PHY 337/637 Analytical Mechanics
12:30-1:45 in Olin103

Lecture notes for Lecture 2
Chapter 5 of Cline

Introduction to the calculus of variations

1. Mathematical construction
2. Practical use
3. Examples

Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	Reading	Topic	PP	Assign
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	Calculus of variation		
4	Th, 9/07/2023	Ch. 5	Calculus of variation		
5	Tu, 9/12/2023				
6	Th, 9/14/2023				

PHY 337/637 -- Assignment #2

Assigned: 8/31/2023 Due: 9/5/2023

Start reading Chapter 5, in **Cline**.

1. Using calculus of variations, find the equation, $y(x)$, of the shortest length "curve" which passes through the points ($x=0$, $y=0$) and ($x=3$, $y=4$). What is the length of this "curve"?

The “calculus of variation” as a mathematical construction.

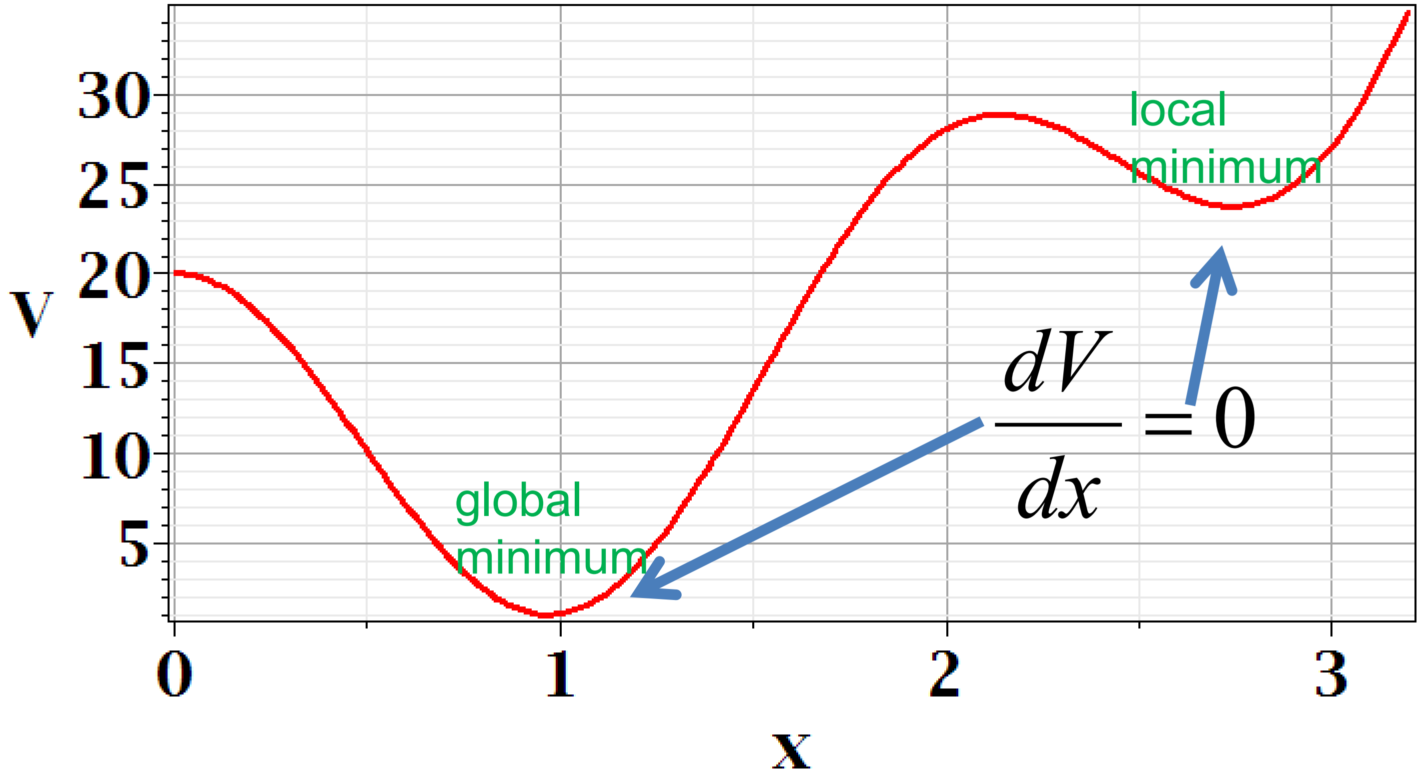
According wikipedia – **Joseph-Louis Lagrange** (born **Giuseppe Luigi Lagrangia** or **Giuseppe Ludovico De la Grange Tournier**; 25 January 1736 – 10 April 1813), also reported as **Giuseppe Luigi Lagrange** or **Lagrangia**, was an Italian mathematician and astronomer, later naturalized French. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

According to Wikipedia –

Leonard Euler (April 7, 1707-September 18, 1783) Swiss mathematician, physicist, astronomer, geographer, logician and engineer who founded the studies of graph theory and topology and made pioneering and influential discoveries in many other branches of mathematics such as analytic number theory, complex analysis, and infinitesimal calculus. He introduced much of modern mathematical terminology and notation, including the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory.

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

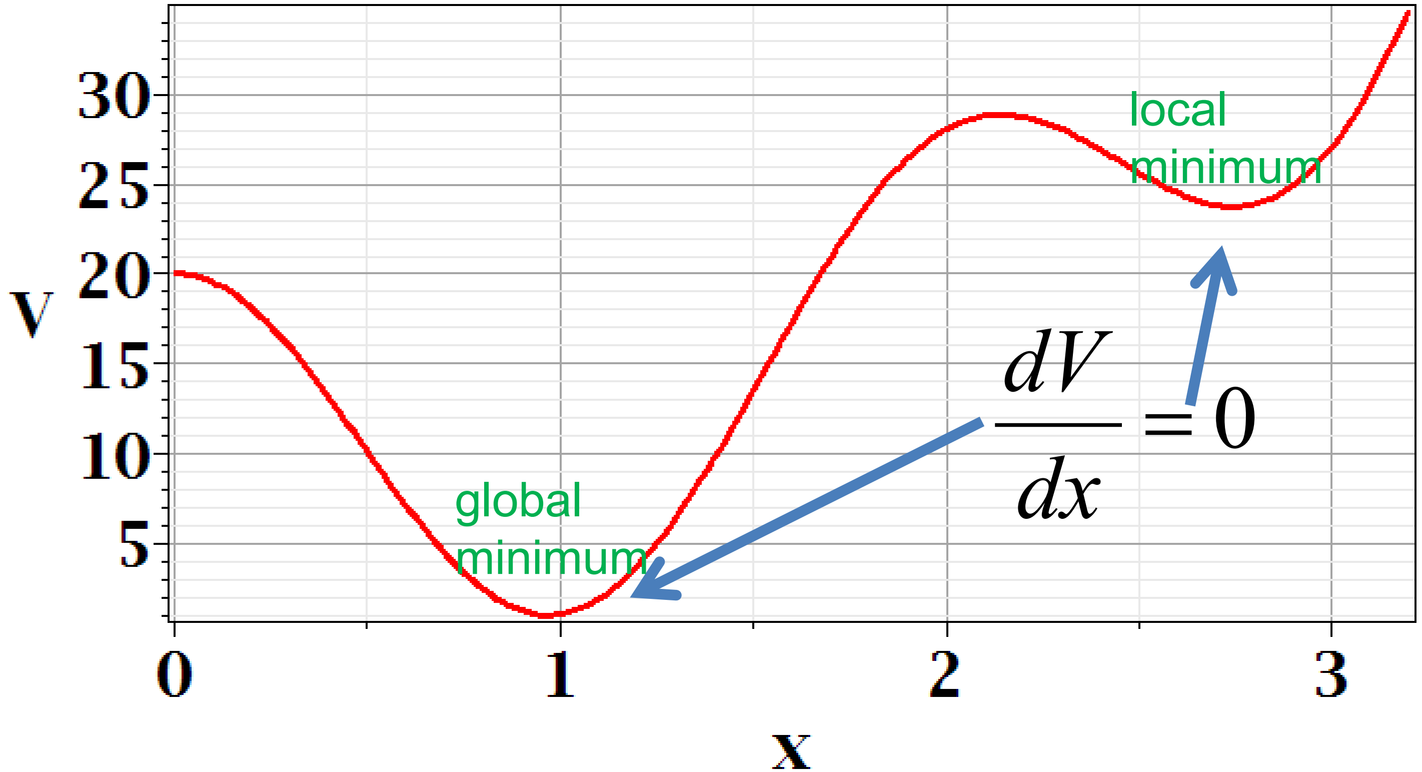
Minimization of a simple function



Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx} = 0$



Functional minimization of an integral relationship

Consider a family of functions $y(x)$, with fixed end points

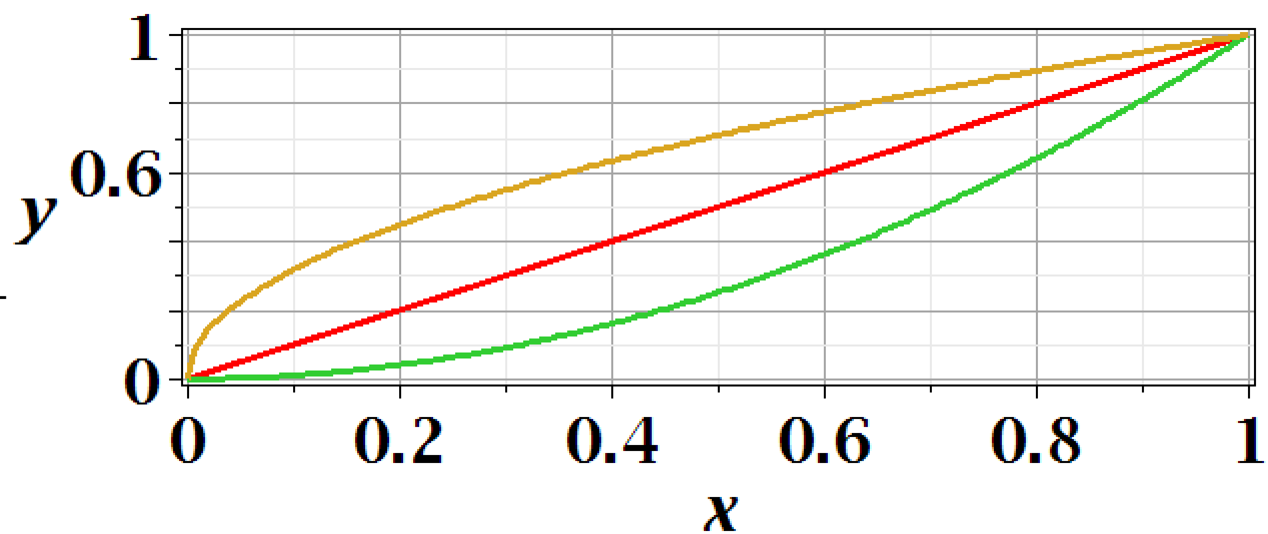
$y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral form $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Difference between minimization of a function V(x) and the minimization in the calculus of variation.

Minimization of a function – V(x)
➔Know V(x)   ➔Find x_0 such that $V(x_0)$ is a minimum.

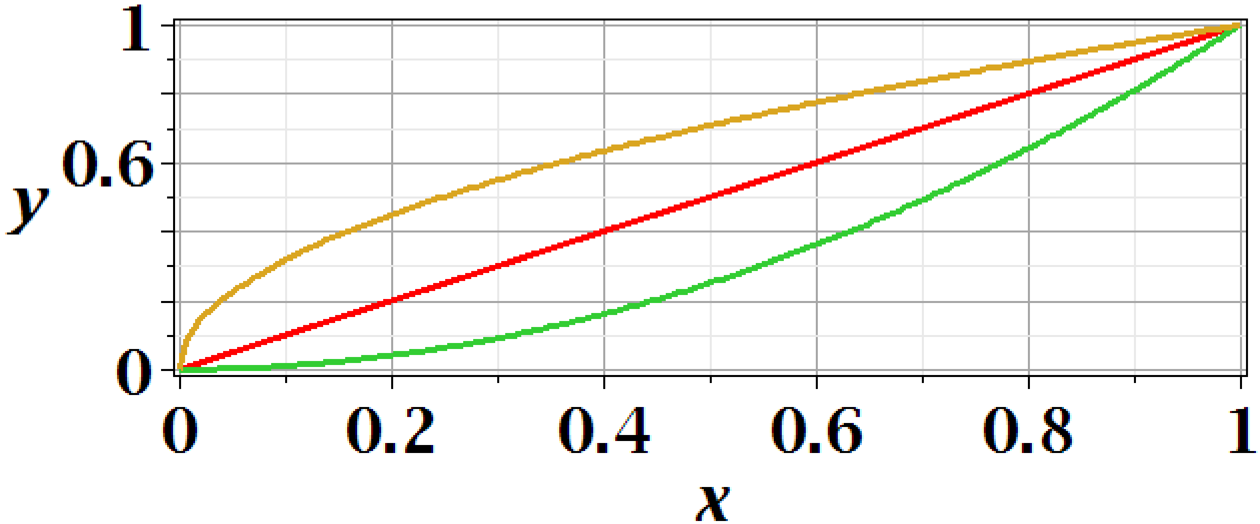
Calculus of variation

For $x_i \le x \le x_f$ want to find a function $y(x)$ that minimizes an integral that depends on $y(x)$.
The analysis involves deriving and solving a differential equation for the function $y(x)$.

Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Sample functions :

$y_1(x) = \sqrt{x}$ $\qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$

$y_2(x) = x$ $\qquad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$

$y_2(x) = x^2$ $\qquad L = \int_0^1 \sqrt{1+4x^2}\, dx = 1.4789$

Calculus of variation example for a pure integral function

$$\text{Find the function } y(x) \text{ which extremizes } L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$$

$$\text{where} \quad L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$$

$$\text{Necessary condition:} \quad \delta L = 0$$

$$\text{At any } x, \text{ let } y(x) \rightarrow y(x) + \delta y(x)$$

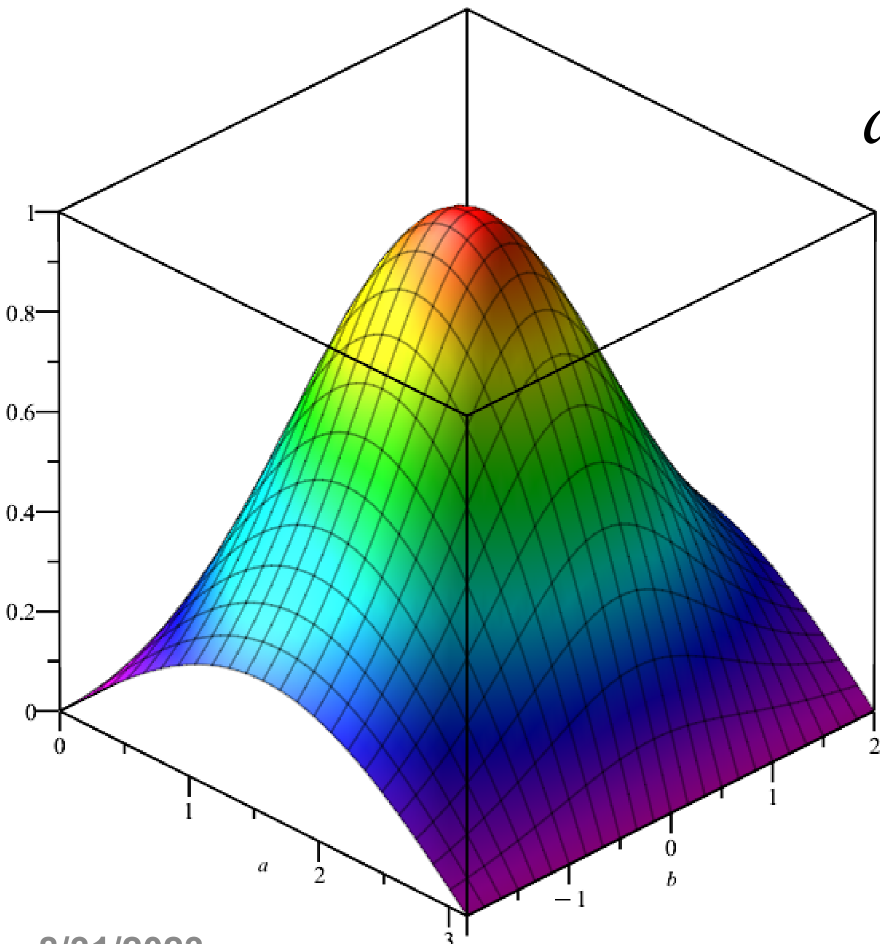
$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx.$$

Comment on partial derivatives -- function $f(a,b)$

$$\frac{\partial f}{\partial a} \equiv \lim_{da \to 0}\left(\frac{f(a+da,b)-f(a,b)}{da}\right) \equiv \left.\frac{\partial f}{\partial a}\right|_b$$



$$df = \left(\frac{\partial f}{\partial a}\right)_b da + \left(\frac{\partial f}{\partial b}\right)_a db$$

Comment about notation concerning functional dependence and partial derivatives

Suppose x, y, z represent independent variables that determine a function f :

We write $f(x, y, z)$. A partial derivative with respect to x implies that we hold y, z fixed and infinitessimally change x

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}\right)$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Note that this is a "total" derivative

“Some” derivations --

Consider the term

$$\int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] dx :$$

If $y(x)$ is a well-defined function, then $\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}\delta y$ *

$$\int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] dx = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \frac{d}{dx} \delta y \right] dx$$

$$= \int_{x_i}^{x_f} \left[\frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] - \frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

Note that the δy notation is meant to imply a general infinitessimal variation of the function $y(x)$

*

Clarification -- what is the meaning of the following statement:

$$\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}\delta y$$

Up to now, the operator δ is not well defined and meant to represent a general infinitessimal difference. Suppose that $\delta y \equiv \frac{dy}{da}$, where a appears in the functional form somehow. For most functional forms that one can think of, $\frac{d^2 y(x,a)}{dxda}=\frac{d^2 y(x,a)}{dadx}$. One can show this to be the case even for $y(x,a)=x^a$ where $\frac{d^2 y(x,a)}{dxda}=\frac{d^2 y(x,a)}{dadx}=x^{a-1}\left(1+a\ln\left(x\right)\right)$.

“Some” derivations (continued)--

$$\int_{x_i}^{x_f} \left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] - \frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

$$= \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \left[\frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

$$= \qquad 0 \qquad - \int_{x_i}^{x_f} \left[\frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

Euler-Lagrange equation:

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Clarfication – Why does this term go to zero?

$$\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]_{x_i}^{x_f}-\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\qquad 0\qquad -\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Answer --

By construction $\delta y(x_i)=\delta y(x_f)=0$

Recap --

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Here we conclude that the integrand has to vanish at every argument in order for the integral to be zero

a. Necessary?
b. Overkill?

Note that Cline uses the notation

$$\frac{dy}{dx} \equiv y' \quad \text{so that Euler's equation reads:}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,y'} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial y'}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Example: End points -- $y(0)=0; y(1)=1$

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

Example from your textbook --

Example 5.3.3: Minimal travel cost

Assume that the cost of flying an aircraft at height z is $e^{-\kappa z}$ per unit distance of flight-path, where κ is a positive constant. Consider that the aircraft flies in the (x, z)-plane from the point $(-a, 0)$ to the point $(a, 0)$ where $z = 0$ corresponds to ground level, and where the z-axis points vertically upwards. Find the extremal for the problem of minimizing the total cost of the journey.

The differential arc-length element of the flight path ds can be written as

$$ds = \sqrt{dx^2 + dz^2} = \sqrt{1 + z'^2}dx$$

where $z' \equiv \frac{dz}{dx}$. Thus the cost integral to be minimized is

$$C = \int_{-a}^{+a} e^{-\kappa z} ds = \int_{-a}^{+a} e^{-\kappa z} \sqrt{1 + z'^2} dx$$

The function of this integral is

$$f = e^{-\kappa z} \sqrt{1 + z'^2}$$

Want to find the function $z(x)$

where $z(-a) = 0 = z(a)$

$$f(z,z';x) = \mathrm{e}^{-\kappa z}\sqrt{1+z'^2}$$

$$\left(\frac{\partial f}{\partial z}\right)_{x,z'} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial z'}\right)_{x,z}\right] = 0 \quad \text{for all } -a \le x \le a$$

$$\left(\frac{\partial f}{\partial z}\right)_{x,z'} = -\kappa \mathrm{e}^{-\kappa z}\sqrt{1+z'^2}$$

$$\left(\frac{\partial f}{\partial z'}\right)_{x,z} = \mathrm{e}^{-\kappa z}\frac{z'}{\sqrt{1+z'^2}}$$

Therefore Euler's equation equals

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} = -\kappa e^{-\kappa z}\sqrt{1+z'^2} - \frac{z'' e^{-\kappa z}}{\sqrt{1+z'^2}} + \frac{\kappa z'^2 e^{-\kappa z}}{\sqrt{1+z'^2}} + \frac{z'' z'^2 e^{-\kappa z}}{\left(1+z'^2\right)^{3/2}} = 0$$

This can be simplified by multiplying the radical to give

$$-\kappa - 2\kappa z'^2 - \kappa z'^4 - z'' - z'' z'^2 + \kappa z'^2 + \kappa z'^4 + z'' z'^2 = 0$$

Cancelling terms gives

$$z'' + \kappa\left(1+z'^2\right) = 0$$

Need to solve this differential equation for *z(x)* --

$$z'' + \kappa\left(1 + z'^2\right) = 0$$

$$\frac{z''}{\left(1+z'^2\right)} = -\kappa \qquad \Rightarrow \frac{\frac{dz'}{dx}}{\left(1+z'^2\right)} = -\kappa \qquad \Rightarrow \frac{dz'}{\left(1+z'^2\right)} = -\kappa dx$$

$$\int^x \frac{dz'}{\left(1+z'^2\right)} = -\int^x \kappa dx$$

$$\operatorname{atan}\left(z'\right) = -\kappa x + c_1 \qquad \Rightarrow z' = \tan\left(-\kappa x + c_1\right)$$

$$z(x) = \int^x \tan\left(-\kappa u + c_1\right) du + c_2 = \frac{\ln\left(1 + \tan^2\left(-\kappa x + c_1\right)\right)}{-2\kappa} + c_2$$

$$= \frac{1}{\kappa}\ln\left(\cos\left(-\kappa x + c_1\right)\right) + c_2$$

$$z(x) = \frac{\ln\left(1+\tan^2\left(-\kappa x + c_1\right)\right)}{-2\kappa} + c_2 = \frac{1}{\kappa}\ln\left(\cos\left(-\kappa x + c_1\right)\right) + c_2$$

Choosing constants c_1 and c_2 so that $z(-a) = z(a) = 0$

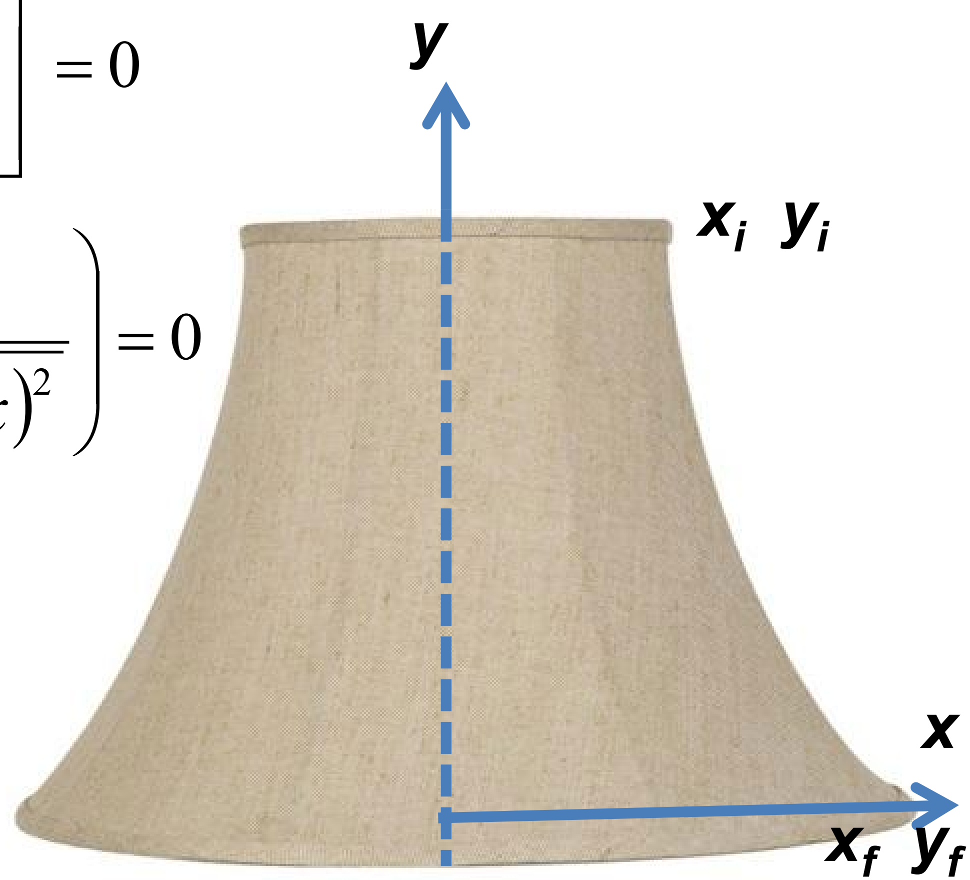
$$z(x) = \frac{1}{\kappa}\ln\left(\frac{\cos(\kappa x)}{\cos(\kappa a)}\right)$$

Example: Lamp shade shape *y(x)*

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$



$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}=K_1$$

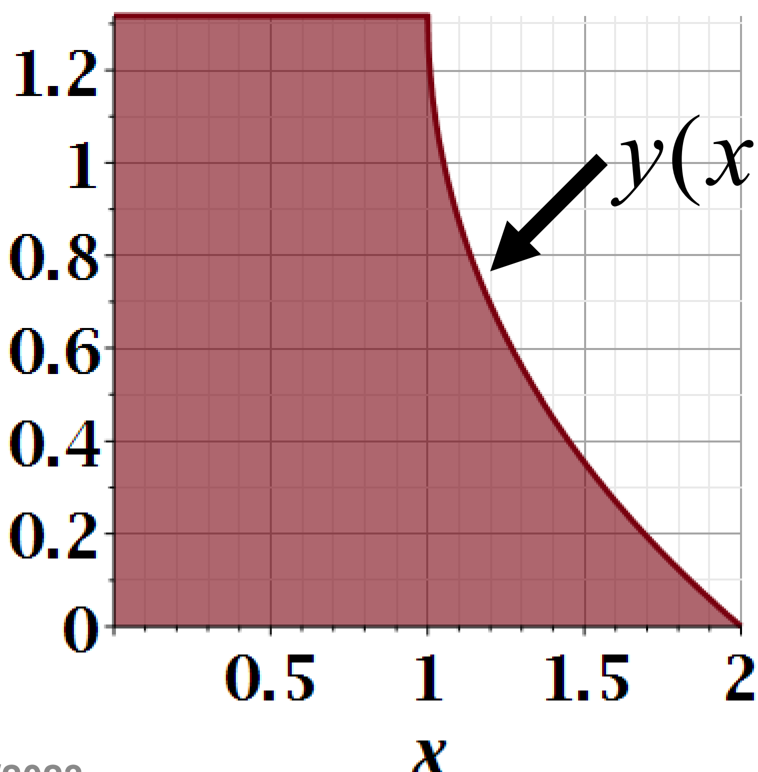
$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

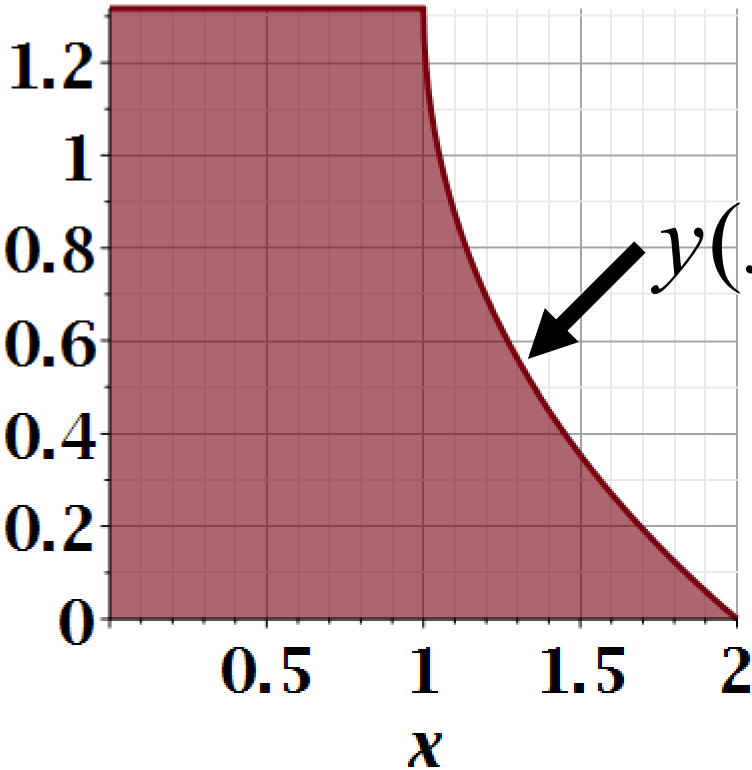
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose $K_1 = 1$ and $K_2 = 2 + \sqrt{3}$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Another example:

(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0)=0$ and $y(1)=1$

that minimize the integral :

$$I = \int_0^1 \left(\left(\frac{dy}{dx} \right)^2 - ay^2 \right) dx \quad \text{for constant } a > 0$$

Euler - Lagrange equation :

$$\frac{d^2y}{dx^2} + ay = 0$$

$$\Rightarrow y(x) = \frac{\sin\left(\sqrt{a}x\right)}{\sin\left(\sqrt{a}\right)}$$

Review : for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation