PHY 337/637 Analytical Mechanics
12:30-1:45 PM TR in Olin 103

Notes for Lecture 14
Course review

1. Calculus of variation methodology
2. Lagrangian formalism
3. Hamiltonian formalism
4. Dynamics of rigid bodies
5. Scattering theory

Exams returned at the end of class

Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	Reading	Topic	PP	Assign
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 6	Lagrangian mechanics	6E	#4
5	Tu, 9/12/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	#5
6	Th, 9/14/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	
7	Tu, 9/19/2023	Ch. 13	Dynamics of rigid bodies	13E	#6
8	Th, 9/21/2023	Ch. 13	Dynamics of rigid bodies	13E	#7
9	Tu, 9/26/2023	Ch. 13 & 11	Review of rigid bodies and intro to scattering	11E	#8
10	Th, 9/28/2023	Ch. 11	Scattering theory	11E	#9
11	Tu, 10/3/2023	Ch. 11	Scattering theory	11E	
12	Th, 10/5/2023		Summary and examples		Take home exam start
13	Tu, 10/10/2023		Summary and examples		Take home exam due
	Th, 10/12/2023	Fall Break			
14	Tu, 10/17/2023		Summary and examples		

Physics and Chemistry Joint Colloquium

Thursday

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October 19th, 2023

Molecular Photovoltaics and the Advent of Perovskite Solar Cells

Photovoltaic cells using molecular dyes, semiconductor quantum dots or perovskite pigments as light harvesters have emerged as credible contenders to conventional devices. Dye sensitized solar cells (DSCs) use a three-dimensional nanostructured junction for photovoltaic electricity production and currently reach a power conversion efficiency (PCE) of 15.2 % in full sunlight and over 30 % in ambient light. They possess unique practical advantages in terms of particularly high effective electricity production from ambient light, ease of manufacturing, flexibility and transparency, bifacial light harvesting, and aesthetic appeal, which have fostered large scale industrial production and commercial applications. They served as a launch pad for perovskite solar cells (PSCs) which are presently being intensively investigated as one of the most promising future PV technologies, the PCE of solution processed laboratory cells having currently reached 25.7%. Present research focuses on their scale up to as well as ascertaining their long-term operational stability. This lecture will cover the most recent findings in these revolutionary photovoltaic domains.

Michael Grätzel

EPFL

Switzerland

Note that lecture location is in Salem Hall ➔

4 pm - Salem 012

Refreshments served prior to seminar

Review of the concept of the calculus of variation

- Based on the notion of minimization, but applied to an integral form
- Optimization performed to find a function – such as $y(x)$.
- Uses –
  - Various optimization problems in a variety of applications
  - Optimizing the “action integral” (Hamilton’s principle)
  - Richard Feynman applied it to develop an alternative approach to quantum mechanics called path integrals

Functional minimization of an integral relationship

Consider a family of functions $y(x)$, with fixed end points

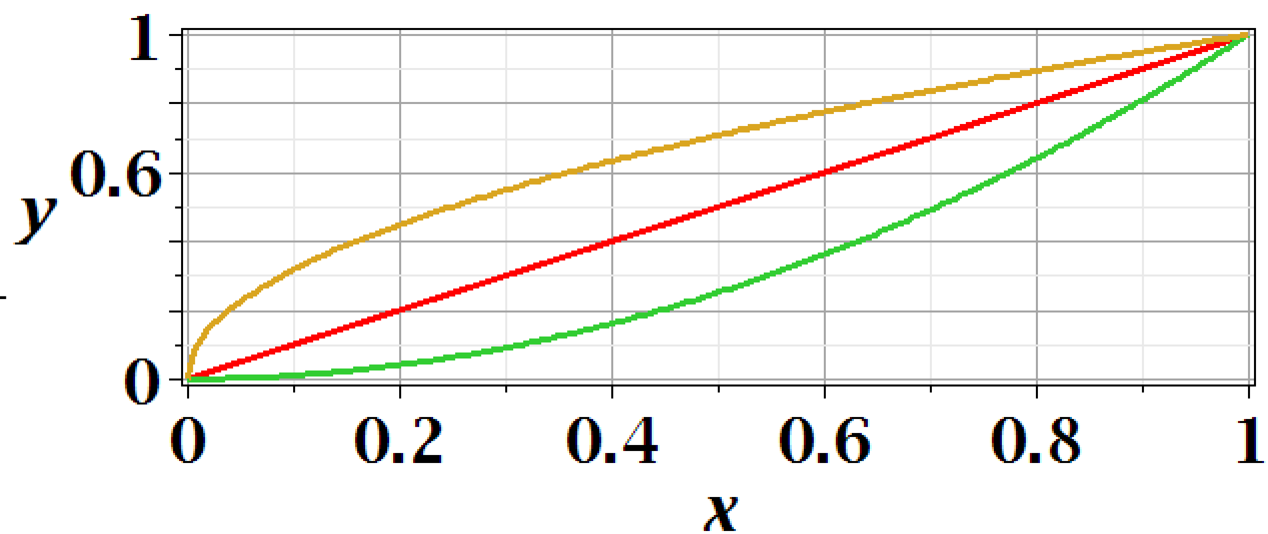
$y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral form $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

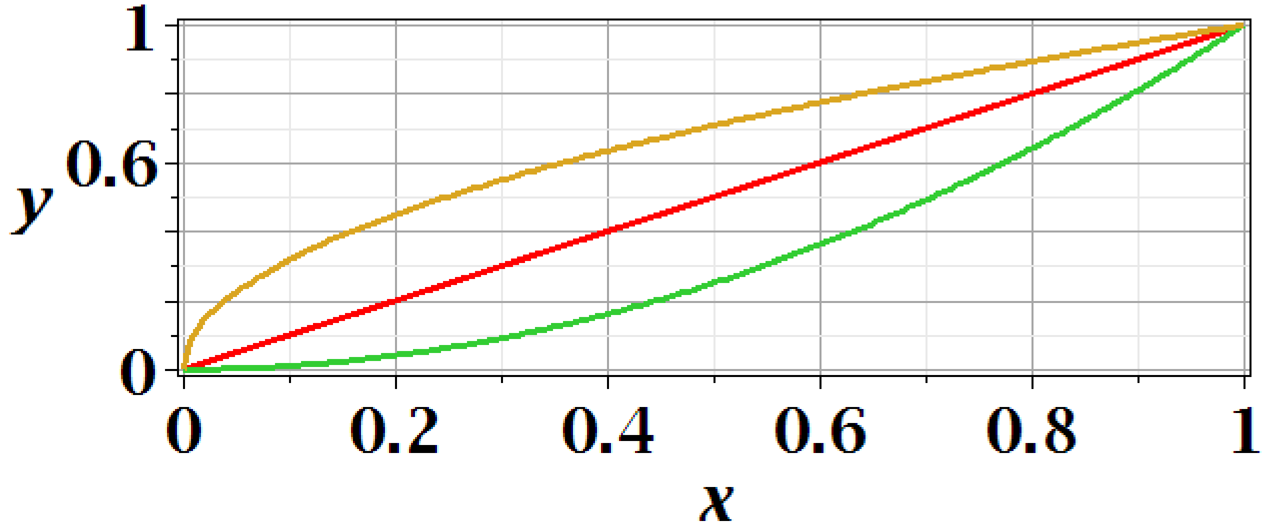
$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



Sample functions:

$y_1(x) = \sqrt{x}$ $\qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}} dx = 1.4789$

$y_2(x) = x$ $\qquad L = \int_0^1 \sqrt{1+1} dx = \sqrt{2} = 1.4142$

$y_3(x) = x^2$ $\qquad L = \int_0^1 \sqrt{1 + 4x^2} dx = 1.4789$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x,\frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x,\frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y \, dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x,\frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Note that this is a "total" derivative

Euler-Lagrange equation
➔ Differential equation to find *y(x)*

Comment on partial derivatives versus total derivatives.

For a simple function $y(t)$, the notation means that y is a function only of t so that $\frac{dy}{dt}$ is well defined.

For a more complicated function of several variables, $f(a(t),b(t),t)$ the notion of partial and total derivatives needs to be considered.

$$\frac{df(a(t),b(t),t)}{dt} =$$

$$\left(\frac{\partial f(a(t),b(t),t)}{\partial a}\right)_{b,t}\frac{da}{dt}+\left(\frac{\partial f(a(t),b(t),t)}{\partial b}\right)_{a,t}\frac{db}{dt}+\left(\frac{\partial f(a(t),b(t),t)}{\partial t}\right)_{a,b}$$

Also note that

$$\int_{t_i}^{t_f} \left(\frac{df(a(t),b(t),t)}{dt} \right) dt = f(a(t_f),b(t_f),t_f) - f(a(t_i),b(t_i),t_i)$$

We are now going to shift notation in order to apply the calculus of variation formalism to Hamilton's principle and Lagrangian mechanics.

$$x \rightarrow t$$

$$y(x) \rightarrow q(t)$$

$$\frac{dy}{dx} \rightarrow \dot{q}(t)$$

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$S = \int_{t_1}^{t_2} L\left(q,\dot{q},t\right)dt \equiv \int_{t_1}^{t_2} L\left(y,\frac{dy}{dt},t\right)dt \quad \text{where} \quad L = T - U$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g.*

Newton's formulation: $m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $y(t) = y_i + v_i t - \frac{1}{2}gt^2$

Lagrangian for this case:

$$L = T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy (T)

Potential energy (U)

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) dt$$ is minimized for physical $y(t)$:

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Extension of these ideas to multiple coordinates due to multiple dimensions and/or multiple particles.

1 particle with 1 degree of freedom ➔ **many particles with multiple degrees of freedom**

$$S = \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt \quad \Rightarrow \quad S = \int_{t_1}^{t_2} L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) dt$$

for example: $L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) \equiv L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$

1 particle+3 Cartesian dimensions

or $L(\{x_i\}, \{y_i\}, \{z_i\}, \{\dot{x}_i\}, \{\dot{y}_i\}, \{\dot{z}_i\}, t)$

N particles+3 Cartesian dimensions

Introducing the Hamiltonian --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

2. Compute generalized momenta: $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$

3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$

4. Form Hamiltonian function: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

5. Analyze canonical equations of motion:

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Example

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)\equiv T-U=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2-mgy$$

$$p_y=m\frac{dy}{dt}$$

$$H=p_y\frac{dy}{dt}-\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2-mgy\right)=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2+mgy$$

$$H(y,p_y)=\frac{p_y^2}{2m}+mgy=T+U$$

Note that D'Alembert's "justification" of the Lagrangian approach relied on the potential not depending on the particle velocities. We "derived" an exception to this for treating forces due to magnetic fields and/or the vector potential **A**(**r**,*t*)

$$U = U_{mechanical}(\mathbf{r}) + q\Phi\left(\mathbf{r},t\right) - q\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$$

Note that not all velocity-dependent forces can be treated withing the Lagrangian formalism.

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$ (in SI units)

$x-$component : $F_x = q\left(E_x + \left(\mathbf{v}\times\mathbf{B}\right)_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Note: Here we are using cartesian coordinates for convenience.

x-component: $\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0$

Apparently: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

Answer: $U = q\Phi\left(\mathbf{r},t\right) - q\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$

where $\mathbf{E}\left(\mathbf{r},t\right) = -\nabla\Phi\left(\mathbf{r},t\right) - \frac{\partial \mathbf{A}\left(\mathbf{r},t\right)}{\partial t}$ $\mathbf{B}\left(\mathbf{r},t\right) = \nabla\times\mathbf{A}\left(\mathbf{r},t\right)$

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Suppose $\mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0 \hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2} B_0 y\right) - \frac{q}{2} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2} B_0 x\right) + \frac{q}{2} B_0 \dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt} m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2} B_0 y\right) - \frac{q}{2} B_0 \dot{y} = 0 \qquad \Rightarrow m\ddot{x} - qB_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2} B_0 x\right) + \frac{q}{2} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + qB_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +qB_0\dot{y}$$

$$m\ddot{y} = -qB_0\dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained from direct application of Newton's laws:

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times B_0\hat{\mathbf{z}}$$

Example Lorentz force -- continued

Evaluation of equations:

$$m\ddot{x} - qB_0\dot{y} = 0$$

$$m\ddot{y} + qB_0\dot{x} = 0$$

$$m\ddot{z} = 0$$

$$\dot{x}(t) = V_0 \sin\left(\tfrac{qB_0}{m}t + \phi\right)$$

$$\dot{y}(t) = V_0 \cos\left(\tfrac{qB_0}{m}t + \phi\right)$$

$$\dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \tfrac{m}{qB_0}V_0 \cos\left(\tfrac{qB_0}{m}t + \phi\right)$$

$$y(t) = y_0 + \tfrac{m}{qB_0}V_0 \sin\left(\tfrac{qB_0}{m}t + \phi\right)$$

$$z(t) = z_0 + V_{0z}t$$

Constructing the Hamiltonian for this case

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2}B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$p_x = m\dot{x} - \frac{q}{2}B_0 y \qquad p_y = m\dot{y} + \frac{q}{2}B_0 x \qquad p_z = m\dot{z} = 0$$

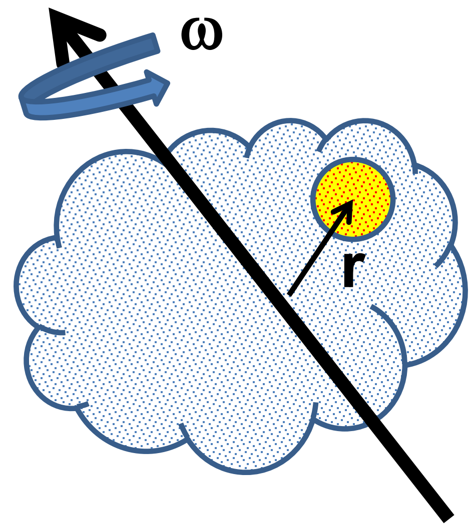
$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \left(\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2}B_0\left(-\dot{x}y + \dot{y}x\right)\right)$$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Canonical form:

$$H(x, y, z, p_x, p_y, p_z) = \frac{(p_x + qB_0 y/2)^2}{2m} + \frac{(p_y - qB_0 x/2)^2}{2m} + \frac{p_z^2}{2m}$$

Kinetic energy of rigid body, rotating at angular velocity $\boldsymbol{\omega}$



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{r}$$

=0 for rigid body

$$\Rightarrow \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

$$T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_p\cdot\mathbf{r}_p\right) - \left(\mathbf{r}_p\cdot\boldsymbol{\omega}\right)^2\right]$$

$$T = \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2 \right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Moment of inertia tensor:

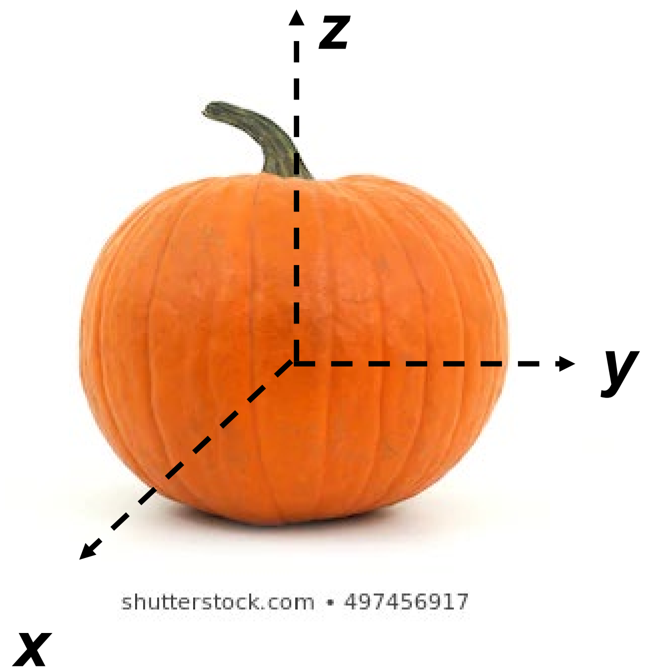
$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

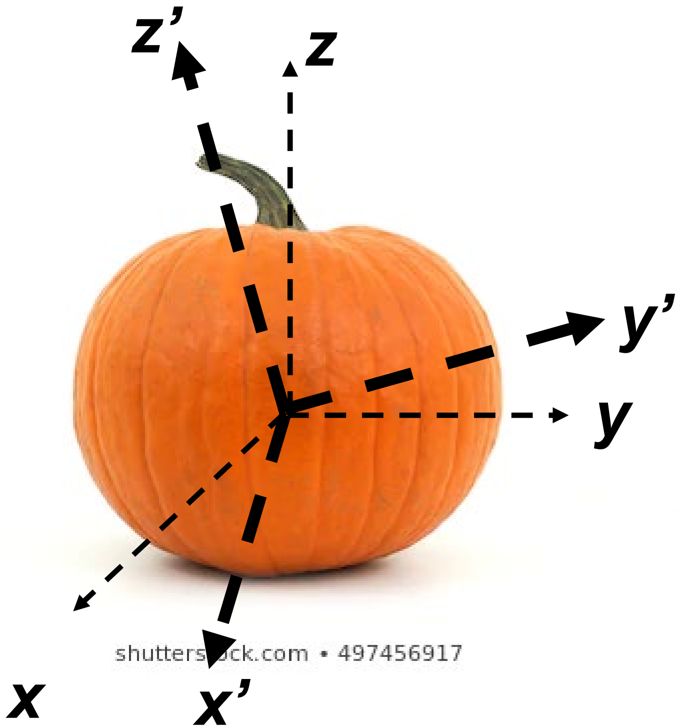
Note: For a given object and a given coordinate system, one can find the moment of inertia matrix

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$





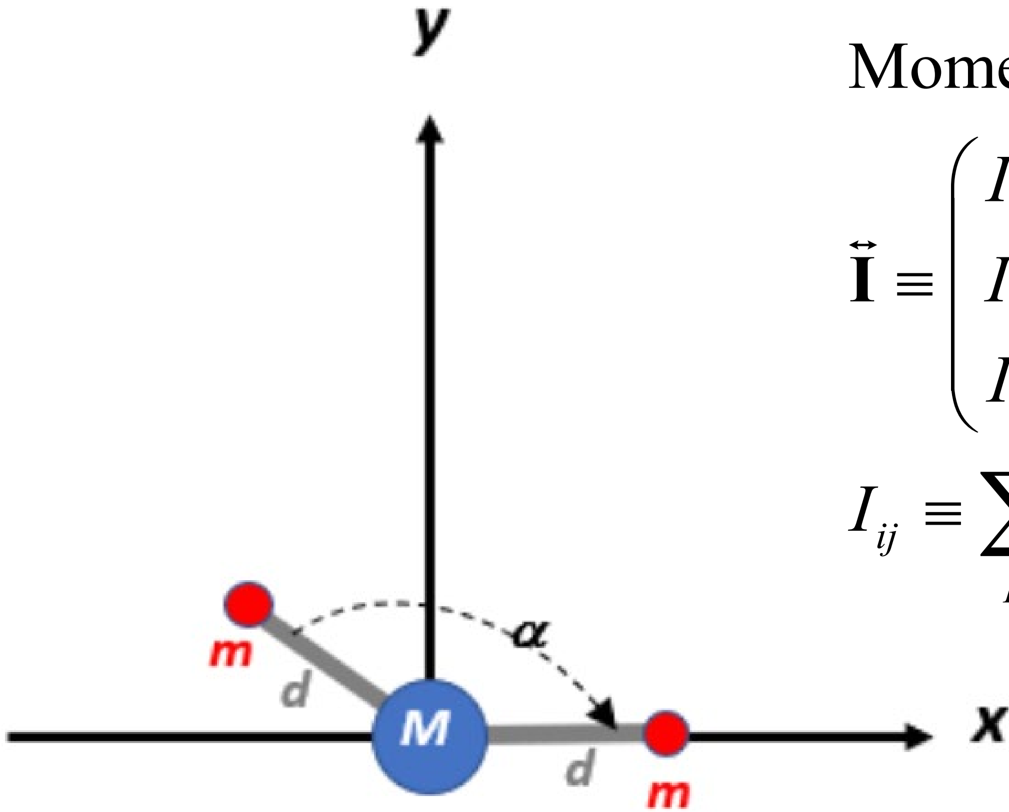
Moment of inertia in original coordinates

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Moment of inertia in principal axes ***(x',y',z')***

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$



Moment of inertia matrix

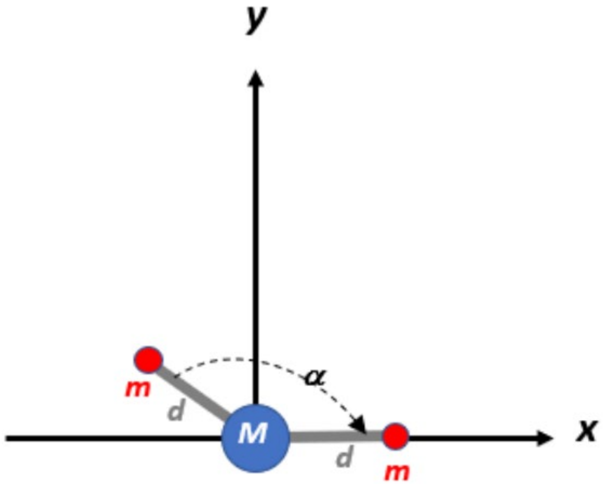
$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

$$I_{xx} = md^2 \sin^2\alpha \qquad I_{xy} = -md^2 \cos\alpha \, \sin\alpha = I_{yx}$$

$$I_{yy} = md^2 \left(1 + \cos^2\alpha\right)$$

$$I_{zz} = 2md^2$$



$$I = md^2 \begin{pmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & 1+\cos^2\alpha & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\tilde{I} = md^2 \begin{pmatrix} 1+\cos\alpha & 0 & 0 \\ 0 & 1-\cos\alpha & 0 \\ 0 & 0 & 2 \end{pmatrix}$$