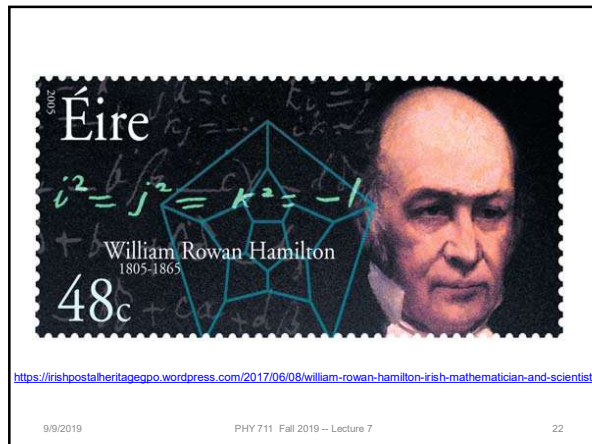
Home
Biography
Mathematical studies
Optics and Dynamics
Quaternions
Quotations
Hamilton Key Dates
Hamilton Links
Graphics
Math News

9/9/2019

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21

21



22

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy
Potential energy

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) dt \quad \text{is minimized for physical } y(t) :$$

9/9/2019 PHY 711 Fall 2019 – Lecture 7 23

23

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \quad y(t) = y_i + v_i t - \frac{1}{2}gt^2$$

9/9/2019 PHY 711 Fall 2019 – Lecture 7 24

24

Check:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0$, $y_i = h \equiv \frac{1}{2} g T^2$; $t_f = T$, $y_f = 0$

Trial trajectories: $y_1(t) = \frac{1}{2} g T^2 (1 - t / T) = h - \frac{1}{2} g T t$

$$y_2(t) = \frac{1}{2} g T^2 (1 - t^2 / T^2) = h - \frac{1}{2} g t^2$$

$$y_3(t) = \frac{1}{2} g T^2 (1 - t^3 / T^3) = h - \frac{1}{2} g t^3 / T$$

Maple says:

$$S_1 = -0.125 m g^2 T^3$$

$$S_2 = -0.167 m g^2 T^3$$

$$S_3 = -0.150 m g^2 T^3$$

9/9/2019

PHY 711 Fall 2019 – Lecture 7

25

25
