PHY 337/637 Analytical Mechanics
12:30-1:45 PM TR in Olin103

Lecture notes for Lecture 3
Chapter 5 of Cline

More about the calculus of variations

1. Review examples – Area of lamp shade
2. Brachistochrone problem
3. Calculus of variation with constraints

Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	Reading	Topic	PP	Assign
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 5	Calculus of variation		
5	Tu, 9/12/2023				

PHY 337/637

Assigned: 09/05/2023 Due: 09/12/2023

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Write a formal expression for $\frac{df}{dt}$ in terms of an arbitrary form of $f = f(q, \dot{q}, t)$ and an arbitrary function $q(t)$.

   (b) Now suppose that

   $$f(q, \dot{q}, t) = q\dot{q}^2 t, \quad \text{where} \quad q(t) = \mathrm{e}^{-t/\tau}.$$

   Here τ is a constant. Evaluate df/dt as a function of t using the expression you derived in part (a)..

   (c) Now find the expression for f as an explicit function of t ($f(t)$) and then take its time derivative directly to check your previous result.

Physics Colloquium Series

The originally scheduled colloquium for this week has been rescheduled for December 7, 2023

In order to keep up the departmental good spirits, please join **Physics Reception in the Olin Lobby at 3:30 PM**

WAKE FOREST UNIVERSITY

Comment on "class participation"

- The course webpage mentions this category as part of your grade
- The purpose is to help you get into the practice of formulating and articulating questions about the physics and/or the related mathematics

How to earn "class participation" points

- After reading/skimming lecturenotes posted on the class website, email me questions about the material or suggested discussion points before class.
- Attend one or both course tutorial sessions by graduate student Arezoo Nameny (namea22@wfu.edu) **Tuesdays & Wednesdays at 5:30-7 PM in Olin 107**
- Ask questions as they arise in office hours, office appointments, via email, etc.

More comments about tutorial sessions

- Not required, but hopefully very helpful to you
- Arezoo may be able to explain ideas in a more relatable way
- Focus on “exercise” problems at the end chapters we cover (some of them can give you insight)
- May ask questions about posted HW assignments as well. Arezoo will most likely not work your HW for you, but may point you in a good direction.
- Class participation credit points offered for
  - Attendance
  - Prepared presentations – the thought being that you can learn a lot by explaining to others

Another interesting problem from your textbook (not assigned for points, but might want to discuss/solve in the tutorial sessions…………..

5.E: Calculus of Variations (Exercises)

◀ 5.11: Variational Approach to... | 5.S: Calculus of Variations (S... ▶

Douglas Cline
University of Rochester

1. Find the extremal of the functional

$$J(x) = \int_{1}^{2} \frac{\dot{x}^2}{t^3} dt$$

that satisfies $x(1) = 3$ and $x(2) = 18$. Show that this extremal provides the global minimum of J.

Your questions –

From Alex:

Would you mind explaining what we need to do for the tutorials with Arezoo at the start of class? I am still a bit confused.

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\int_{x_i}^{x_f} f\left(y(x),\frac{dy}{dx};x\right)dx.$$

Find the function $y(x)$ which extremizes $L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$.

$\delta L = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Example: Find minimum curve between points -- $y(0)=0; y(1)=1$

$$L=\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

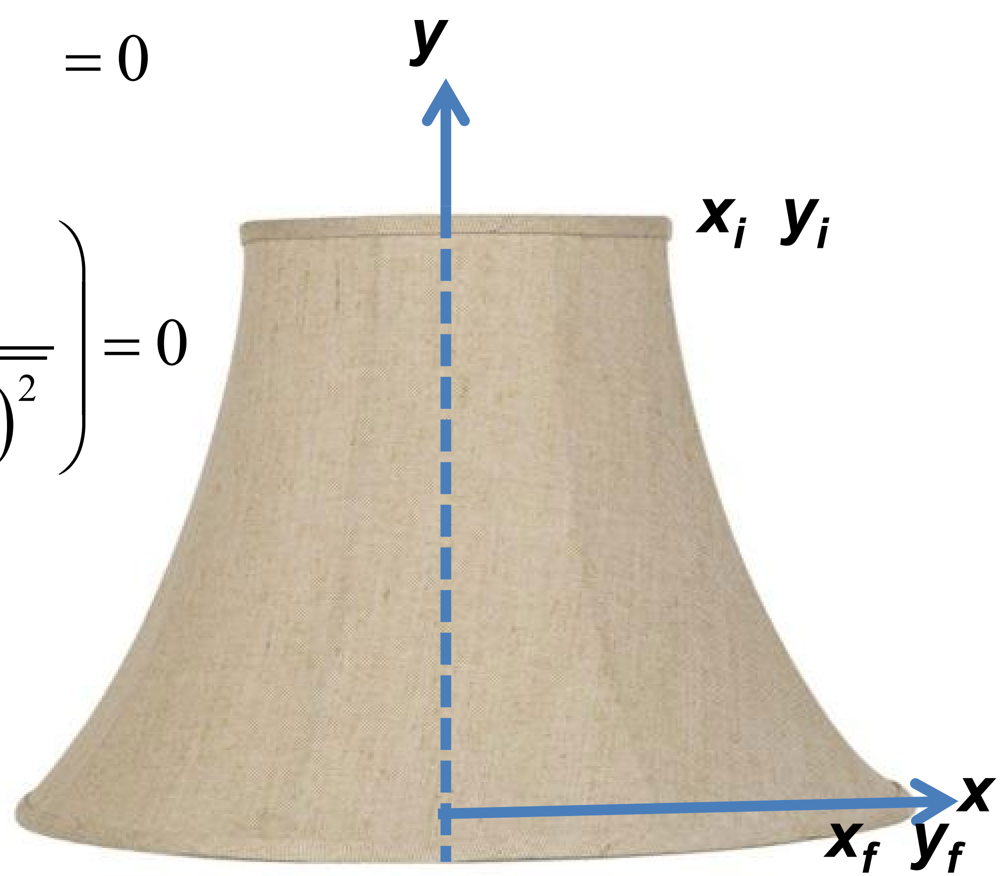
$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

