PHY 337/637 Analytical Mechanics
12:30-1:45 PM TR in Olin 103

Notes for Lecture 10: Scattering analysis – Chap. 11 (Cline; especially 11.12)

1. Definition of differential scattering cross section
2. Calculation of particle trajectories for a central potential
3. Relation of particle trajectories to the differential scattering cross section
4. Example of Rutherford scattering

Physics Colloquium

Thursday

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September 28th, 2023

Culture-Based Approaches to Physics Instruction

Physics faculty and secondary teachers have a variety of methods at their disposal to embrace a culturally based approach to teaching. In this context, we want to spotlight the endeavors of professional learning communities composed of physics instructors. These educators have made deliberate efforts to infuse culture-based pedagogical elements into their classrooms, focusing on three key areas: (1) cultivating a strong physics teacher identity, (2) designing effective curricula, and (3) assessing the impact of curriculum materials on students' comprehension of physics concepts. The outcomes of this study reveal that these instructors exhibit notable signs of employing culturally relevant pedagogy. They exhibit a willingness to tackle sensitive subjects, consistently motivate students to strive for academic excellence, and adapt their curricula to incorporate students' strengths. Despite these positive aspects, instructors still grapple with certain challenges. These challenges include encouraging students to adopt a critical stance towards physics knowledge and effectively managing students' ability to take charge of their own learning process. The implications of the research findings are significant for physics instructors aiming to implement culturally relevant pedagogy in their teaching methodologies.

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Physics
Lyman Briggs College
Assistant Professor
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Department of Teacher Education
Michigan State University

4 pm - Olin 101

Refreshments will be served in Olin Lobby beginning at 3:30pm.

Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	**Reading**	**Topic**	**PP**	**Assign**
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 6	Lagrangian mechanics	6E	#4
5	Tu, 9/12/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	#5
6	Th, 9/14/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	
7	Tu, 9/19/2023	Ch. 13	Dynamics of rigid bodies	13E	#6
8	Th, 9/21/2023	Ch. 13	Dynamics of rigid bodies	13E	#7
9	Tu, 9/26/2023	Ch. 13 & 11	Review of rigid bodies and intro to scattering	11E	#8
10	Th, 9/28/2023	Ch. 11	Scattering theory	11E	#9
11	Tu, 10/3/2023	Ch. 11	Scattering theory	11E	
12	Th, 10/5/2023				
13	Tu, 10/10/2023				
	Th, 10/12/2023	Fall Break			
14	Tu, 10/17/2023				

PHY 337/637 – Assignment #9

Assigned: 09/28/2023 Due: 10/03/2023

1. Suppose that a particle is scattered by a very massive target particle such that energy and angular momentum are conserved. The trajectory of the scattering particle is found to have an impact parameter b which depends on the scattering angle θ according to the formula

$$b(\theta) = K \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|,$$

where K denotes a constant which depends on energy and other parameters. What is the differential cross section for this process?

Scattering theory:

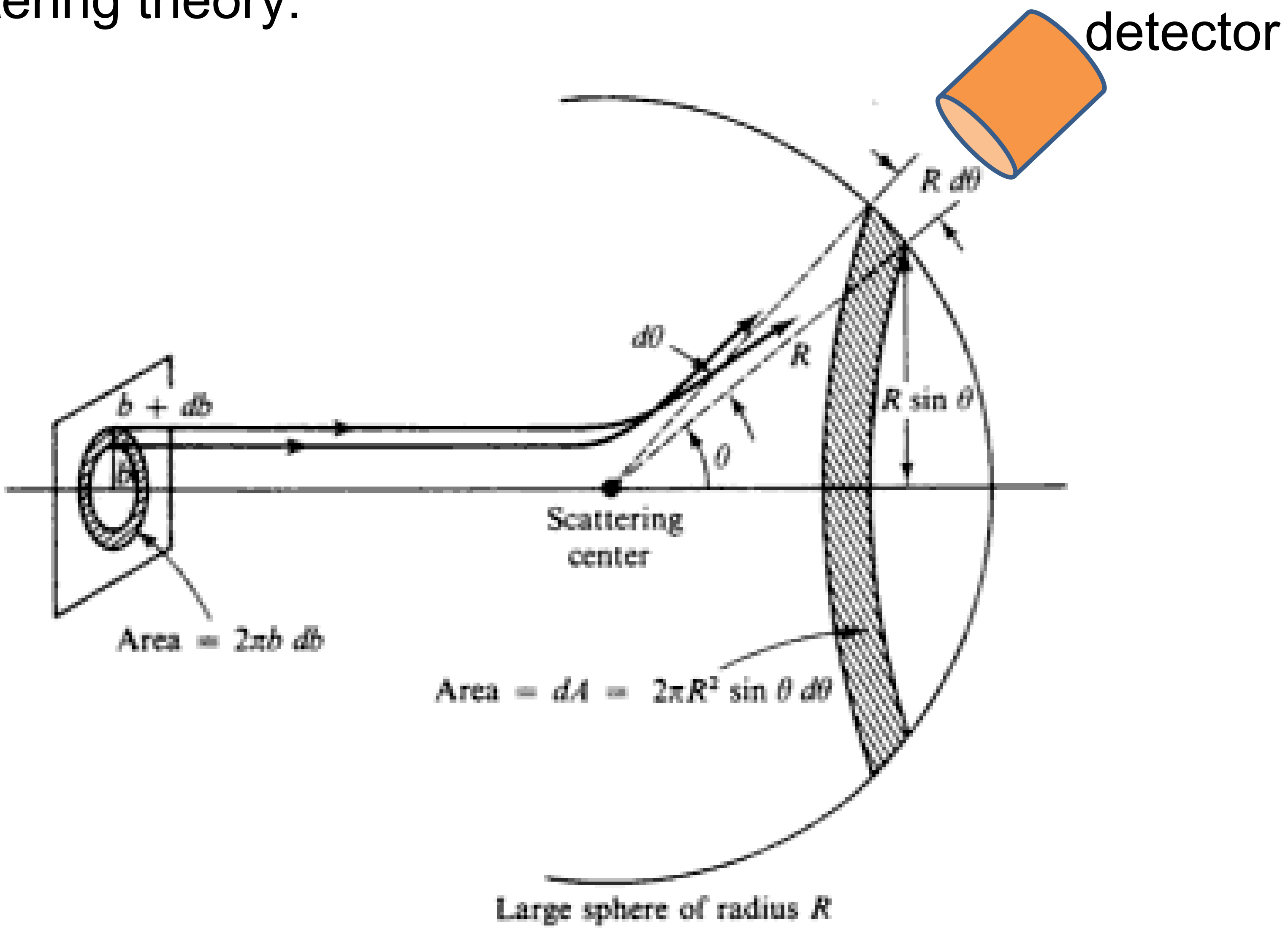


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Scattering theory:

