PHY 337/637 Analytical Mechanics
12:30-1:45 PM TR in Olin 103

Notes for Lecture 11 -- Chap. 11.12 of Cline

- Summary of scattering theory of a single particle from a stationary target
- Analysis of two particle system; center of mass and laboratory frames
- Differential cross section for in the center of mass reference frame

Physics Colloquium

Thursday

October 5th, 2023

Adaptive Optics and Interference Theory Enable Measurement of Retinal Function

Imaging of the retina has long been part of an ophthalmic exam, but the optics of the eye have aberrations that limit the quality of those images. Using adaptive optics, a technique originating in astronomy, researchers can measure and correct for the eye's optical aberrations thereby enabling diffraction limited imaging of the living retina. With this technology, individual photoreceptors and other retinal cells can be visualized noninvasively in the living human eye. My talk will provide an overview of adaptive optics imaging and will discuss how adaptive optics in combination with interference of light waves allows assessments of photoreceptor function.

4 pm - Olin 101

Refreshments will be served in the Olin lobby beginning at 3:30 pm

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Scheie Eye Institute

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Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	**Reading**	**Topic**	**PP**	**Assign**
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 6	Lagrangian mechanics	6E	#4
5	Tu, 9/12/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	#5
6	Th, 9/14/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	
7	Tu, 9/19/2023	Ch. 13	Dynamics of rigid bodies	13E	#6
8	Th, 9/21/2023	Ch. 13	Dynamics of rigid bodies	13E	#7
9	Tu, 9/26/2023	Ch. 13 & 11	Review of rigid bodies and intro to scattering	11E	#8
10	Th, 9/28/2023	Ch. 11	Scattering theory	11E	#9
11	Tu, 10/3/2023	Ch. 11	Scattering theory	11E	
12	Th, 10/5/2023		Summary and examples		Take home exam start
13	Tu, 10/10/2023		Summary and examples		Take home exam due
	Th, 10/12/2023	Fall Break			
14	Tu, 10/17/2023		Summary and examples		

Scattering theory:

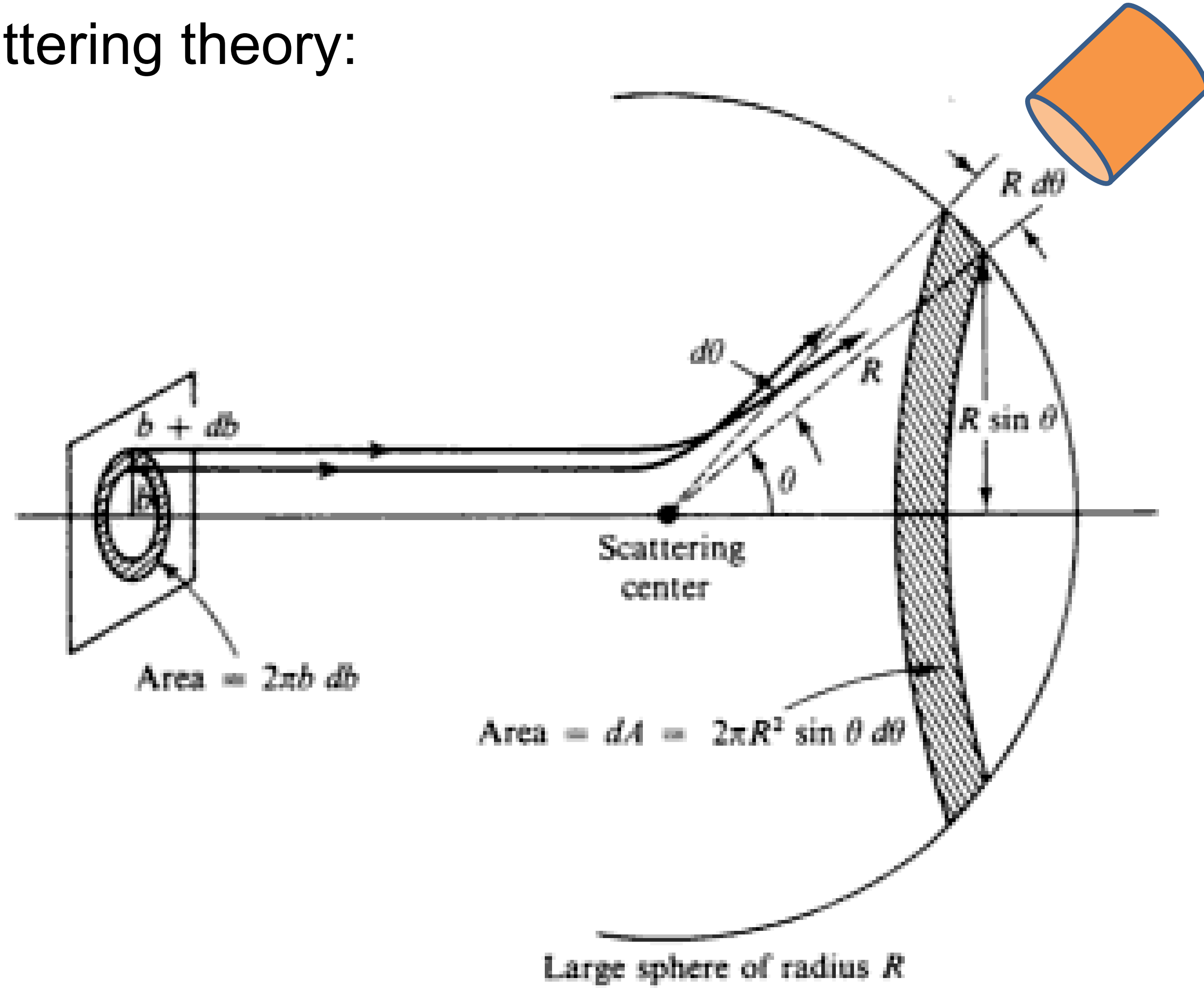


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Can you think of examples of such an experimental setup?

Other experimental designs –

At CERN https://home.cern/science/experiments/totem

the study of highly energetic proton-proton scattering is designed in the center of mass frame of reference by accelerating two proton beams focused to collide head on in the Large Hadron Collider LHC facility.

