

PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 8:
Continue reading Chapter 3

- 1. D'Alembert's principle**
- 2. Hamilton's principle**
- 3. Lagrange's equation in generalized coordinates**

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Physics Colloquium – Wednesday, Sept. 11, 2019
Wake Forest College & Graduate School of Arts and Sciences

WFU Physics

Events

Colloquium "Dielectrophoresis and Electrophysiology" – Wednesday Sept. 11, 2019 at 3 PM
Professor Erin Henklein Department of Engineering Wake Forest University Winston-Salem, NC George P. Williams, Jr. Lecture Hall, (Olin 101) Wednesday, Sept. 11, 2019, at 3:00 PM. There will be a ...

News

Tenure-Track Faculty Position in Theoretical / Computational Condensed Matter / Materials Physics

Now Accepting

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

Date	F&W Reading	Topic	Assignment Due
1 Mon, 8/26/2019	Chap. 1	Introduction	#1
2 Wed, 8/28/2019	Chap. 1	Scattering theory	#2
3 Fri, 8/30/2019	Chap. 1	Scattering theory	#3
4 Mon, 9/02/2019	Chap. 1	Scattering theory	#4
5 Wed, 9/04/2019	Chap. 2	Non-inertial coordinate systems	#5
6 Fri, 9/06/2019	Chap. 3	Calculus of Variation	#6
7 Mon, 9/09/2019	Chap. 3	Calculus of Variation	#7
8 Wed, 9/11/2019	Chap. 3	Lagrangian Mechanics	

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Jean d'Alembert 1717-1783
French mathematician and philosopher



"Deriving" Lagrangian mechanics from Newton's laws.

The Lagrangian function is:

$$L\left(\left\{q_i(t)\right\}, \left\{\frac{dq_i}{dt}\right\}, t\right) = T - U \quad q_i(t) \text{ are generalized coordinates}$$

Hamilton's principle states:

$$S \equiv \int_{t_1}^{t_2} L\left(\left\{q_i(t)\right\}, \left\{\frac{dq_i}{dt}\right\}, t\right) dt \quad \text{is minimized for physical } y(t):$$

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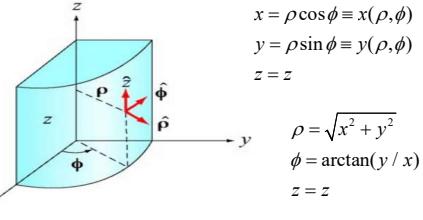
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Digression -- notion of generalized coordinates
Referenced to cartesian coordinates: $\mathbf{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$

Cylindrical coordinates



$$\begin{aligned} x &= \rho \cos \phi \equiv x(\rho, \phi) \\ y &= \rho \sin \phi \equiv y(\rho, \phi) \\ z &= z \\ \rho &= \sqrt{x^2 + y^2} \\ \phi &= \arctan(y/x) \\ z &= z \end{aligned}$$

Figure B.2.4 Cylindrical coordinates
(Figure taken from 8.02 handout from MIT.)

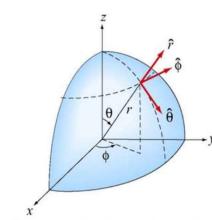
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Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi \equiv x(r, \theta, \phi) \\ y &= r \sin \theta \sin \phi \equiv y(r, \theta, \phi) \\ z &= r \cos \theta \equiv z(r, \theta, \phi) \\ r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi &= \arctan(y/x) \end{aligned}$$

Figure B.3.1 Spherical coordinates

(Figure taken from 8.02 handout from MIT.)

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D'Alembert's principle:

Generalized coordinates:
 $q_\sigma(\{x_i\})$

Note that: $ds = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

Newton's laws: $\mathbf{F} \cdot d\mathbf{s} = \sum \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$

$\mathbf{F} \cdot d\mathbf{s} = \sum \sum_i F_i \frac{\partial \dot{x}_i}{\partial q_\sigma} \delta q_\sigma$

For a conservative force: $F_i = -\frac{\partial U}{\partial x_i}$

$\mathbf{F} \cdot d\mathbf{s} = -\sum \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$

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Generalized coordinates:
 $q_\sigma(\{x_i\})$

Newton's laws:
 $\mathbf{F} \cdot d\mathbf{s} = \sum \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$

$\mathbf{F} \cdot d\mathbf{s} = \sum \sum_i m\ddot{x}_i \left(\frac{d}{dt} \left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - m\dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$

Claim: $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$\mathbf{F} \cdot d\mathbf{s} = \sum \sum_i \left(\frac{d}{dt} \left(\frac{\partial \left(\frac{1}{2} m\dot{x}_i^2 \right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial \left(\frac{1}{2} m\dot{x}_i^2 \right)}{\partial q_\sigma} \right) \delta q_\sigma$

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$x_i = x_i(\{q_\sigma(t)\}, t)$

Claim: $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$

Details: $\dot{x}_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial x_i}{\partial t}$ Therefore: $\frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \frac{\partial x_i}{\partial q_\sigma}$

Claim: $\frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$\sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_\sigma \partial q_{\sigma'}} \dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial t \partial q_\sigma} \quad \sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_\sigma \partial q_{\sigma'}} \dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial q_\sigma \partial t}$

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Generalized coordinates:
 $q_\sigma(\{x_i\})$



$$m\mathbf{a} \cdot d\mathbf{s} = \sum_{\sigma} \sum_i \left(\frac{d}{dt} \left(\frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (\frac{1}{2} m \dot{x}_i^2)}{\partial q_\sigma} \right) \delta q_\sigma$$

Define -- kinetic energy : $T \equiv \sum_i \frac{1}{2} m \dot{x}_i^2$

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_{\sigma} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

Recall:

$$\mathbf{F} \cdot d\mathbf{s} = - \sum_{\sigma} \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = - \sum_{\sigma} \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = - \sum_{\sigma} \frac{\partial U}{\partial q_\sigma} \delta q_\sigma - \sum_{\sigma} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

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Generalized coordinates:
 $q_\sigma(\{x_i\})$



$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = - \sum_{\sigma} \frac{\partial U}{\partial q_\sigma} \delta q_\sigma - \sum_{\sigma} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$= - \sum_{\sigma} \left(\frac{d}{dt} \frac{\partial (T - U)}{\partial \dot{q}_\sigma} - \frac{\partial (T - U)}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$= - \sum_{\sigma} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$L(q_\sigma, \dot{q}_\sigma; t) = T - U$

Note: This is only true if
 $\frac{\partial U}{\partial q_\sigma} = 0$

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Generalized coordinates:
 $q_\sigma(\{x_i\})$



Define-- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$(\mathbf{F}-m\mathbf{a}) \cdot d\mathbf{s} = - \sum_{\sigma} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

\Rightarrow Minimization integral: $S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

→ Hamilton's principle

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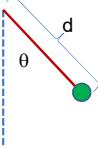
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Euler – Lagrange equations : $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta, \dot{\theta}) = \frac{1}{2}md^2\dot{\theta}^2 - mg(d - d\cos\theta)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \Rightarrow \frac{d}{dt} md^2\dot{\theta} + mgd\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{d}\sin\theta$$

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Another example : $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2}I_1(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2}I_3(\dot{\alpha} \cos \beta + \dot{\gamma})^2 - Mgd \cos \beta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}(I_1 \dot{\alpha} \sin^2 \beta + I_3(\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta) = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}(I_1 \dot{\beta}) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}(I_3(\dot{\alpha} \cos \beta + \dot{\gamma})) = 0$$

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Example – simple harmonic oscillator

$$T = \frac{1}{2}m\dot{x}^2$$

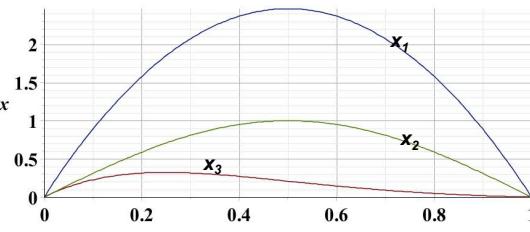
$$U = \frac{1}{2}m\omega^2 x^2$$

Assume $x(0) = 0$ and $\dot{x}(\frac{\pi}{\omega}) = 0$

$$S = \int_0^{\frac{\pi}{\omega}} (T - U) dt$$

Trial functions

$x_1(t) = A \sin(\omega t)$	$S_1 = 0$
$x_2(t) = A \omega t \cdot (\pi - \omega t)$	$S_2 = 0.067 A^2 m \omega^2$
$x_3(t) = A e^{-\omega t} \sin(\omega t)$	$S_3 = 0.062 A^2 m \omega^2$



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Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$,

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

Euler-Lagrange equations:

$$\sum_{\sigma} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

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Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

\Rightarrow How can we represent velocity - dependent forces?

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Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$:

Lorentz force: $\mathbf{F} = q(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B})$

x -component: $F_x = q(E_x + \frac{1}{c}(v_x B_z - v_z B_x))$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$x\text{-component: } \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) = 0$$

$$\text{Apparently: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$\text{Answer: } \boxed{U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)}$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

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Lorentz forces, continued:

$$x - \text{component of Lorentz force: } F_x = q(E_x + \frac{1}{c}(\mathbf{v} \times \mathbf{B})_x)$$

$$\text{Suppose: } U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{Consider: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} A_x(\mathbf{r}, t)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \frac{d A_x(\mathbf{r}, t)}{dt} = -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

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Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \dot{y} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_y(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_y(\mathbf{r}, t)}{\partial z} \right) \\ &= qE_x(\mathbf{r}, t) + \frac{q}{c} (\dot{y}B_z(\mathbf{r}, t) - \dot{z}B_y(\mathbf{r}, t)) = qE_x(\mathbf{r}, t) + \frac{q}{c} (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))_x \end{aligned}$$

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Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

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Example Lorentz force

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Suppose $\mathbf{E}(\mathbf{r}, t) \equiv 0$, $\mathbf{B}(\mathbf{r}, t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{2}B_0(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{xy} + \dot{yx})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0y\right) - \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \quad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0x\right) + \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \quad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

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Example Lorentz force -- continued

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{xy} + \dot{yx})$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0y\right) - \frac{q}{2c}B_0\dot{y} = 0 \quad \Rightarrow m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0x\right) + \frac{q}{2c}B_0\dot{x} = 0 \quad \Rightarrow m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0$$

$$\frac{d}{dt}m\dot{z} = 0 \quad \Rightarrow m\ddot{z} = 0$$

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Example Lorentz force -- continued

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{xy} + \dot{yx})$$

$$m\ddot{x} = +\frac{q}{c}B_0\dot{y}$$

$$m\ddot{y} = -\frac{q}{c}B_0\dot{x}$$

$m\ddot{z} = 0$ Note that same equations are obtained from direct application of Newton's laws :

$$m\ddot{\mathbf{r}} = \frac{q}{c}\dot{\mathbf{r}} \times B_0\hat{\mathbf{z}}$$

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Example Lorentz force -- continuedConsider formulation with different Gauge : $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

$$\begin{aligned} L &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c} B_0 \dot{x} y \\ \frac{d}{dt} \left(m \dot{x} - \frac{q}{c} B_0 y \right) &= 0 \quad \Rightarrow m \ddot{x} - \frac{q}{c} B_0 \dot{y} = 0 \\ \frac{d}{dt} (m \dot{y}) + \frac{q}{c} B_0 \dot{x} &= 0 \quad \Rightarrow m \ddot{y} + \frac{q}{c} B_0 \dot{x} = 0 \\ \frac{d}{dt} m \dot{z} &= 0 \quad \Rightarrow m \ddot{z} = 0 \end{aligned}$$

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Example Lorentz force -- continued

Evaluation of equations :

$$\begin{aligned} m \ddot{x} - \frac{q}{c} B_0 \dot{y} &= 0 & \dot{x}(t) &= V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right) \\ m \ddot{y} + \frac{q}{c} B_0 \dot{x} &= 0 & \dot{y}(t) &= V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right) \\ m \ddot{z} &= 0 & \dot{z}(t) &= V_{0z} \end{aligned}$$

$$\begin{aligned} x(t) &= x_0 - \frac{mc}{qB_0} V_0 \cos\left(\frac{qB_0}{mc} t + \phi\right) \\ y(t) &= y_0 + \frac{mc}{qB_0} V_0 \sin\left(\frac{qB_0}{mc} t + \phi\right) \\ z(t) &= z_0 + V_{0z} t \end{aligned}$$

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