Another example: Lamp shade shape $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$

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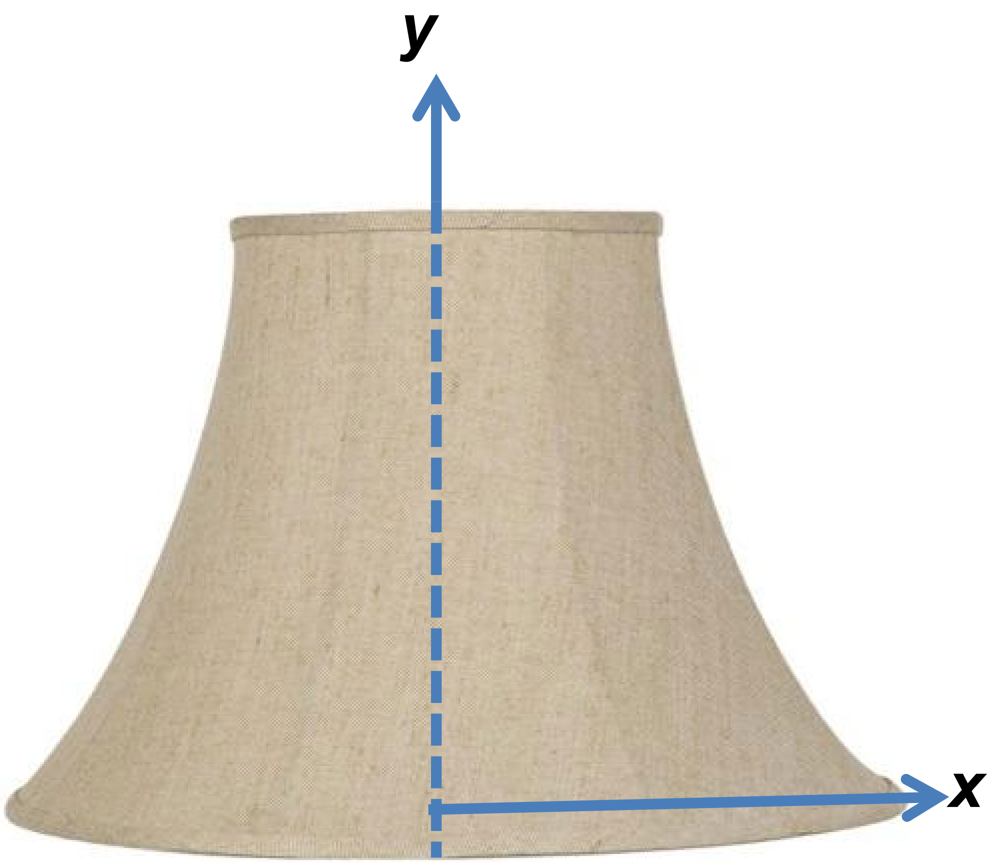
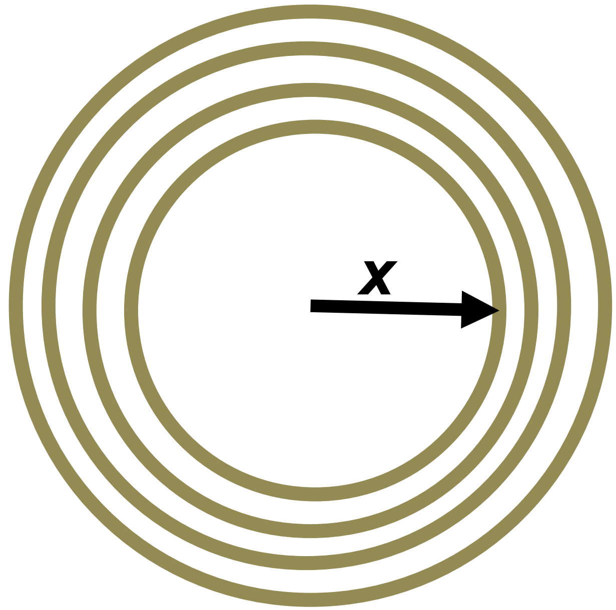