Figure from CERN website

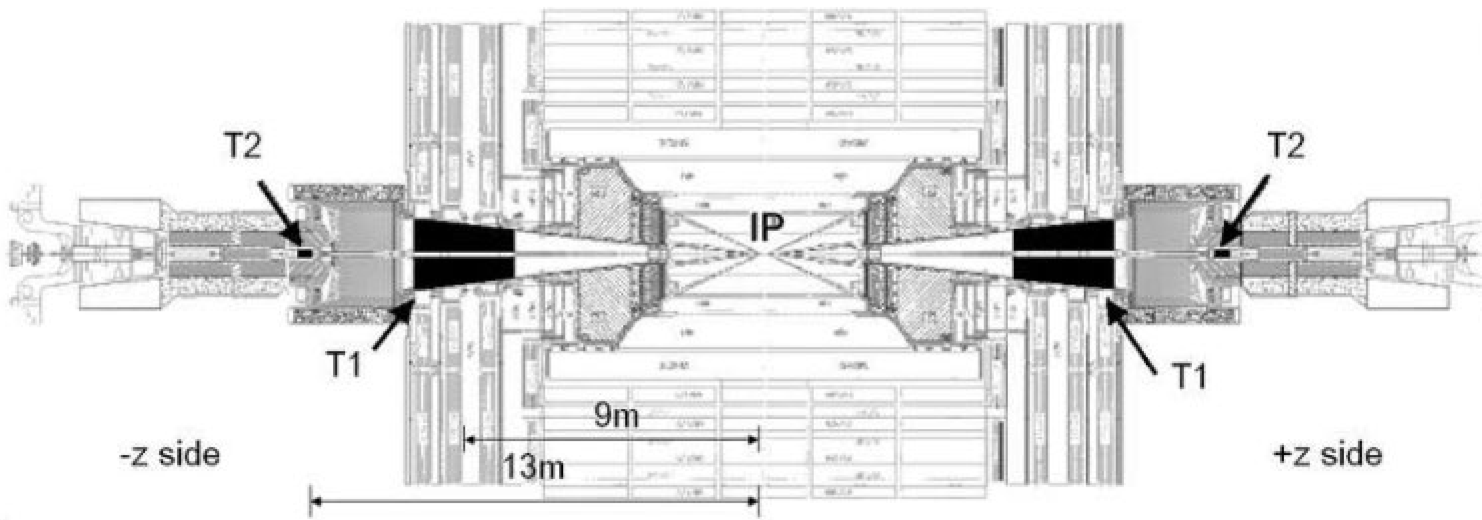


Figure 1.17: View of the inelastic forward trackers T1 and T2 inside the CMS detector.

What might be the advantage/disadvantage of this design?

What are the benefits/disadvantages of expressing the scattering cross section in the laboratory frame of reference vs center of mass frame of reference? (When or why to use a particular frame of reference)

Advantages of Lab frame

1. Natural experimental design.
2. Some targets are more naturally at rest.
3. ??

Advantages of CM frame

1. Analysis is done in CM frame.
2. Experiment is more energy efficient.
3. ??

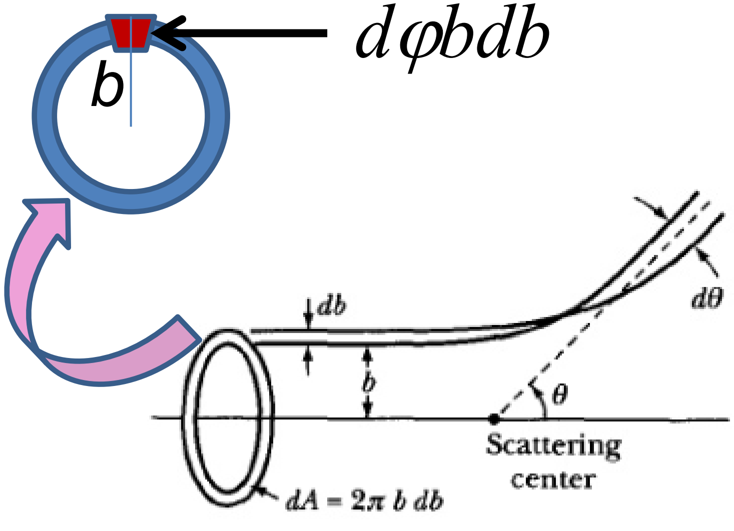
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector}$$

$$\text{at angle } \theta$$

$$d\varphi b db$$



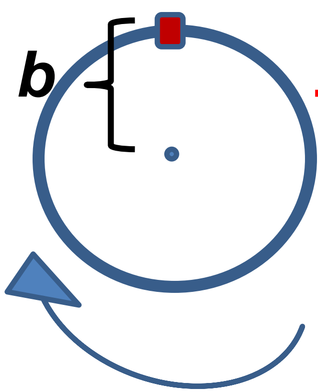
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

More details --

We imagine that the beam of particles has a cylindrical geometry and that the physics is totally uniform in the azimuthal direction. The cross section of the beam is a circle.

View of beam cross section:

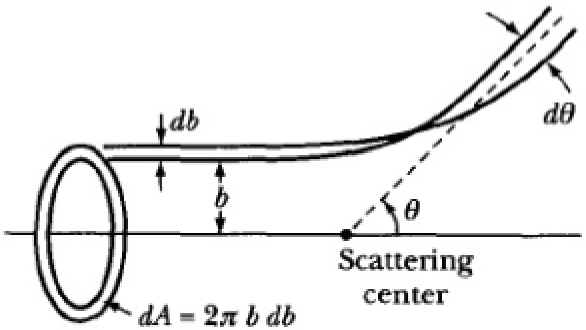


←This piece of the beam scatters into the detector at angle θ

This logic leads to the notion that b is a function of theta and we will try to find b(θ) for various cases.

