PHY 337/637 Analytical Mechanics
12:30-1:45 PM TR in Olin 103

Notes for Lecture 12
Review

1. Calculus of variation methodology
2. Lagrangian formalism
3. Hamiltonian formalism

PHYSICS COLLOQUIUM

THURSDAY

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OCTOBER 5TH, 2023

Adaptive Optics and Interference Theory Enable Measurement of Retinal Function

Imaging of the retina has long been part of an ophthalmic exam, but the optics of the eye have aberrations that limit the quality of those images. Using adaptive optics, a technique originating in astronomy, researchers can measure and correct for the eye's optical aberrations thereby enabling diffraction limited imaging of the living retina. With this technology, individual photoreceptors and other retinal cells can be visualized noninvasively in the living human eye. My talk will provide an overview of adaptive optics imaging and will discuss how adaptive optics in combination with interference of light waves allows assessments of photoreceptor function.

4 pm - Olin 101

Refreshments will be served in the Olin lobby beginning at 3:30 pm

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Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	Reading	Topic	PP	Assign
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 6	Lagrangian mechanics	6E	#4
5	Tu, 9/12/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	#5
6	Th, 9/14/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	
7	Tu, 9/19/2023	Ch. 13	Dynamics of rigid bodies	13E	#6
8	Th, 9/21/2023	Ch. 13	Dynamics of rigid bodies	13E	#7
9	Tu, 9/26/2023	Ch. 13 & 11	Review of rigid bodies and intro to scattering	11E	#8
10	Th, 9/28/2023	Ch. 11	Scattering theory	11E	#9
11	Tu, 10/3/2023	Ch. 11	Scattering theory	11E	
12	Th, 10/5/2023		Summary and examples		Take home exam start
13	Tu, 10/10/2023		Summary and examples		Take home exam due
	Th, 10/12/2023	Fall Break			
14	Tu, 10/17/2023		Summary and examples		

Review of the concept of the calculus of variation

- Based on the notion of minimization, but applied to an integral form
- Optimization performed to find a function – such as *y(x)*.
- Uses –
  - Various optimization problems in a variety of applications
  - Optimizing the “action integral” (Hamilton’s principle)
  - Richard Feynman applied it to develop an alternative approach to quantum mechanics called path integrals

Functional minimization of an integral relationship

Consider a family of functions $y(x)$, with fixed end points

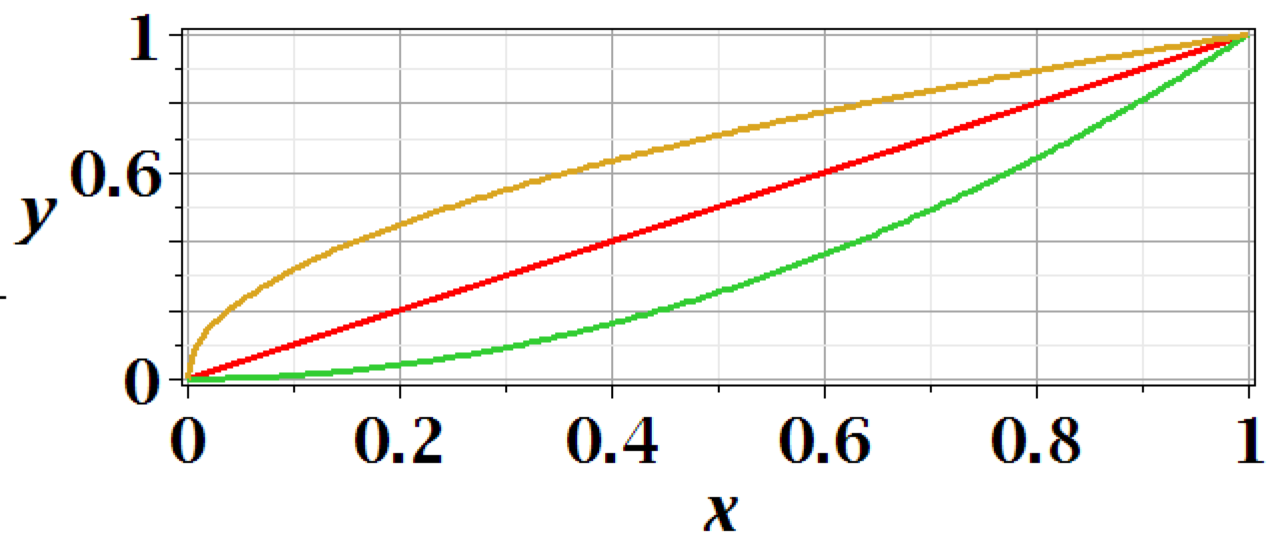
$y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral form $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Difference between minimization of a function V(x) and the minimization in the calculus of variation.

Minimization of a function – V(x)

➔Know V(x) ➔Find x_0 such that $V(x_0)$ is a minimum.

Calculus of variation

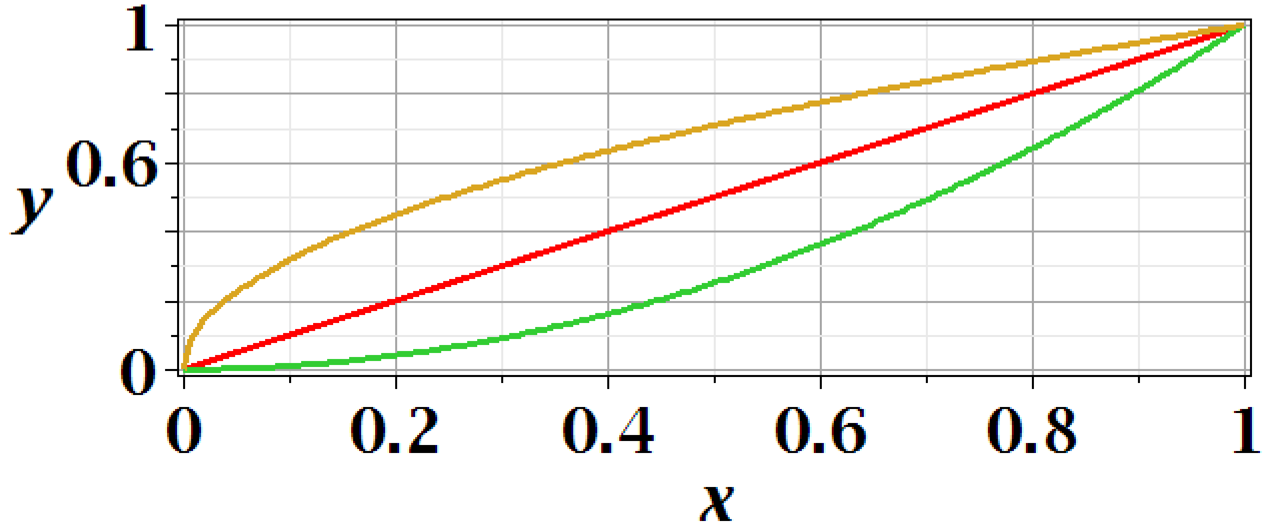
For $x_i \leq x \leq x_f$ want to find a function $y(x)$ that minimizes an integral that depends on $y(x)$.

The analysis involves deriving and solving a differential equation for the function $y(x)$.

Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Sample functions:

$$y_1(x) = \sqrt{x} \qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$$

$$y_2(x) = x \qquad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$$

$$y_3(x) = x^2 \qquad L = \int_0^1 \sqrt{1+4x^2}\, dx = 1.4789$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Note that this is a
"total" derivative

Example: End points -- $y(0)=0;\ y(1)=1$

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

Review: for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation

Comment on partial derivatives versus total derivatives.

For a simple function $y(t)$, the notation means that y is a function only of t so that $\frac{dy}{dt}$ is well defined.

For a more complicated function of several variables, $f(a(t),b(t),t)$ the notion of partial and total derivatives needs to be considered.

$$\frac{df(a(t),b(t),t)}{dt} =$$

$$\left(\frac{\partial f(a(t),b(t),t)}{\partial a}\right)_{b,t}\frac{da}{dt}+\left(\frac{\partial f(a(t),b(t),t)}{\partial b}\right)_{a,t}\frac{db}{dt}+\left(\frac{\partial f(a(t),b(t),t)}{\partial t}\right)_{a,b}$$

$\Rightarrow$ What happens when $b(t)=\frac{da}{dt}$?

a. No problem
b. Solvable problem

Also note that

$$\int_{t_i}^{t_f} \left(\frac{df(a(t),b(t),t)}{dt} \right) dt = f(a(t_f),b(t_f),t_f) - f(a(t_i),b(t_i),t_i)$$

We are now going to shift notation in order to apply the calculus of variation formalism to Hamilton's principle and Lagrangian mechanics.

$$x \rightarrow t$$

$$y(x) \rightarrow q(t)$$

$$\frac{dy}{dx} \rightarrow \dot{q}(t)$$

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$S = \int_{t_1}^{t_2} L\left(\{q,\dot{q}\};t\right)dt \equiv \int_{t_1}^{t_2} L\left(\left\{y,\frac{dy}{dt}\right\};t\right)dt$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g.*

Newton's formulation: $\quad m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $\quad y(t) = y_i + v_i t - \frac{1}{2}gt^2$

Lagrangian for this case:

$$L = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) dt$$ is minimized for physical $y(t)$:

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Extension of these ideas to multiple coordinates due to multiple dimensions and/or multiple particles.

1 particle with 1 degree of freedom ➔ **many particles with multiple degrees of freedom**

$$S = \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt \qquad \Rightarrow \qquad S = \int_{t_1}^{t_2} L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) dt$$

for example: $L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) \equiv L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$

1 particle+3 Cartesian dimensions

or $\quad L(\{x_i\}, \{y_i\}, \{z_i\}, \{\dot{x}_i\}, \{\dot{y}_i\}, \{\dot{z}_i\}, t)$

N particles+3 Cartesian dimensions

Another example: $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}\left(I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}\left(I_1\dot{\beta}\right) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}\left(I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\right) = 0$$

Do you remember this example?

a. With fondness
b. Without fondness
c. With different notation

Here we can see some possible benefits of this approach --

$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma})=\tfrac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+\tfrac{1}{2}I_3\left(\dot{\alpha}\cos\beta+\dot{\gamma}\right)^2-Mgd\cos\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}}=\frac{d}{dt}\underbrace{\left(I_1\dot{\alpha}\sin^2\beta+I_3\left(\dot{\alpha}\cos\beta+\dot{\gamma}\right)\cos\beta\right)}_{\text{Constant in time}}=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\beta}}=\frac{d}{dt}\left(I_1\dot{\beta}\right)=\frac{\partial L}{\partial\beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\gamma}}=\frac{d}{dt}\underbrace{\left(I_3\left(\dot{\alpha}\cos\beta+\dot{\gamma}\right)\right)}_{\text{Constant in time}}=0$$

Introducing the Hamiltonian --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $\quad H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

2. Compute generalized momenta: $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$

3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$

4. Form Hamiltonian function: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

5. Analyze canonical equations of motion:

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad\qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Questions for discussion

a. Given that we need to start with the Lagrangian, do we really need the Hamiltonian
b. Given that for all of the examples we have discussed, we get the same answers from analysis with Newton’s laws and Lagrangian and Hamiltonian formalism,   how does one pick the appropriate approach?