Top view



$$A = 2\pi \int_{x_i y_i}^{x_f y_f} x\sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi \int_{x_i}^{x_f} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Lamp shade area

$2\pi x dL$ where $dL = \sqrt{(dx)^2 + (dy)^2}$

$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}=K_1$$

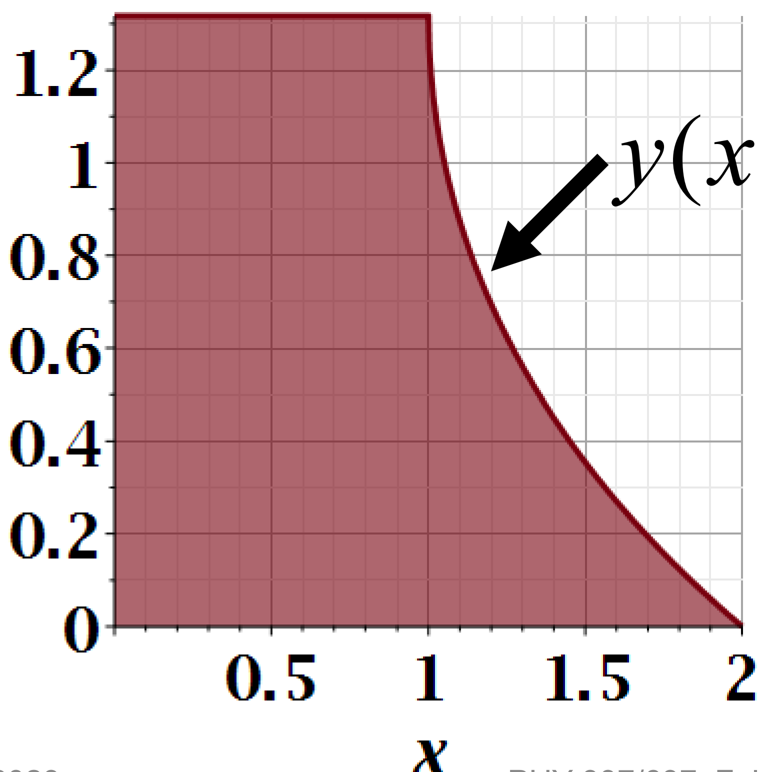
$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

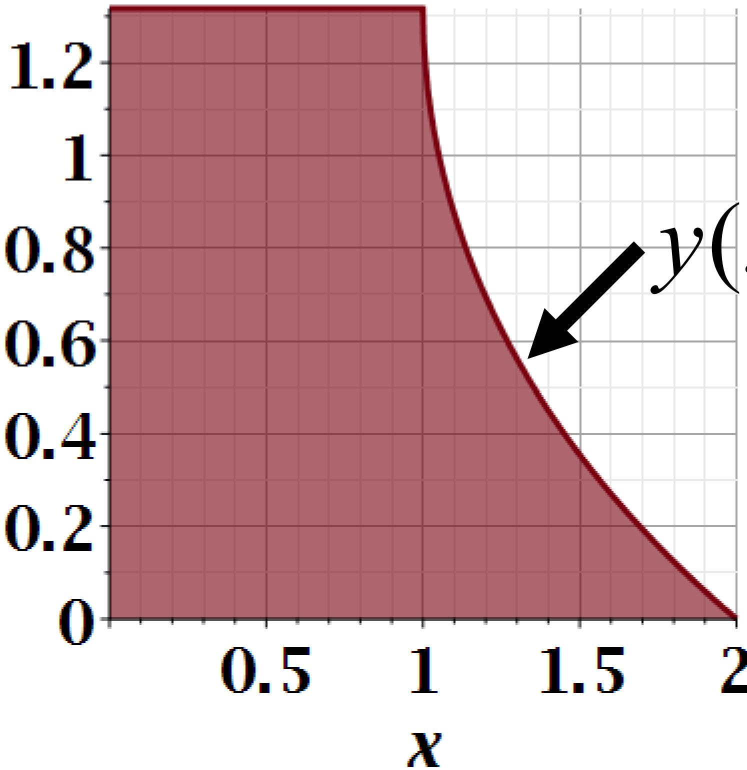
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose $K_1 = 1$ and $K_2 = \ln\left(2+\sqrt{3}\right)$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Review: for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation

A few more steps --

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$ and $\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$

$$\frac{df}{dx}=\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

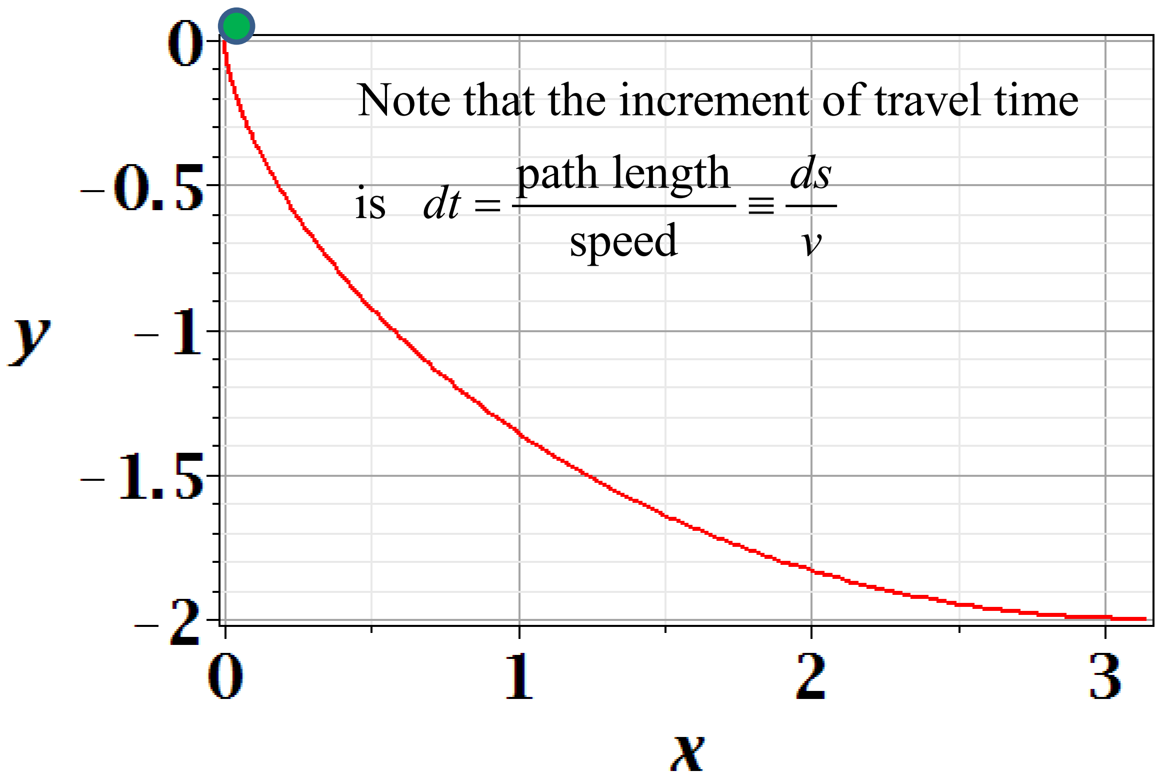
$$=\left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

$$=\left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)\right)+\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



Note that the increment of travel time is $dt = \frac{\text{path length}}{\text{speed}} \equiv \frac{ds}{v}$

A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

$$E = \frac{1}{2}mv^2 + mgy \qquad \text{with } y(t=0)=0 \text{ and } \dot{y}(t=0)=0$$

With this choice of initial conditions, $E = 0$

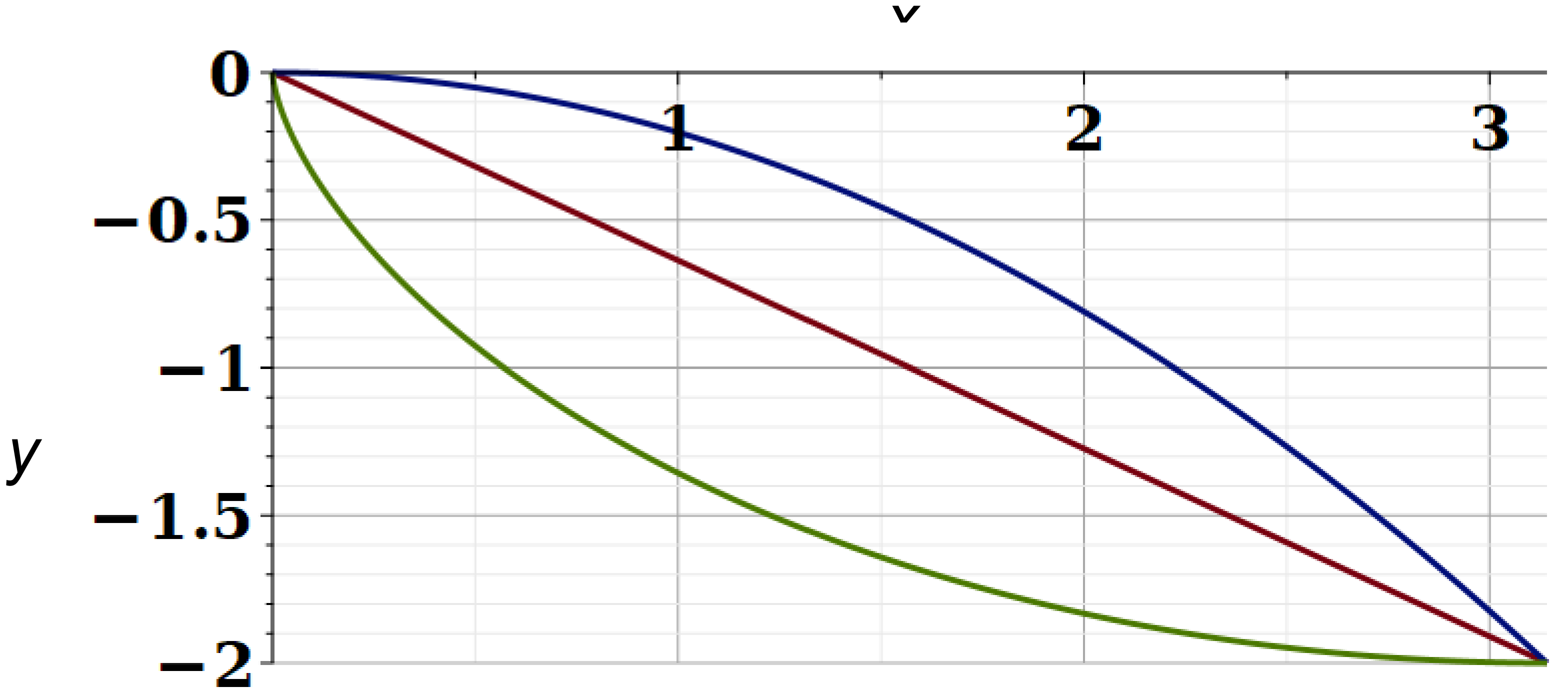
Note that the increment of travel time

is $dt = \frac{\text{path length}}{\text{speed}} \equiv \frac{ds}{v}$

Alternatively --

$$v = \frac{ds}{dt} \qquad \Rightarrow dt = \frac{ds}{v}$$

Vote for your favorite path



Which gives the shortest time?

a. Green
b. Red
c. Blue

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that your textbook applied a different trick to simplify the equations which also works, effectively switching the role of $x \leftarrow\rightarrow y$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

Question – why this choice?
Answer – because the answer will be more beautiful. (Be sure that was not my cleverness.)

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

Let $\quad y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

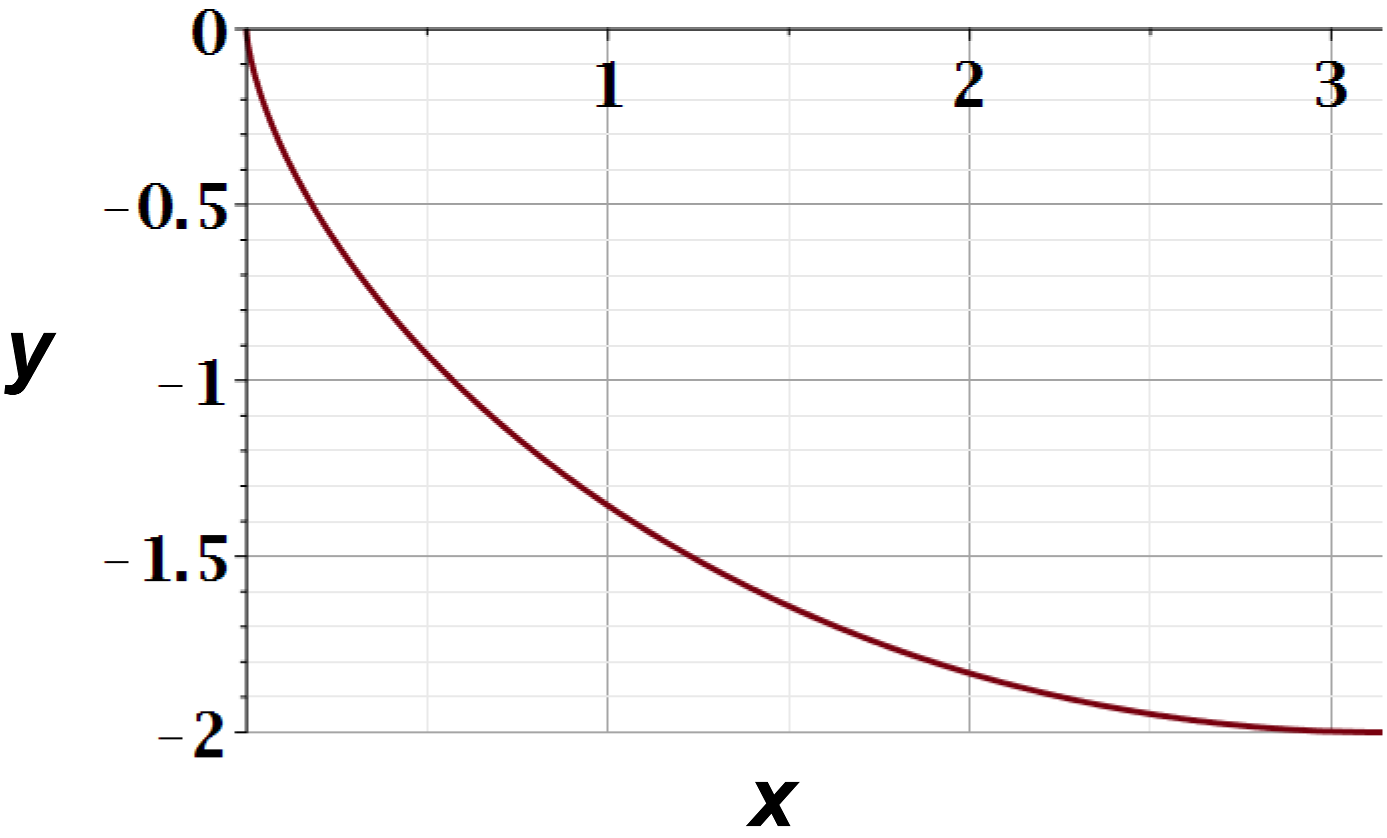
$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$

Parametric plot --

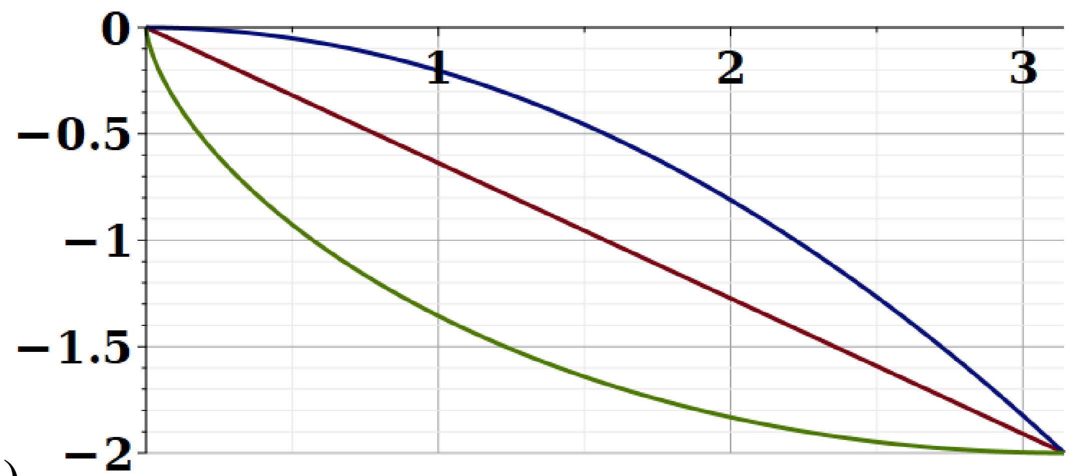
plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



Checking the results

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\, dx$$

units of $\frac{1}{\sqrt{(2g)}}$; $(0,0) \rightarrow (\pi,-2)$



T=4.4429 $\quad x = \theta - \sin\theta \quad y = \cos\theta - 1$

T=5.2668 $\quad y(x) = -2x/\pi$

T=infinite $\quad y(x) = -2x^2/\pi^2$

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Euler-Lagrange equation:

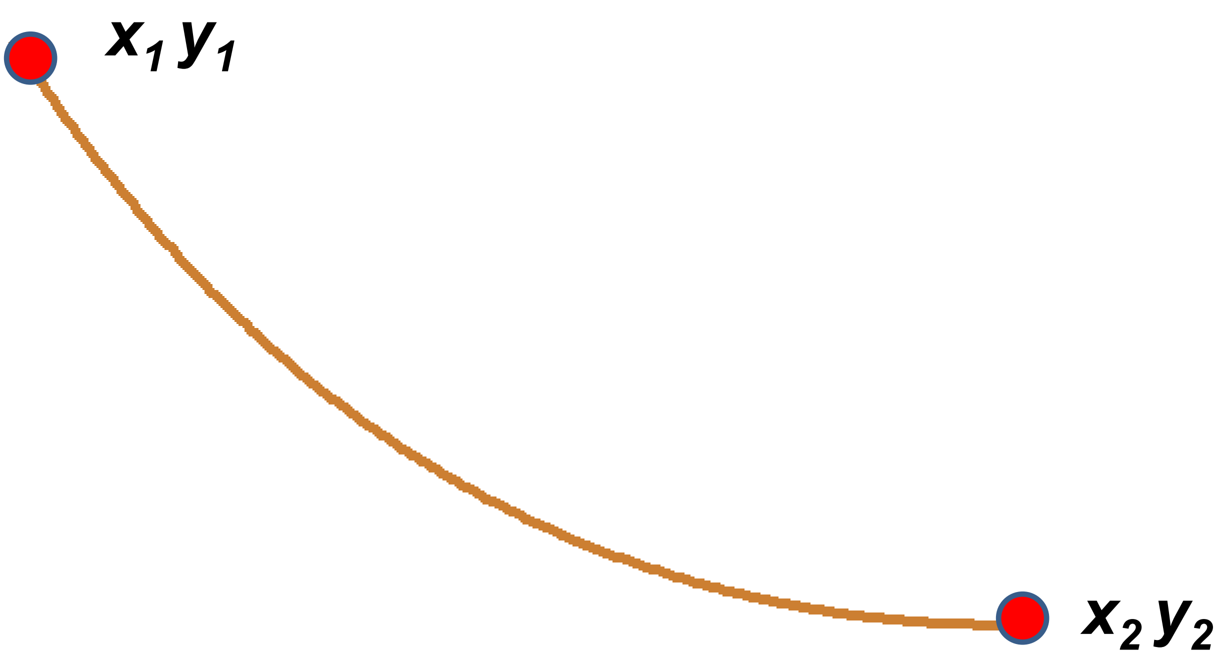
$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density ρ hanging between two points



Example from internet --

Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

$$\delta W = 0 = \delta E + \lambda \delta L \text{ for fixed } \lambda$$

is a very clever mathematical trick to help solve the minimization and constraint at the same time.

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right) = K$$

$$\left(\rho g y+\lambda\right)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$\left(\rho g y+\lambda\right)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$y(x)=-\frac{1}{\rho g}\left(\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants: K, a, λ

Constraints: $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = L$$

Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx \qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\right] = 0$$

or $$\frac{d}{dx}\left(f - \frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Chapter 5 of your textbook covers a number of extensions of these ideas, to multiple degrees of freedom and to application of constraints in additional ways. We will instead now jump to the application of the calculus of variation to its use in the analysis of the motion of particles covered in Chapter 6, where the multiple degrees of freedom and other extensions will then be treated.

Application to particle dynamics

$$x \rightarrow t \qquad \text{(time)}$$

$$y \rightarrow q \qquad \text{(generalized coordinate)}$$

$$f \rightarrow L \qquad \text{(Lagrangian)}$$

$$I \rightarrow A \text{ or } S \qquad \text{(action)}$$

Denote: $\dot{q} \equiv \frac{dq}{dt}$

$$S = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt$$

http://www-history.mcs.st-and.ac.uk/Biographies/Hamilton.html

Sir William Rowan Hamilton

Wednesday, September 11th, 2013

Sitemap

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- Mathematical studies
- Optics and Dynamics
- Quaternions
- Quotations
- Hamilton Key Dates
- Hamilton Links
- Graphics
- Math News

Tribute to Sir William Hamilton (1805–1865)

Hello and welcome! This page is dedicated to the life and work of Sir William Rowan Hamilton.

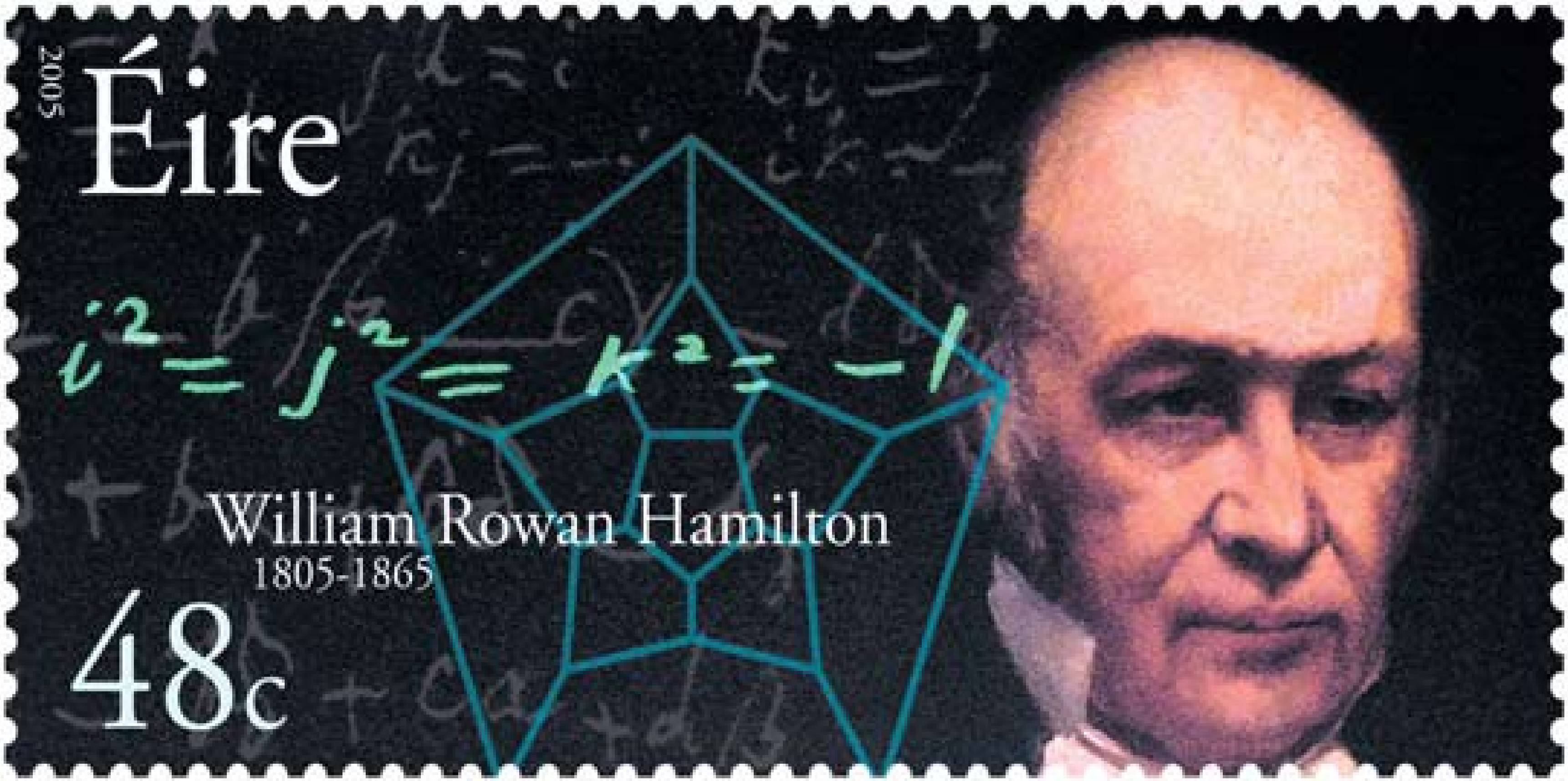
William Rowan Hamilton was Ireland's greatest scientist. He was an mathematician, physicist, and astronomer and made important works in optics, dynamics, and algebra.

His contribution in dynamics plays a important role in the later developed quantum mechanics. His name was perpetuated in one of the fundamental concepts in quantum mechanics, called "Hamiltonian".

The Discovery of Quaternions is probably is his most familiar invention today.

2005 was the Hamilton Year, celebrating his 200th birthday. The year was dedicated to celebrate Irish Science. 2005 was called the Einstein year also, reminding of three great papers of the year 1905. So UNESCO designated 2005 to the World Year of Physics

Thanks for visiting this site! Please enjoy your stay while browsing through the pages.



https://irishpostalheritagegpo.wordpress.com/2017/06/08/william-rowan-hamilton-irish-mathematician-and-scientist/

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$S = \int_{t_1}^{t_2} L\left(\{q,\dot{q}\};t\right)dt \equiv \int_{t_1}^{t_2} L\left(\left\{y,\frac{dy}{dt}\right\};t\right)dt$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g*.

Newton's formulation: $m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $y(t) = y_i + v_i t - \frac{1}{2}gt^2$

Lagrangian for this case:

$$L = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Note that we have not yet justified this!

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)\equiv T-U$$

Kinetic energy (T)

Potential energy (U)

In our example:

$$L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)\equiv T-U=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2-mgy$$

Hamilton's principle states:

$$S\equiv\int_{t_i}^{t_f}L\left(\left\{y(t),\frac{dy}{dt}\right\},t\right)dt \quad \text{is minimized for physical } y(t):$$

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Perhaps looks familiar?

Check:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0, \;\; y_i = h \equiv \frac{1}{2} g T^2; \quad t_f = T, \;\; y_f = 0$

Trial trajectories: $y_1(t) = \frac{1}{2} g T^2 \left(1 - t/T\right) = h - \frac{1}{2} gTt$

$$y_2(t) = \frac{1}{2} g T^2 \left(1 - t^2/T^2\right) = h - \frac{1}{2} g t^2$$

$$y_3(t) = \frac{1}{2} g T^2 \left(1 - t^3/T^3\right) = h - \frac{1}{2} g t^3/T$$

Maple says:

$$S_1 = -0.125 m g^2 T^3$$

$$S_2 = -0.167 m g^2 T^3$$

$$S_3 = -0.150 m g^2 T^3$$