$$\varphi \equiv \text{azimuthal angle}$$

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



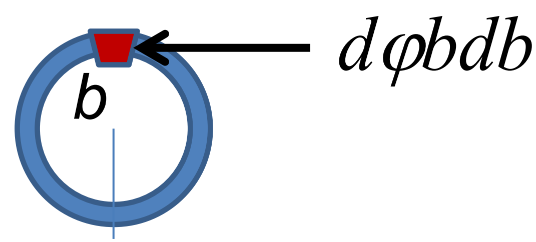
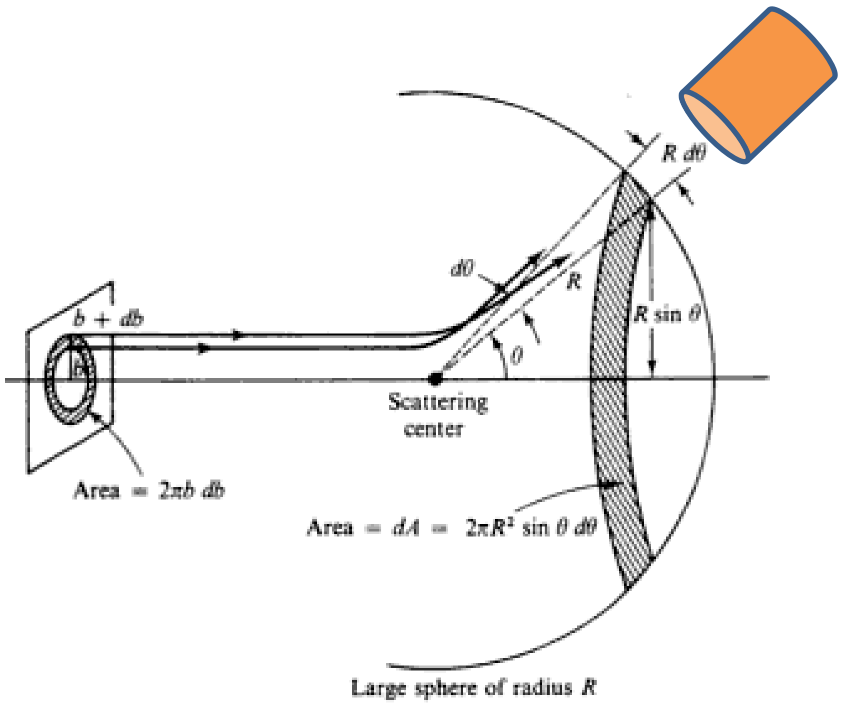


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

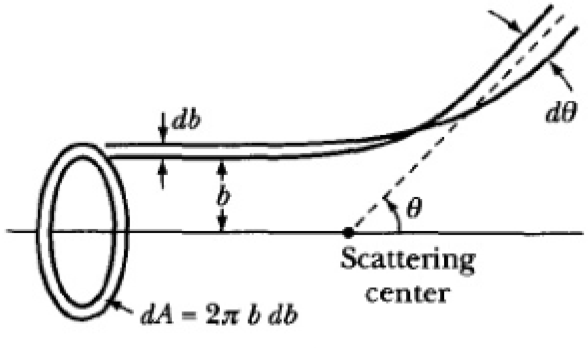
Note: We are assuming that the process is isotropic in ϕ

Elaboration on how we know that $b\, db\, d\phi$ is the relevant piece of beam ending up in our detector?



Comment: The interaction potential will determine the detailed shape of the particle trajectory which we can express as $r(\theta)$, which in principle can be related to the impact parameter as a function of scattering angle $b(\theta)$.

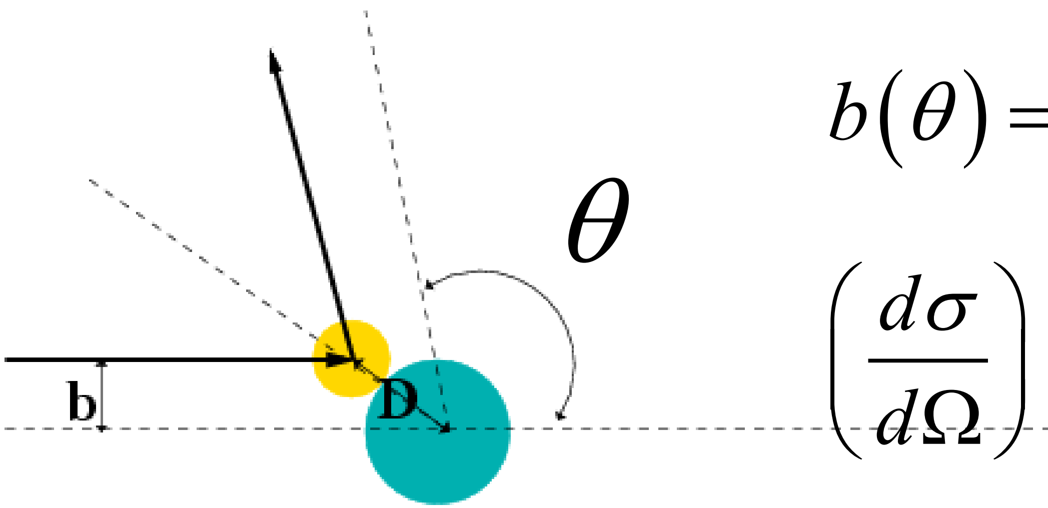
Simple example – collision of hard spheres