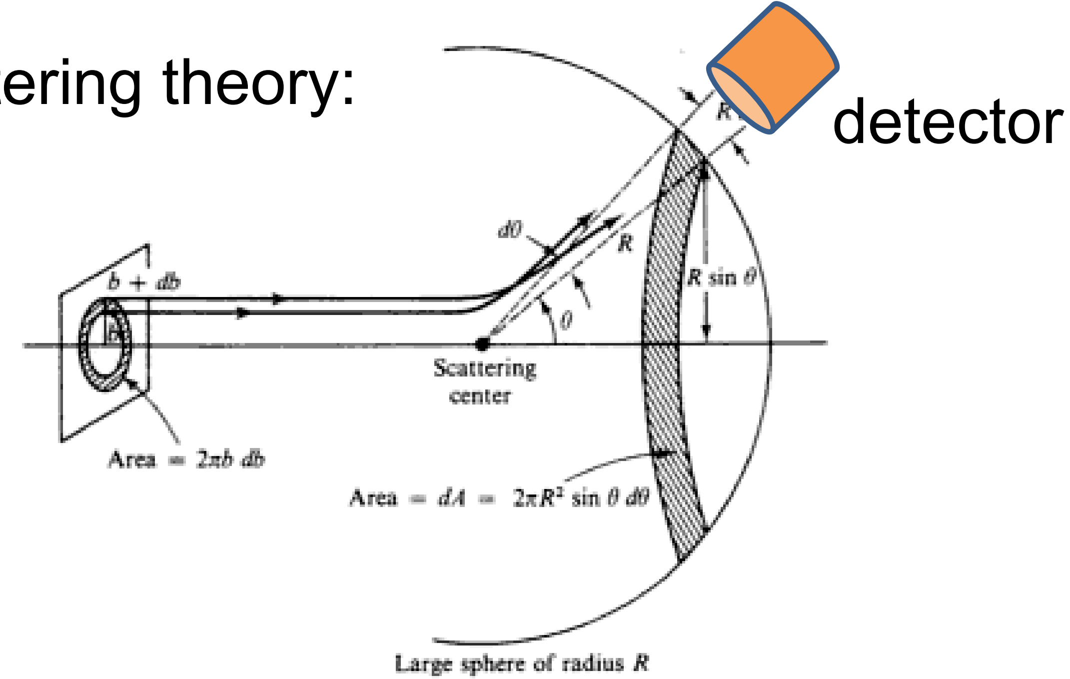


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Some reasons that scattering theory is useful:

1. It allows comparison between measurement and theory
2. The analysis depends on knowledge of the scattering particles when they are far apart
3. The scattering results depend on the interparticle interactions

Standard measure of differential cross section

Differential cross section

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle per unit time}}{\text{Number of incident particles per unit area per unit time per solid angle}}$$

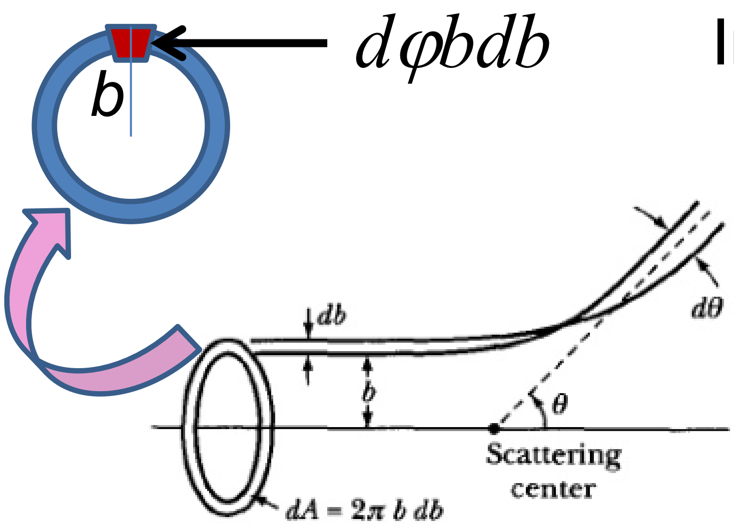
$$= \text{Area of incident beam that is scattered into detector per solid angle at angle } \theta$$

Standardization of scattering experiments --

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle per unit time}}{\text{Number of incident particles per unit area per unit time}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$

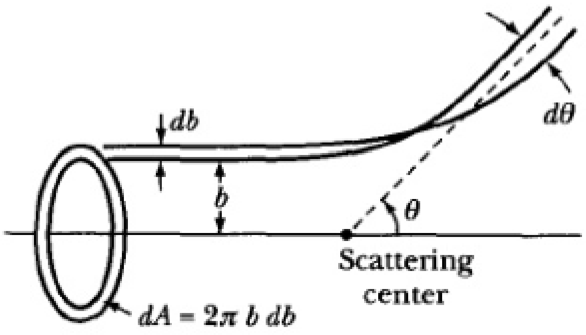


Impact parameter: *b*

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi\, b\, db}{d\phi\, \sin\theta\, d\theta} = \left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



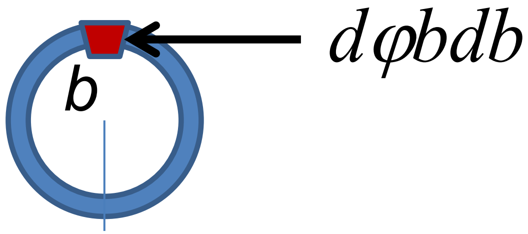


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi\ b\ db}{d\phi\ \sin\theta\ d\theta} = \left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in φ

https://www.aps.org/publications/apsnews/200605/history.cfm

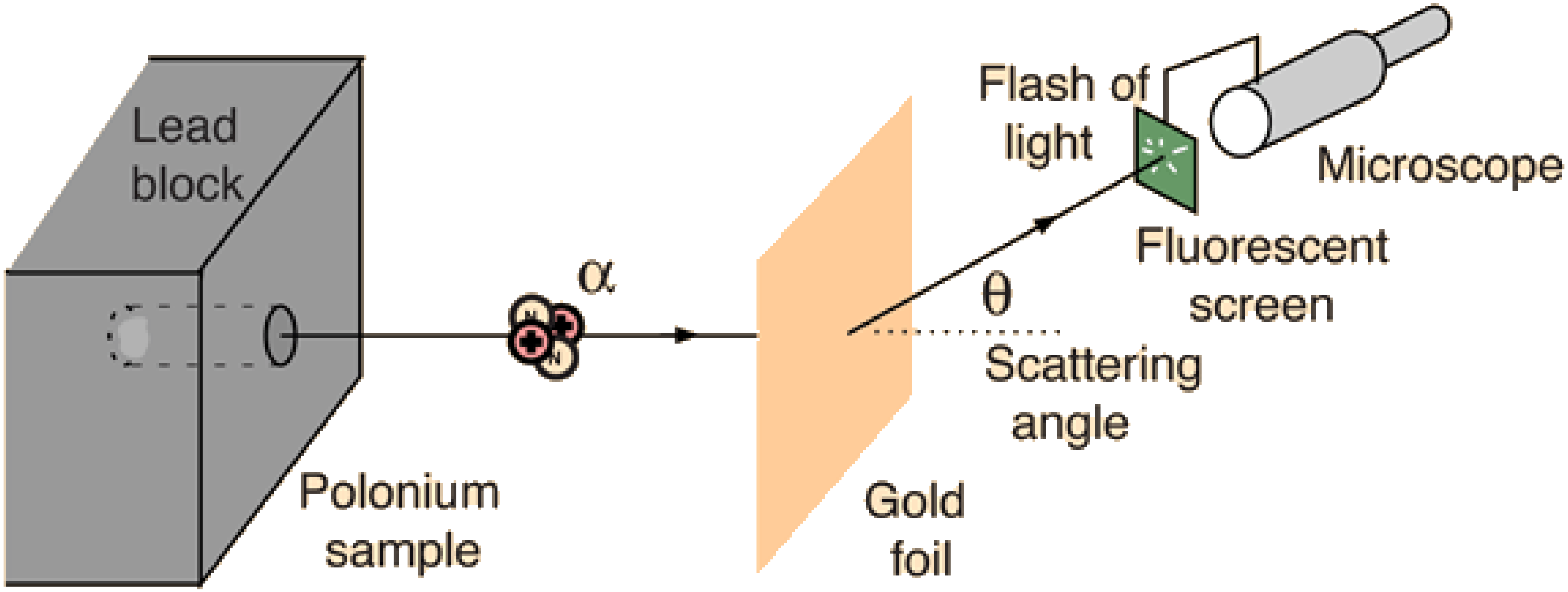
APSNews

May 2006 (Volume 15, Number 5)

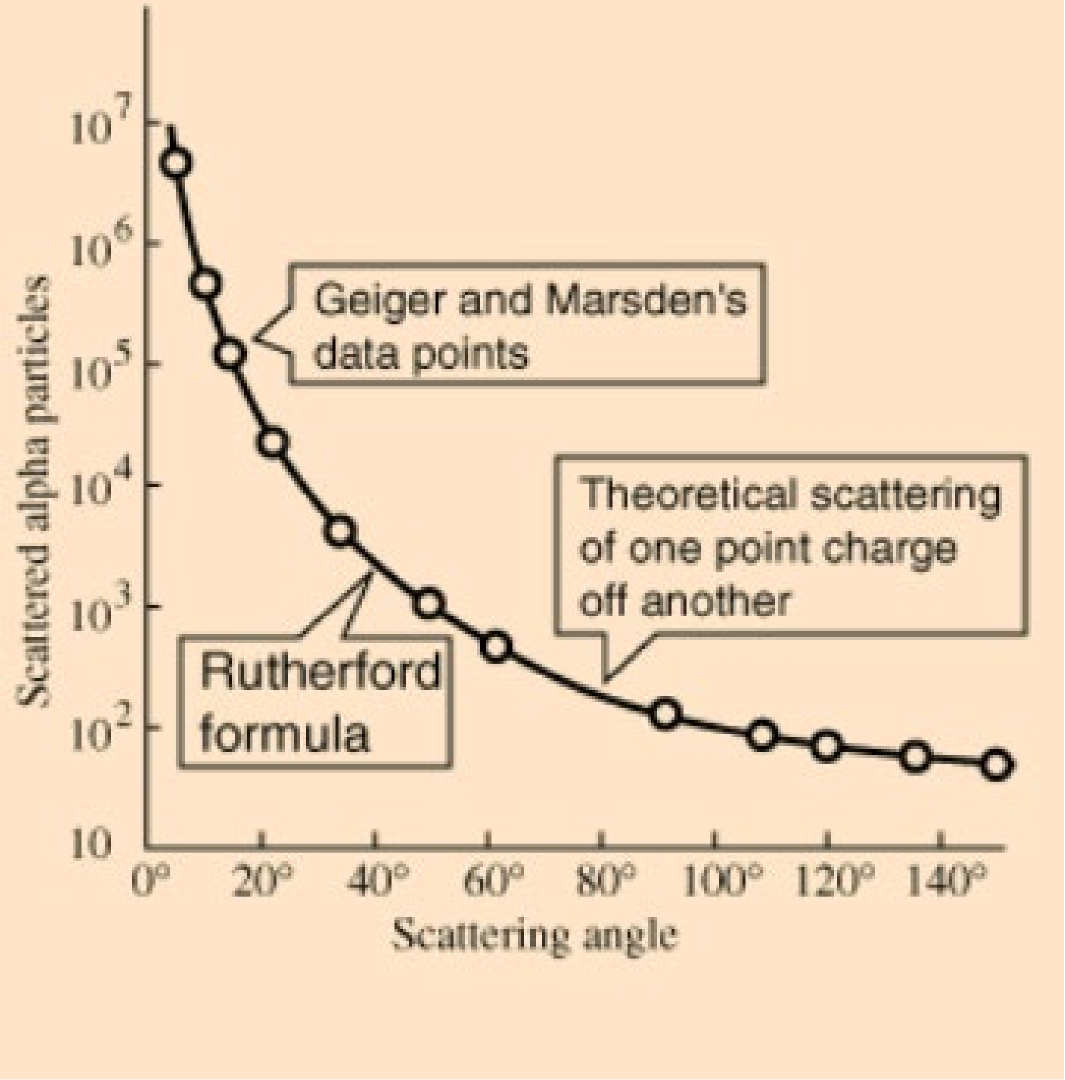
This Month in Physics History

May, 1911: Rutherford and the Discovery of the Atomic Nucleus

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Graph of data from scattering experiment



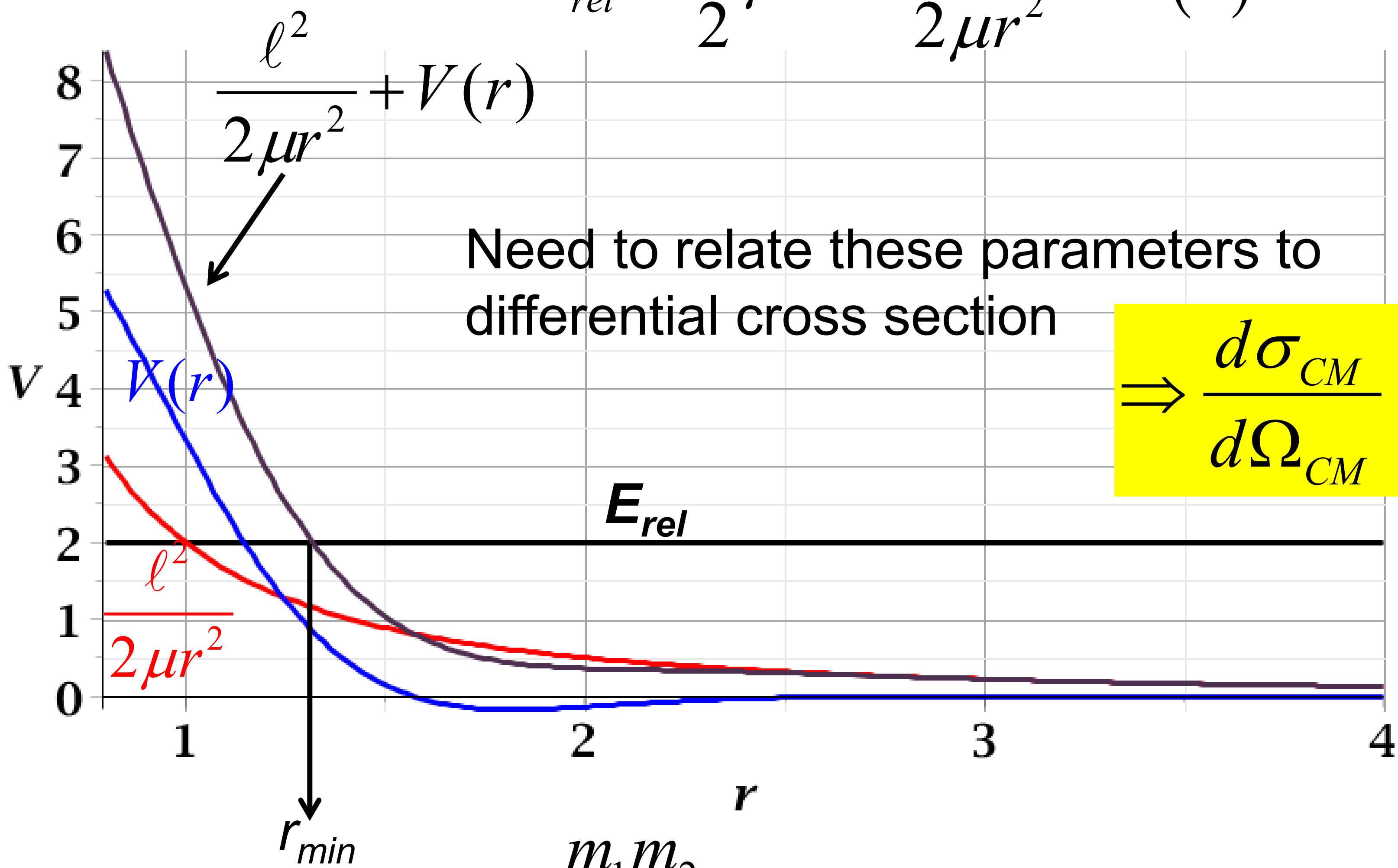
From website: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/rutsca2.html

How can we relate the differential scattering cross section values to information about the interaction potential $V(r)$ (assuming a central force interaction $\Rightarrow$ conserved angular momentum ℓ). In the following we will assume that the target particle is stationary (due to its large mass compared with the scattering particle).

Note that in the following slides, we reference the "center of mass reference frame" which will be discussed next time. For now, we can assume that the scattering particle has mass $\mu = m_1$ and the energy of interest is E_{rel}.

For a continuous potential interaction *V(r)*

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section $\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

More details

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Since **r**(t) represents motion in a plane, we will analyze the system in that plane and use polar coordinates.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$$

$$x(t) = r(t)\cos(\chi(t))$$

$$y(t) = r(t)\sin(\chi(t))$$

$$\text{Note that } |\dot{\mathbf{r}}(t)|^2 = \dot{x}^2(t) + \dot{y}^2(t)$$

$$= \dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)$$

Also note that the relative angular momentum of the system is a constant

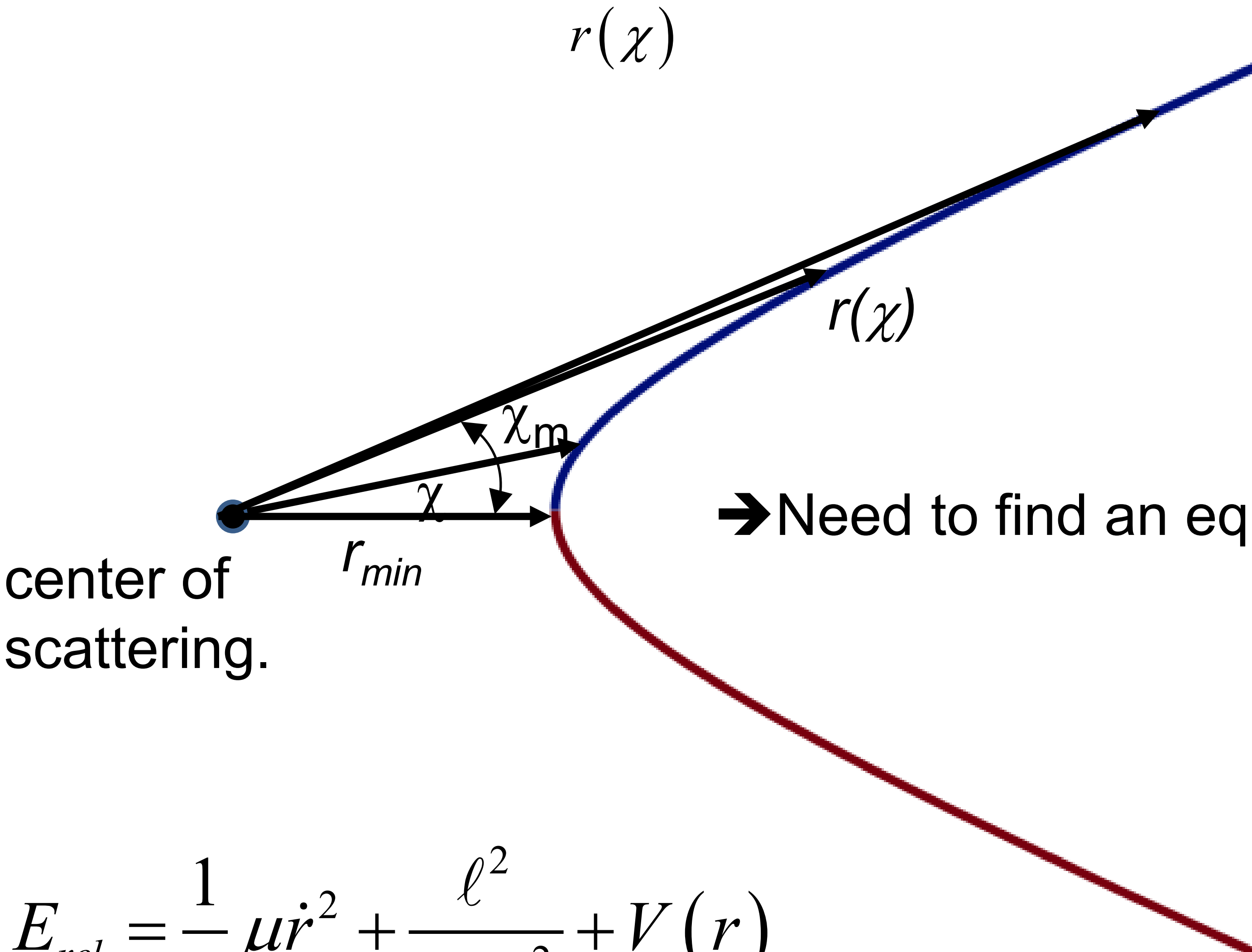
$\ell = \mu r^2 \dot{\chi}$

So that $\frac{1}{2}\mu \; |\dot{\mathbf{r}}(t)|^2 = \frac{1}{2}\mu\left(\dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)\right)$

$$= \frac{1}{2}\mu\dot{r}^2(t) + \frac{\ell^2}{2\mu r^2}$$

➔ $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V\left(r\right)$

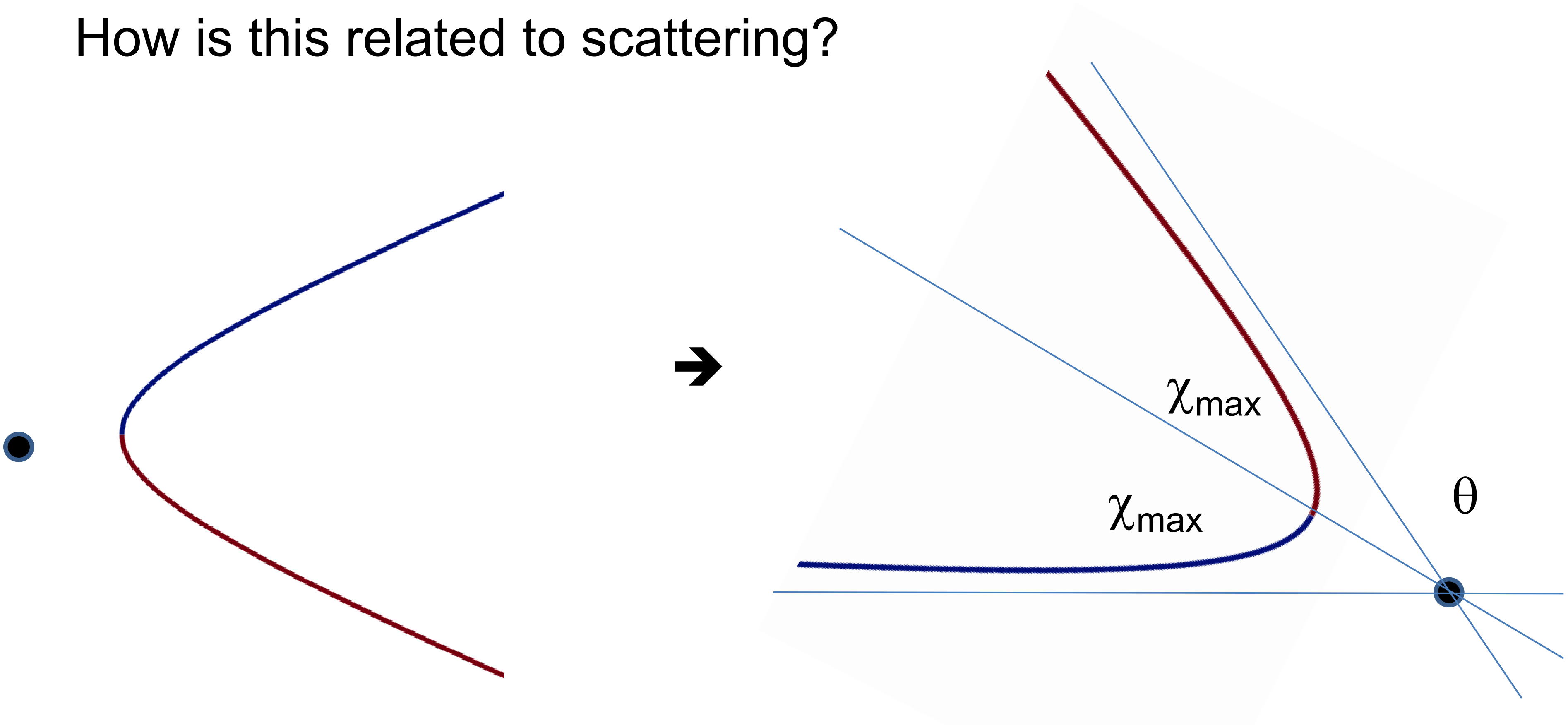
Trajectory of relative vector in center of mass frame



➔Need to find an equation for $r(\chi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

How is this related to scattering?



Note that here θ measures the scattering angle

Evaluation of constants far from scattering center --

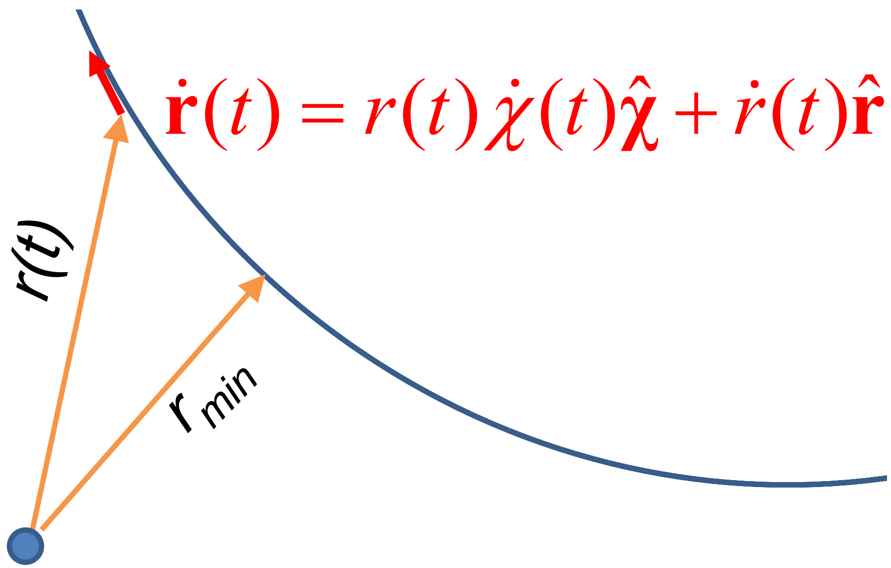
$$\ell = \mathbf{r} \times \left(\mu \dot{\mathbf{r}} \right) = r \mu r \frac{d\chi}{dt} = \mu r^2 \frac{d\chi}{dt}$$

$$\text{also:} \quad \ell = b \mu \dot{r}(t = -\infty)$$

Note that E_{rel} and ℓ can be evaluated from $\dot{r}(t)$ at $t = -\infty$ or $t = \infty$.

$$E_{rel} = \frac{1}{2} \mu \left(\dot{r}(t = -\infty) \right)^2$$

$$\Rightarrow \quad \ell = b\sqrt{2 \mu E_{rel}}$$



Questions:

1. How can we find $r(\chi)$?
2. If we find $r(\chi)$, how can we relate χ to θ?
3. How can we find $b(\theta)$?

Recall --

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables from t to angle:

$$r(t) \Leftrightarrow r(\chi)$$

$$\frac{dr}{dt} = \frac{dr}{d\chi}\frac{d\chi}{dt} = \frac{dr}{d\chi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\chi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\chi) \Leftrightarrow \chi(r)$:

From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\chi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

When the dust clears:

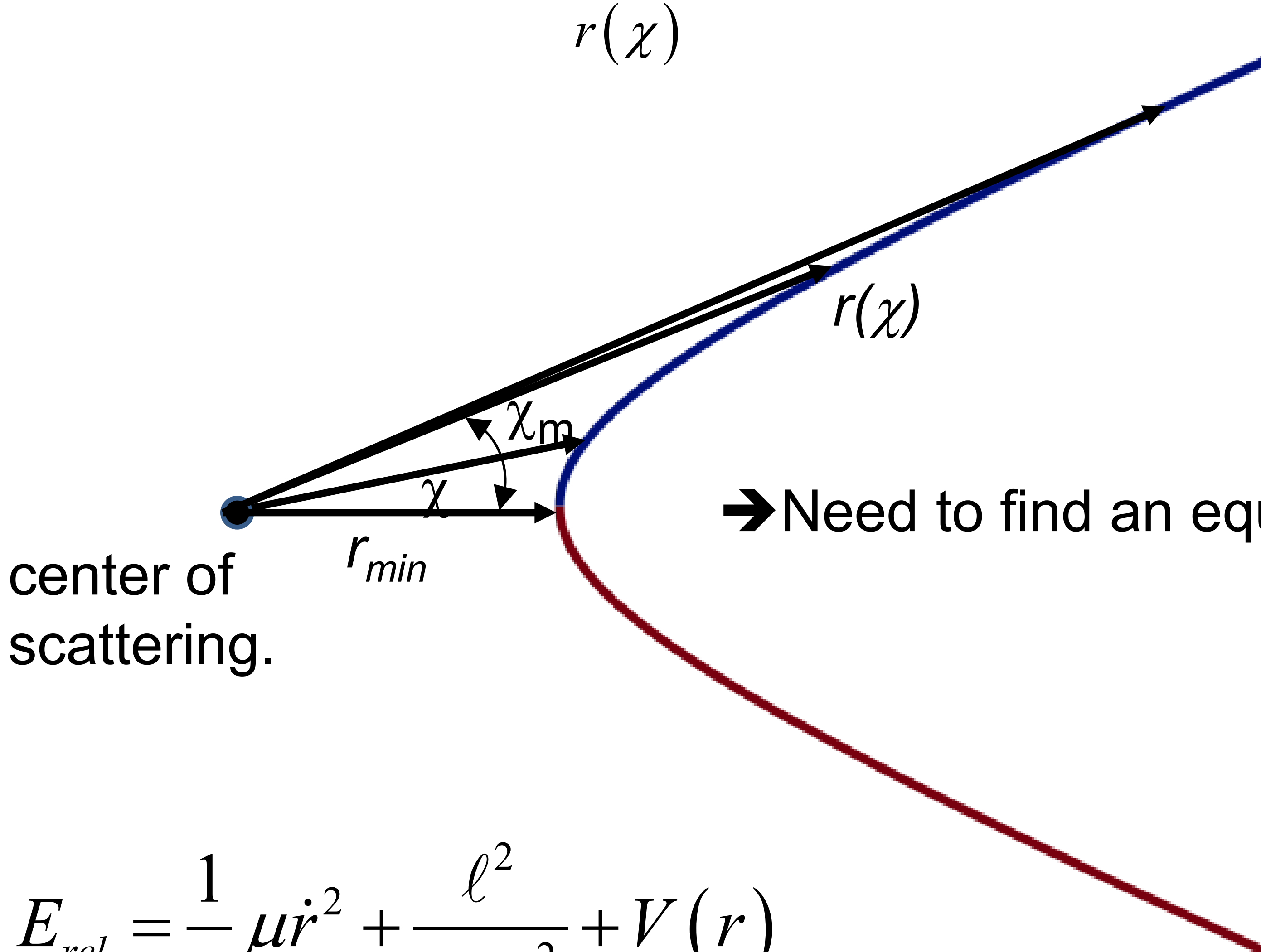
$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$\ell = b\sqrt{2\mu E}$$

$$d\chi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

$$\Rightarrow \chi_{\max}(b,E) = \chi\left(r \to \infty\right) - \chi\left(r = r_{\min}\right)$$

Trajectory of relative vector in center of mass frame



➔Need to find an equation for $r(\chi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

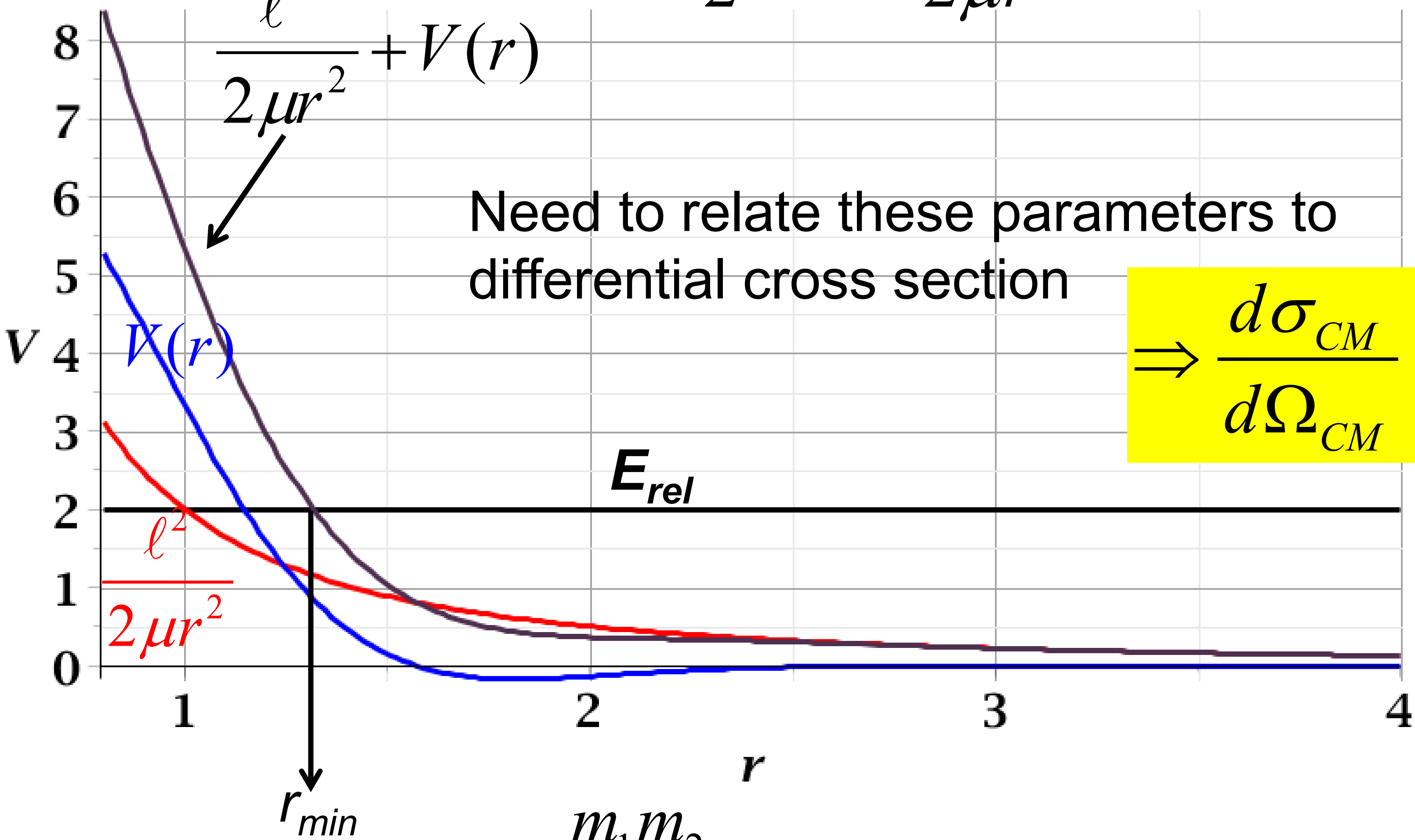
$$\int_0^{\chi_{\max}} d\chi = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

where:

$$1-\frac{b^2}{r_{\min}^2}-\frac{V(r_{\min})}{E}=0$$

For a continuous potential interaction *V(r)*

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

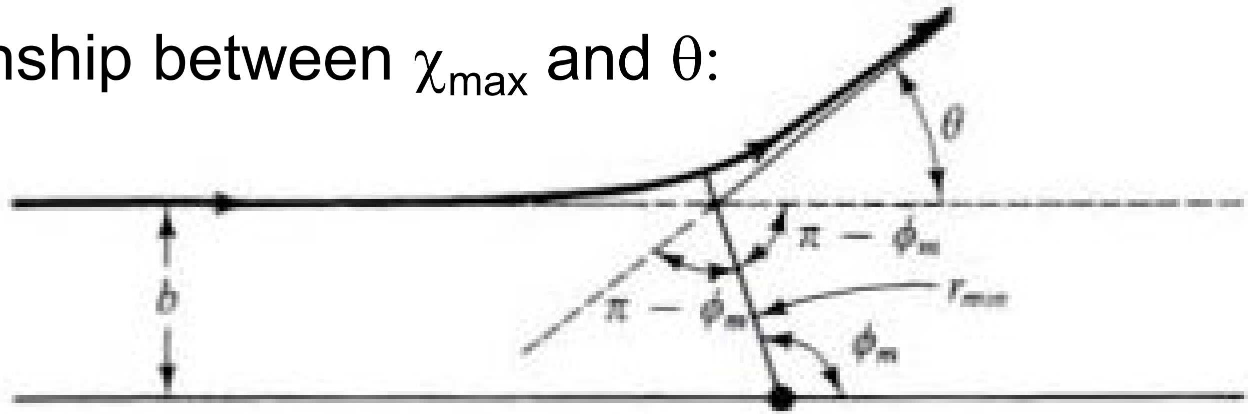
$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

Close up of repulsive interaction for a particular trajectory;
Also visualizing impact parameter *b*

Relationship between χ_{max} and θ:



$$2(\pi - \chi_{\max}) + \theta = \pi$$

$$\Rightarrow \chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

Here, θ represents the scattering angle in the center of mass frame and ϕ is used instead of χ.

General equations for central potential V(r)

$$\chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

➔These equations relate scattering angle to impact parameter *b*

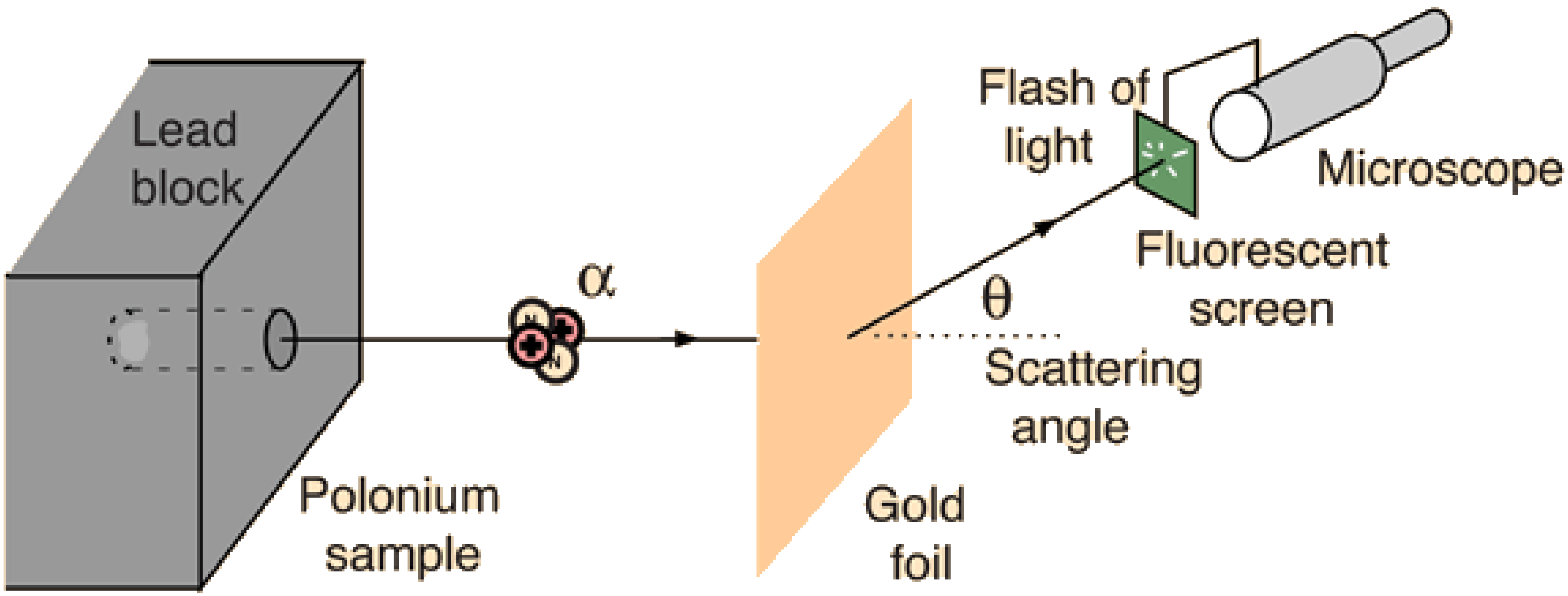
$$\chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

To go further, we need to know *V(r)*

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2\mathrm{b}} + \sqrt{\left(\frac{\kappa}{2\mathrm{b}}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

Some details –

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r}$$

SI units

$$V(r) = \frac{zZe^2}{4\pi\epsilon_0 r}$$ where e represents the elementary charge in Coulombs

r represents the particle separation in meters

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \quad \Rightarrow \kappa = \frac{zZe^2}{4\pi\epsilon_0 E}$$

Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

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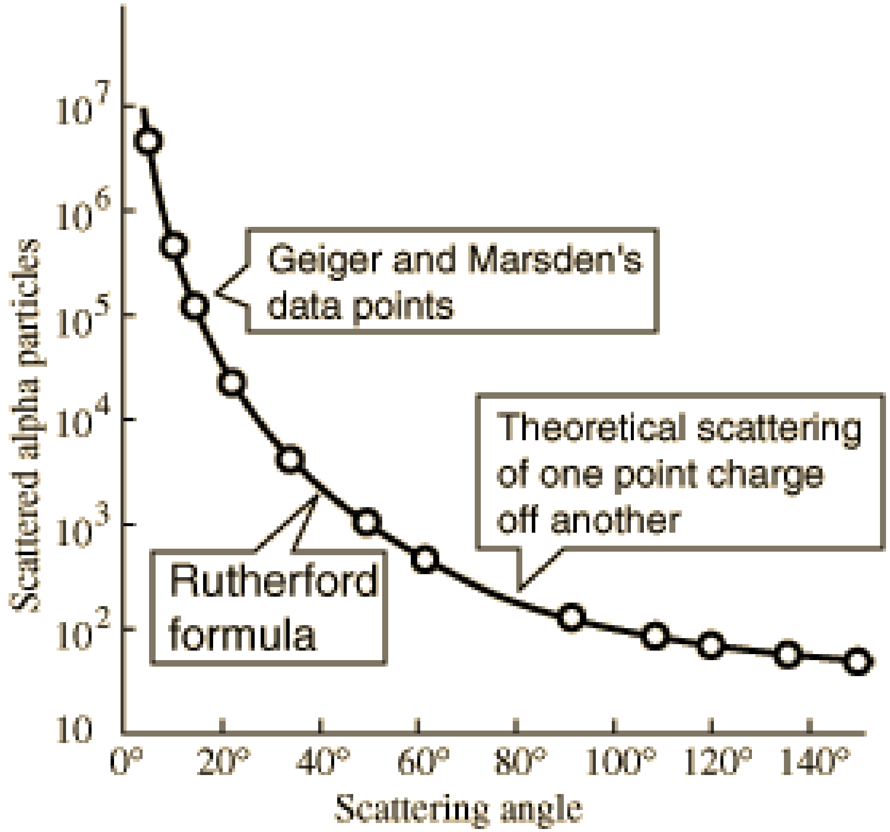

What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

(in SI units)

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\kappa^2}{16}\frac{1}{\sin^4\left(\theta/2\right)}$$

General formula relating b and θ:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

$\Rightarrow$ There are relatively few forms of $V(1/u)$ for which the integral has an analytic result.

A problem mentioned in Cline for an example:

$$V(r) = \frac{\gamma}{r^2} \text{ where } \frac{d\sigma}{d\Omega} = \frac{\gamma\pi^2}{E\sin\theta} \frac{(\pi - \theta)}{\theta^2(2\pi - \theta)^2}$$

More generally, it is possible to use numerical integration methods (with care) to evaluate *b(θ).*

Transformation between lab and center of mass results:

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}$$

where: $$\tan\theta = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

For elastic scattering