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

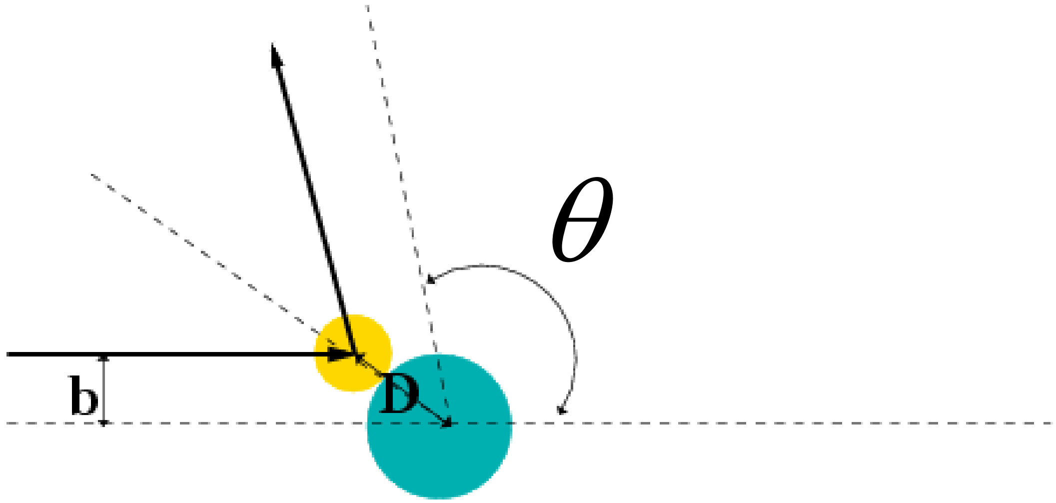
$$b(\theta) = ?$$

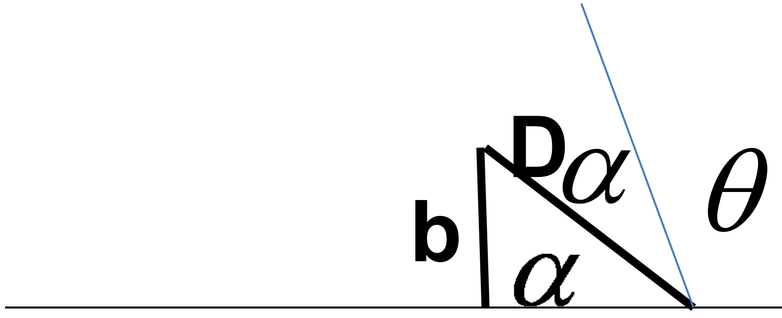


$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

Some more details of form of *b(θ)*

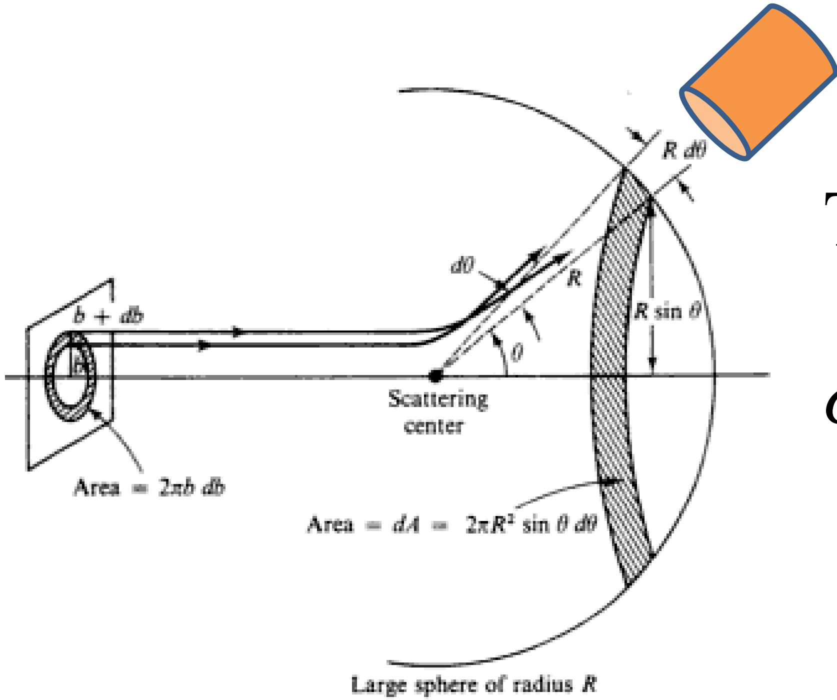




$$b = D\sin\alpha = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$2\alpha + \theta = \pi$$

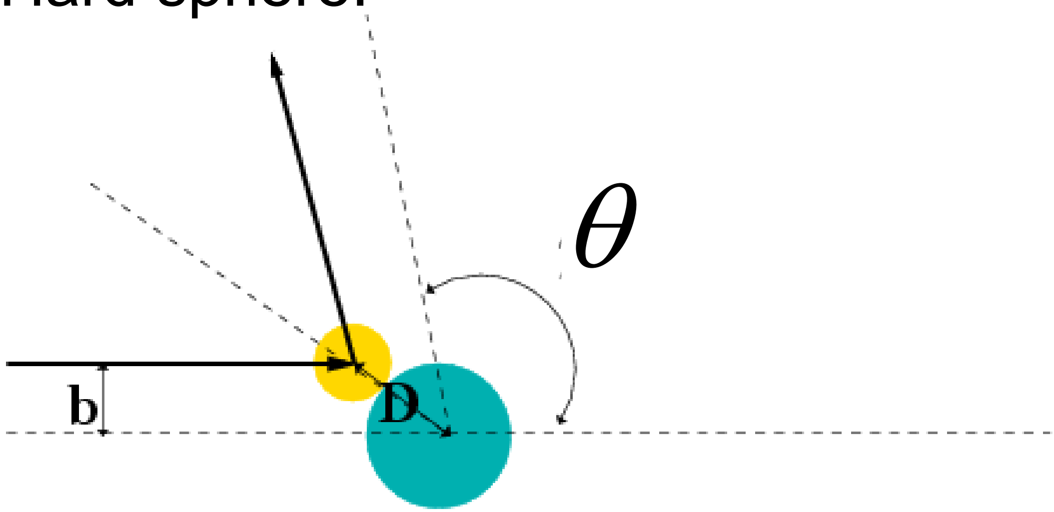
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega} \right) d\Omega$$

Hard sphere:



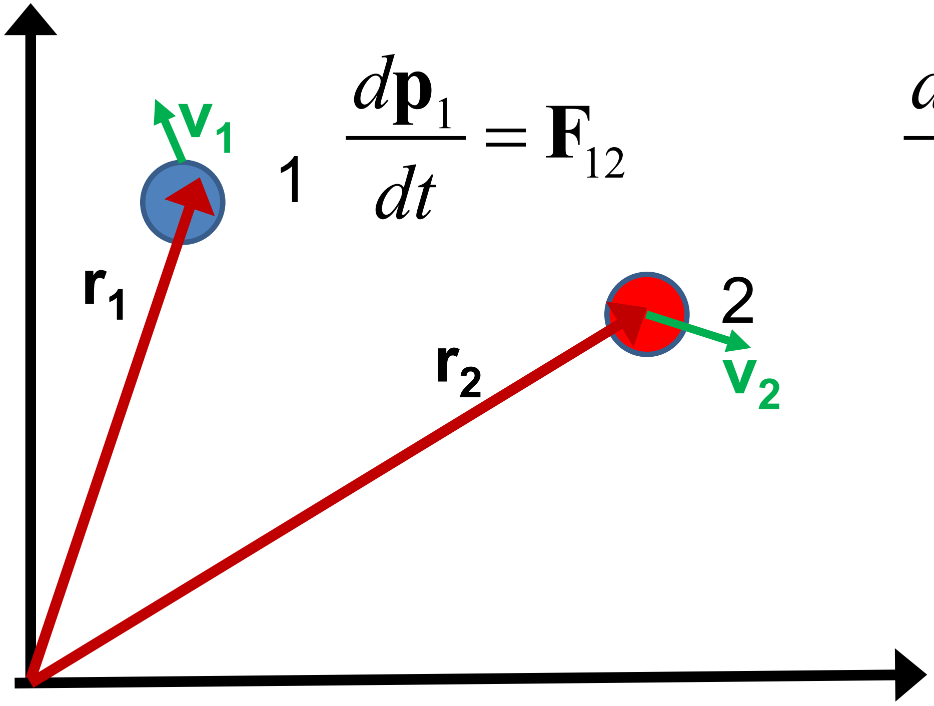
$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

Now consider the more general case of particle interactions and the corresponding scattering analysis.

Scattering theory can help us analyze the interaction potential *V(r).* First, we need to simply the number of variables.

Relationship of scattering cross-section to particle interactions -- Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) \quad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

Relationship between center of mass and laboratory frames of reference. At a time *t*, the following relationships apply --

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \equiv (m_1 + m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 \equiv m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\dot{\mathbf{R}}_{CM} = (m_1 + m_2)\mathbf{V}_{CM}$$

Definition of relative coordinate $\mathbf{r}$

$$\mathbf{r}_1 - \mathbf{r}_2 \equiv \mathbf{r} \qquad\qquad \mathbf{v}_1 - \mathbf{v}_2 \equiv \mathbf{v}$$

Note that: $$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

$$m_1 v_1^2 + m_2 v_2^2 = (m_1 + m_2)\left(\left|\frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}\right|^2\right) + \frac{m_1 m_2}{m_1 + m_2}\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2$$

$$\Rightarrow \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v^2$$

What do you think?

a. No big deal?
b. May be useful…
c. Obviously very clever…

Summary --

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

$$= \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

$$= \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu v^2 + V\left(\mathbf{r}\right)$$

where: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$

What do you think?

a. No big deal?
b. May be useful…
c. Obviously very clever…

Classical mechanics of a conservative 2-particle system -- continued

$$E = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

For central potentials: $V\left(\mathbf{r}_1 - \mathbf{r}_2\right) = V\left(\left|\mathbf{r}_1 - \mathbf{r}_2\right|\right) \equiv V\left(r_{12}\right)$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12} \equiv \mathbf{r}_{12} \times \mu\mathbf{v}_{12}$$

$$E = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu v_{12}^2 + \frac{L_{12}^2}{2\mu r_{12}^2} + V\left(r_{12}\right)$$

Simpler notation:

$$E = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V\left(r\right)$$

Note that we could use the Lagrangian and Hamiltonian formalism to analyze this system and the results would be equivalent.

Simpler notation:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

constants

vary in time

For scattering analysis only need to know trajectory **before** and **after** the collision. We also generally assume that the interaction between particle and target *V(r)* conserves energy and angular momentum.

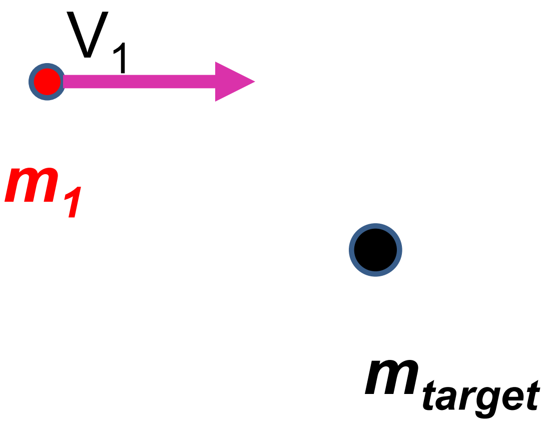
Comment: The impact parameter *b* is a useful concept in the general case and can be analyzed in the relative coordinate system.

$$E_{total} = \underbrace{\frac{1}{2}(m_1 + m_2)V_{CM}^2}_{E_{CM}} + \underbrace{\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)}_{E_{rel}}$$

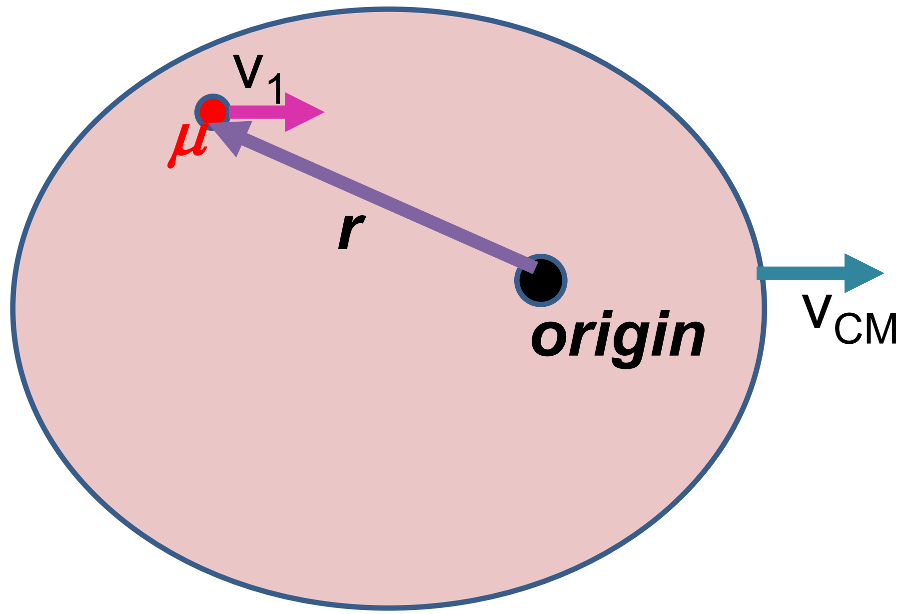
$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{E_{rel}b^2}{r^2} + V(r)$$

Note: The following analysis will be carried out in the center of mass frame of reference.

In laboratory frame:



In center-of-mass frame:



$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = \left| \mathbf{r} \times \mu \mathbf{v}_1 \right|$$

Typically, the laboratory frame is where the data is taken, but the center of mass frame is where the analysis is most straightforward.

Previous equations --

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

constant

relative coordinate system; visualize as “in” CM frame

Note that from the analysis viewpoint, we have reduced a 2-particle system into a single variable problem.

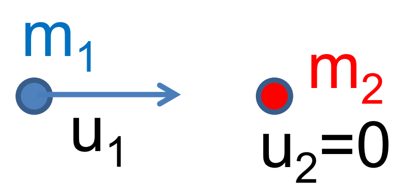
$$\text{Note that } \mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2 \qquad \Rightarrow -\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

It is often convenient to analyze the scattering cross section in the center of mass reference frame.

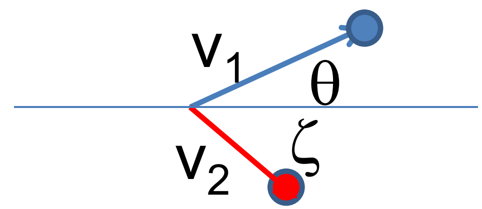
Relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before

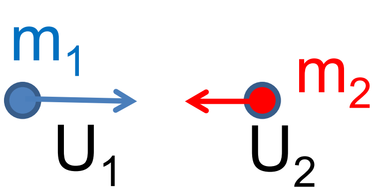


After

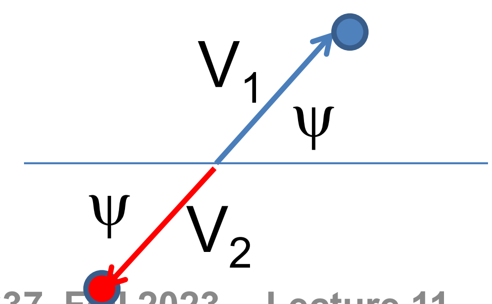


Center of mass reference frame:

Before



After



Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest in lab frame:

Before collision:

$$\mathbf{V}_{CM} = \frac{m_1}{m_1 + m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \qquad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1 + m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

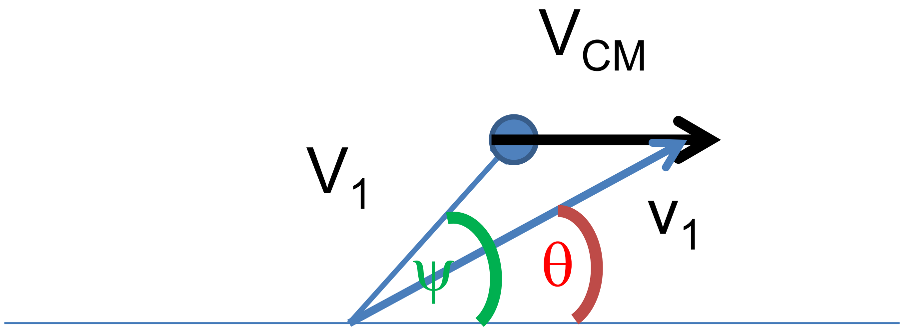
$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \qquad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

After collision:

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

Relationship between center of mass and laboratory frames of reference for the scattering particle 1



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

For elastic scattering

In what situations do particles undergo inelastic scattering, rather than elastic scattering?

Comment – elastic scattering means $E_{initial}=E_{final}$

Typically, elastic scattering occurs when two fundamental particles interact (as long as the final kinetic energy of both particles is taken into account).

Elastically bouncing ball

Inelastically collision

Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2+\tfrac{1}{2}m_2U_2^2+\tfrac{1}{2}(m_1+m_2)V_{CM}^2$$
$$=\tfrac{1}{2}m_1V_1^2+\tfrac{1}{2}m_2V_2^2+\tfrac{1}{2}(m_1+m_2)V_{CM}^2$$

Also note:

$$m_1\mathbf{U}_1+m_2\mathbf{U}_2=0 \qquad\qquad m_1\mathbf{V}_1+m_2\mathbf{V}_2=0$$

$$\mathbf{U}_1=\frac{m_2}{m_1}\mathbf{V}_{CM} \qquad\qquad \mathbf{U}_2=-\mathbf{V}_{CM}$$

$$\Rightarrow |\mathbf{U_1}|=|\mathbf{V_1}| \quad \text{and} \quad |\mathbf{U_2}|=|\mathbf{V_2}|=|\mathbf{V_{CM}}|$$

$$\text{Also note that:} \quad m_1|\mathbf{U_1}|=m_2|\mathbf{U_2}|$$

$$\text{So that:} \quad V_{CM}/V_1=V_{CM}/U_1=m_1/m_2$$

Summary of results --

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM} / V_1} = \frac{\sin\psi}{\cos\psi + m_1 / m_2}$$

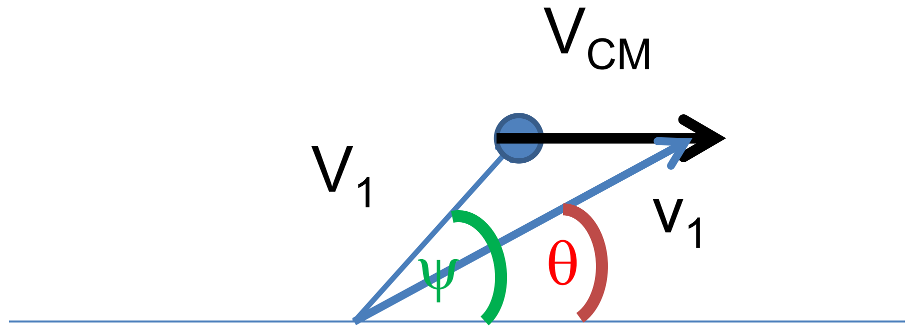
General case

Special case of elastic scattering

For elastic scattering

$$V_{CM} / V_1 = m_1 / m_2$$

Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

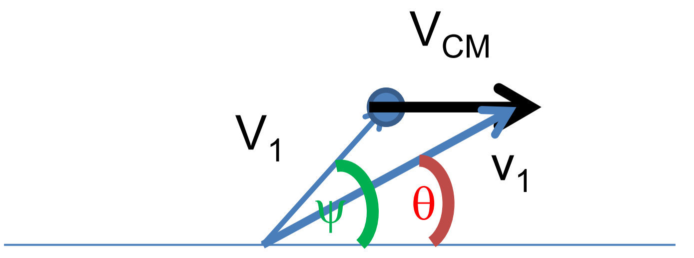
Also: $$\cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\psi + (m_1/m_2)^2}}$$

More details -- from the diagram and equations --

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$



Take the dot product of the first equation with itself

$$v_1^2 = V_1^2 + 2V_1V_{CM}\cos\psi + V_{CM}^2$$

$$\text{or } \frac{v_1}{V_1} = \sqrt{1 + 2\frac{V_{CM}}{V_1}\cos\psi + \frac{V_{CM}^2}{V_1^2}} = \sqrt{1 + 2\frac{m_1}{m_2}\cos\psi + \left(\frac{m_1}{m_2}\right)^2}$$

$$\Rightarrow \quad \cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\psi + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}}=\left|\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta}\right|=\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

Using:

$$\cos\theta=\frac{\cos\psi+m_1/m_2}{\sqrt{1+2(m_1/m_2)\cos\psi+(m_1/m_2)^2}}$$

$$\left|\frac{d\cos\theta}{d\cos\psi}\right|=\frac{(m_1/m_2)\cos\psi+1}{\left(1+2(m_1/m_2)\cos\psi+(m_1/m_2)^2\right)^{3/2}}$$

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}$$

where: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+m_1/m_2}$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}$$

where: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+m_1/m_2}$

Example: suppose $m_1 = m_2$

In this case: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+1} \quad \Rightarrow \theta=\dfrac{\psi}{2}$

note that $0\le\theta\le\dfrac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\theta)}{d\Omega_{CM}}\right)\cdot 4\cos\theta$$

Summary --

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}$$

where: $\tan\theta = \dfrac{\sin\psi}{\cos\psi + m_1/m_2}$

For elastic scattering

Hard sphere example – continued

$m_1 = m_2$

Center of mass frame

Lab frame

$$\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right) = \frac{D^2}{4}$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = D^2\cos\theta \quad \theta = \frac{\psi}{2}$$

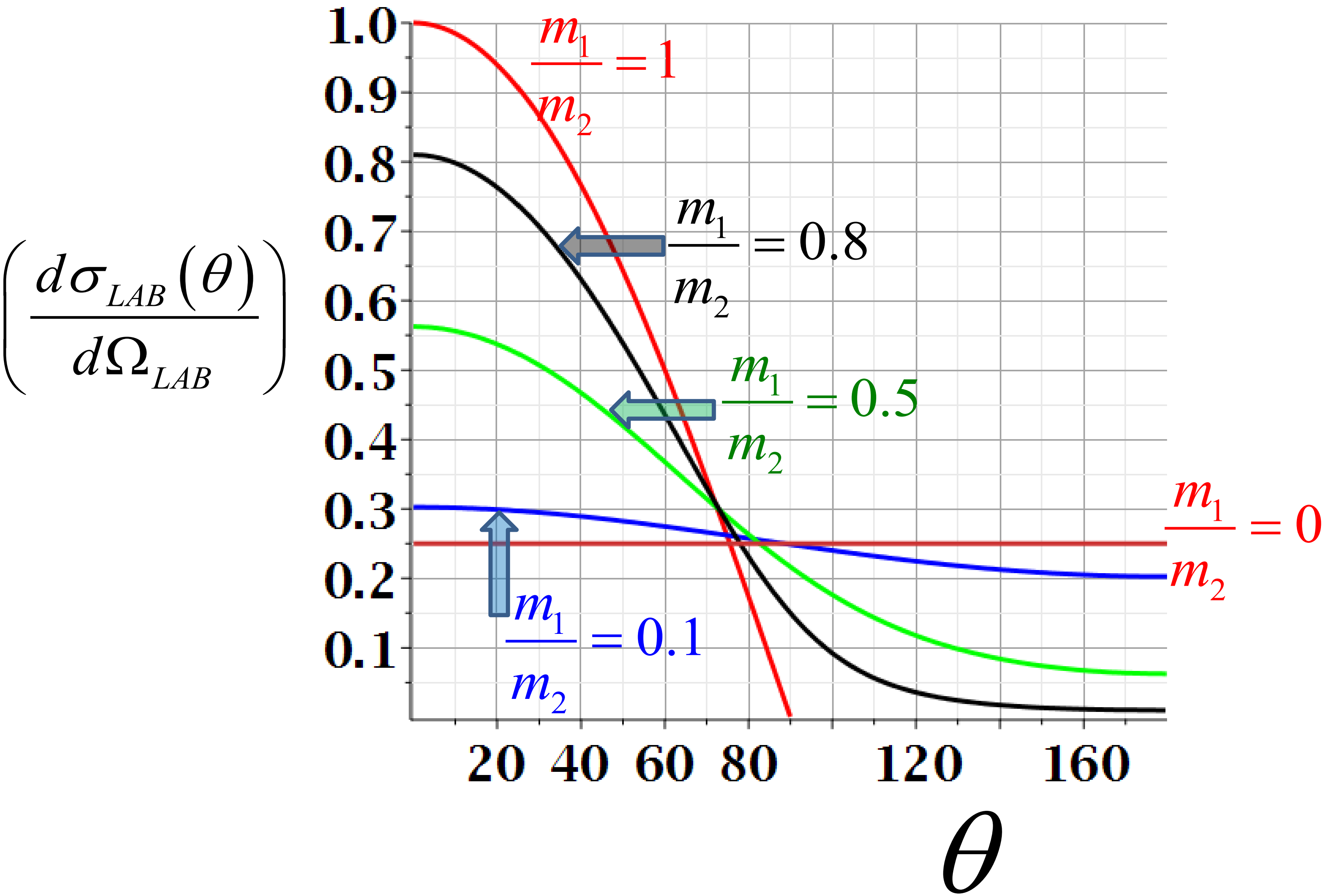
$$\int \frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}} d\Omega_{CM} =$$

$$\frac{D^2}{4} 4\pi = \pi D^2$$

$$\int \frac{d\sigma_{lab}(\theta)}{d\Omega_{lab}} d\Omega_{lab} =$$

$$2\pi D^2 \int_0^{\pi/2} \cos\theta \, \sin\theta d\theta = \pi D^2$$

Scattering cross section for hard sphere in lab frame for various mass ratios:



For visualization, is convenient to make a "parametric" plot of

$$\left(\frac{d\sigma_{LAB}}{d\Omega}(\theta)\right) \text{ vs } \theta(\psi)$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

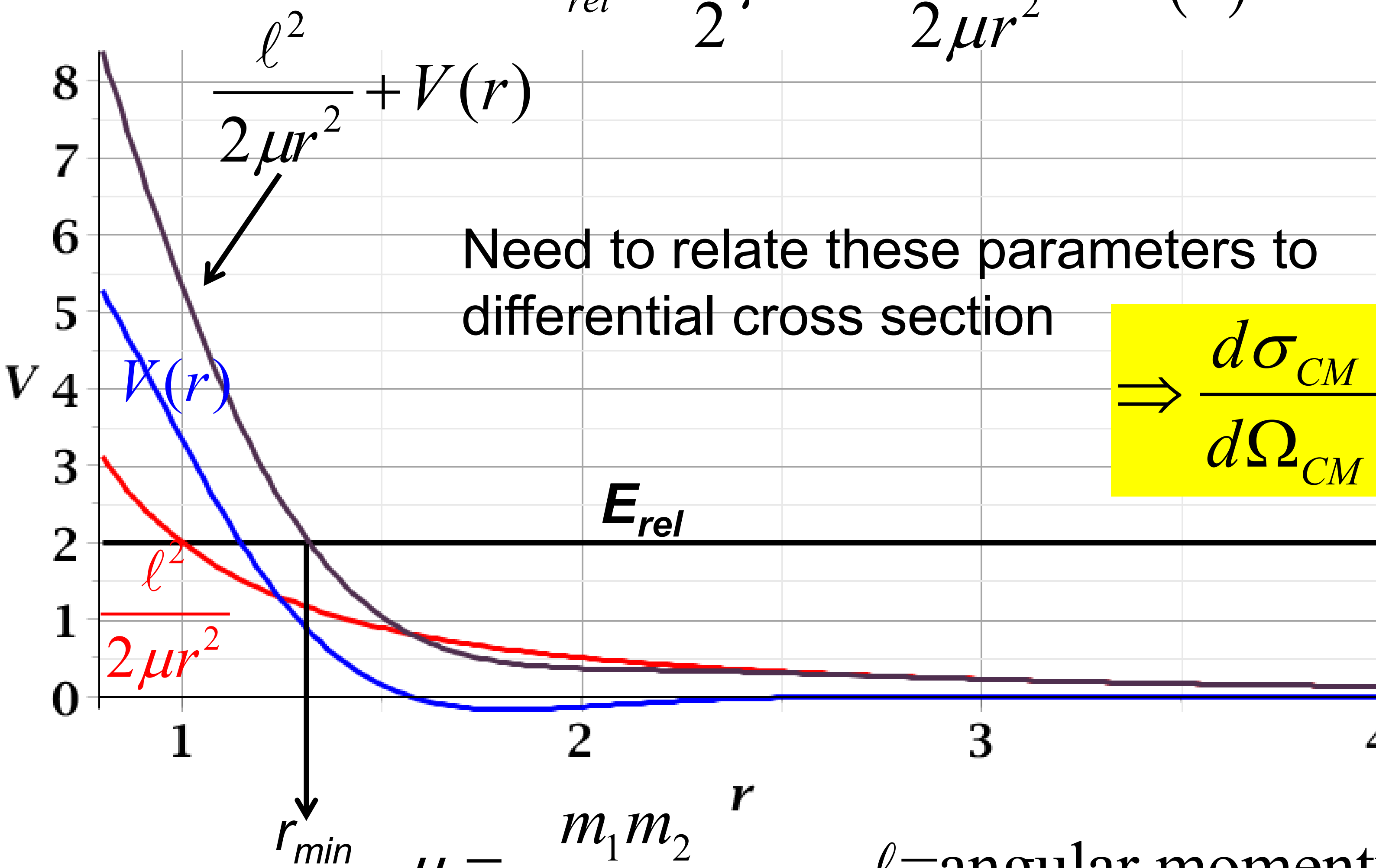
where: $\tan\theta = \dfrac{\sin\psi}{\cos\psi + m_1/m_2}$

Maple syntax:

```
> plot( {[psi(theta, 0), sigma(theta, 0), theta = 0.001 ..3.14], [psi(theta, .1), sigma(theta, .1), theta
     = 0.001 ..3.14], [psi(theta, .5), sigma(theta, .5), theta = 0.001 ..3.14], [psi(theta, .8),
    sigma(theta, .8), theta = 0.001 ..3.14], [psi(theta, 1), sigma(theta, 1), theta = 0.001 ..3.14]},
    thickness = 3, font = ['Times','bold', 24], gridlines = true, color = [red, blue, green, black,
    orange])
```

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum