PHY 337/637 Analytical Mechanics
12:30-1:45 PM MWF in Olin 103

Discussion of Lecture 4 – Chap. 6 in Cline

Calculus of variation applied to classical mechanics

1. Hamilton's principle and introduction to the Lagrangian
2. Extension to multiple and generalized coordinates
3. D'Alembert's principle
4. Velocity dependent forces

Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	**Reading**	**Topic**	**PP**	**Assign**
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 6	Lagrangian mechanics	6E	#4
5	Tu, 9/12/2023				
6	Th, 9/14/2023				

PHY 711 -- Assignment #4

Assigned: 9/07/2023   Due: 9/12/2023

Continue reading Chapter 6, in **Cline**.

Consider a point particle of mass m moving (only) along the x axis according to a force $F_x = -K\,x$, where K is a positive constant. The particle trajectory as a function of time, $x(t)$, has the initial value $x(t=0)=C$, where C denotes a given length, and its initial velocity is 0.

a. Write down and solve Newton's second law for this system, finding the form of the trajectory $x(t)$.
b. Now write down the Lagrangian for this system and solve the Euler-Lagrange equations for the trajectory $x(t)$. How does your answer compare with (a)?

Physics Colloquium Series

The originally scheduled colloquium for this week
has been rescheduled for December 7, 2023

In order to keep up the departmental good spirits, please join
Physics Reception in the Olin Lobby at 3:30 PM

WAKE FOREST
UNIVERSITY

Summary of equations from calculus of variation --

For this class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx \qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

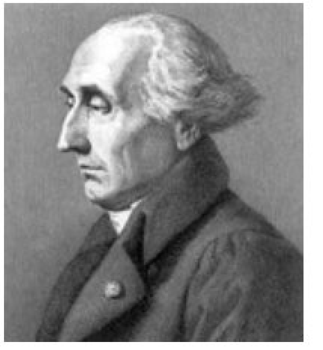
$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\right] = 0$$

or equivalently: $$\frac{d}{dx}\left(f - \frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

➔ differential equation for *y(x)*

Application to particle dynamics

Corresponding chapter in Cline textbook:



6: Lagrangian Dynamics

$x \rightarrow t$ (time)

$y \rightarrow q$ (generalized coordinate)

$f \rightarrow L$ (Lagrangian)

$I \rightarrow A$ or S (action)

Denote: $\dot{q} \equiv \dfrac{dq}{dt}$

$$S = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt \qquad \text{where } L = \text{Kinetic energy-Potential energy}$$

Thanks to Hamilton's principle

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$S = \int_{t_1}^{t_2} L\left(\{q,\dot{q}\};t\right)dt \equiv \int_{t_1}^{t_2} L\left(\left\{y,\frac{dy}{dt}\right\};t\right)dt$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g.*

Newton's formulation: $m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $y(t) = y_i + v_i t - \frac{1}{2}gt^2$

Lagrangian for this case:

$$L = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) dt$$ is minimized for physical $y(t)$:

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Perhaps looks familiar?

Note that, showing that our construction is consistent with Newton's laws **is not a proof**. You will get the chance to consider another example to check if that works (or not) as well.

Digression on multiple coordinates due to multiple dimensions and/or multiple particles.

1 particle+1 degree of freedom

$$S = \int_{t_1}^{t_2} L\left(q,\dot{q},t\right)dt \qquad \Rightarrow \qquad S = \int_{t_1}^{t_2} L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$$

1 particle+3 Carthesian dimensions

$$\text{for example:} \quad L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv L(x,y,z,\dot{x},\dot{y},\dot{z},t)$$

$$\text{or} \qquad L(\{x_i\},\{y_i\},\{z_i\},\{\dot{x}_i\},\{\dot{y}_i\},\{\dot{z}_i\},t)$$

N particles+3 Carthesian dimensions

Note that the notion of "generalized coordinates" could be a single coordinate for a single particle in one dimension, d coordinates for a single particle in d dimensions, or dN coordinates for N particles in d dimensions. Cartesian coordinates are also "generalized coordinates".

Digression -- notion of generalized or curvilinear coordinates

Referenced to cartesian coordinates: $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$

Cylindrical coordinates

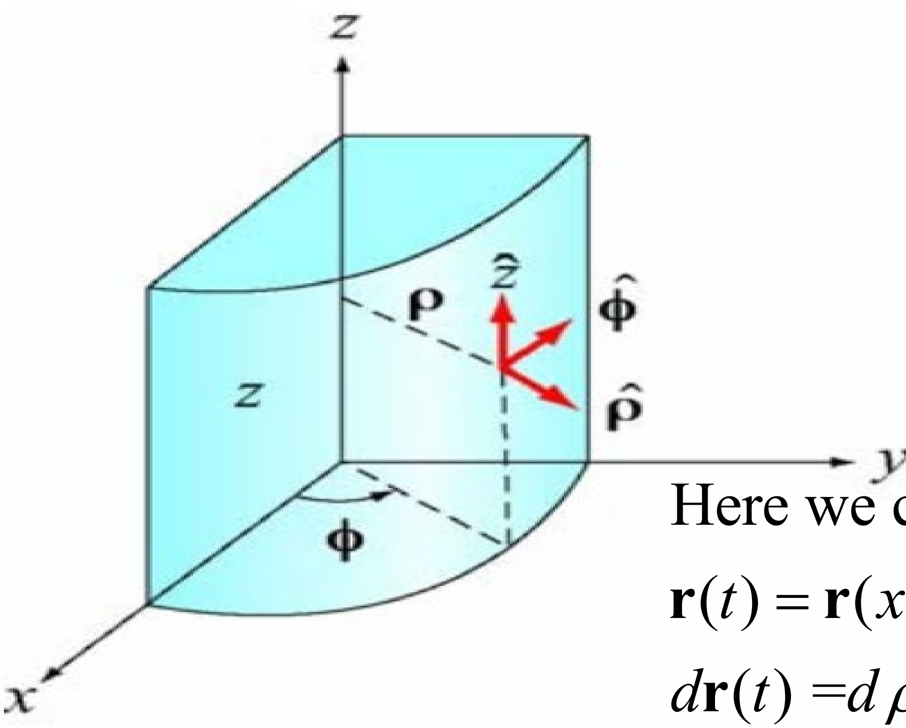


Figure B.2.4 Cylindrical coordinates

$$x = \rho\cos\phi \equiv x(\rho,\phi)$$
$$y = \rho\sin\phi \equiv y(\rho,\phi)$$
$$z = z$$
$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \arctan(y / x)$$
$$z = z$$

Here we can write

$$\mathbf{r}(t) = \mathbf{r}(x(t), y(t), z(t)) = \mathbf{r}(\rho(t), \phi(t), z(t))$$
$$d\mathbf{r}(t) = d\rho(t)\hat{\boldsymbol{\rho}}(t) + \rho(t)d\phi(t)\hat{\boldsymbol{\varphi}}(t) + z(t)\hat{\mathbf{z}}$$

(Figure taken from 8.02 handout from MIT.)

Spherical coordinates

$$x = r\sin\theta\cos\phi \equiv x(r,\theta,\phi)$$

$$y = r\sin\theta\sin\phi \equiv y(r,\theta,\phi)$$

$$z = r\cos\theta \equiv z(r,\theta,\phi)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\phi = \arctan(y / x)$$

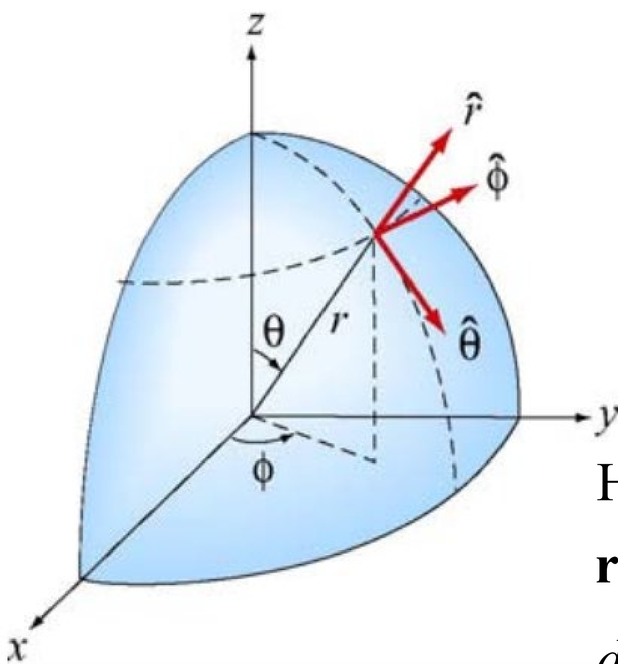


Figure B.3.1 Spherical coordinates

Here we can write

$$\mathbf{r}(t) = \mathbf{r}(x(t), y(t), z(t)) = \mathbf{r}(r(t), \theta(t), \phi(t))$$

$$d\mathbf{r}(t) = dr(t)\hat{\mathbf{r}}(t) + r(t)d\theta(t)\hat{\boldsymbol{\theta}}(t) + r(t)\sin\theta(t)d\phi(t)\hat{\boldsymbol{\varphi}}(t)$$

(Figure taken from 8.02 handout from MIT.)

Jean d'Alembert 1717-1783
French mathematician and philosopher

"Deriving" Lagrangian mechanics from Newton's laws.

The Lagrangian function is:

$$L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right) \equiv T - U \qquad q_i(t) \text{ are generalized coordinates}$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right) dt \qquad \text{is minimized for physical } q_i(t):$$

Generalized coordinates:

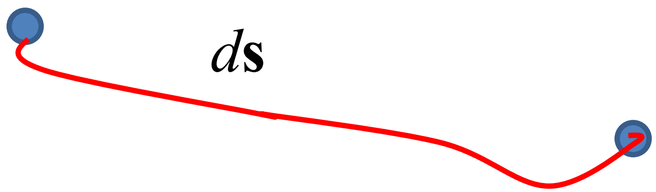
$$q_\sigma(\{x_i\}) \leftrightarrow x_i(\{q_\sigma\})$$

Note that

$q_\sigma(t)$ can be $x(t), \theta(t)$....

$$dx \equiv dx_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

D'Alembert's principle:



Note that: $d\mathbf{s} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

Newton's laws:

$$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a}) \cdot d\mathbf{s} = 0$$

This is D'Alembert's principle.

$$\mathbf{F} \cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

For a conservative force: $F_i = -\frac{\partial U}{\partial x_i}$

$$\mathbf{F} \cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

You might ask why we need "generalized" coordinates. In fact, Cartesian coordinates are often just fine, but using the more flexible possibilities reveals important aspects of the formalism. Cartesian coordinates are a special case of generalized coordinates.

Comment on notation -- $d\mathbf{s} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

For convenience let $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1, \quad \hat{\mathbf{y}} = \hat{\mathbf{x}}_2, \quad \hat{\mathbf{z}} = \hat{\mathbf{x}}_3$

Then $\mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{3} F_i dx_i$

But now we want to change coordinates $q_\sigma\left(\{x_i\}\right) \leftrightarrow x_i(\{q_\sigma\})$

$$dx_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma \qquad \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{3} F_i dx_i = \sum_\sigma \sum_{i=1}^{3} F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

Summary up to now --

$$\mathbf{F}\cdot d\mathbf{s}=\sum_{\sigma}\sum_{i}F_i\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

For a conservative force: $F_i=-\dfrac{\partial U}{\partial x_i}$

$$\mathbf{F}\cdot d\mathbf{s}=-\sum_{\sigma}\sum_{i}\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma=-\sum_{\sigma}\frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

here, we use the identity:

$$\sum_{i}\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}=\frac{\partial U}{\partial q_\sigma}$$

Generalized coordinates:

$$q_\sigma\left(\{x_i\}\right)$$

$$x \Leftrightarrow x_1$$

$$y \Leftrightarrow x_2$$

$$z \Leftrightarrow x_3$$

Newton's laws:

$$\mathbf{F} - m\mathbf{a} = 0 \qquad \Rightarrow \left(\mathbf{F} - m\mathbf{a}\right)\cdot d\mathbf{s} = 0$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt}\left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma}\right) - m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}\right)\delta q_\sigma$$

Claim: $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma}\frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial \dot{q}_\sigma}\right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial q_\sigma}\right)\delta q_\sigma$$

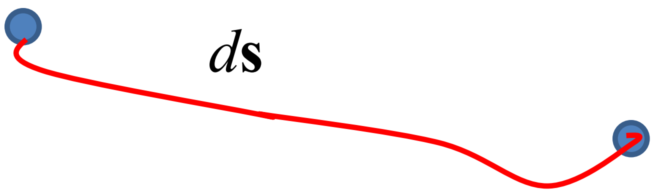
Some details

$$\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} = \frac{d\dot{x}_i}{dt}\frac{\partial x_i}{\partial q_\sigma} = \frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - \dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}$$

You may be still wondering why we need to introduce "generalized" coordinates when cartesian coordinates are an example. What the generalized coordinates allow us to show is that

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

where $T \equiv \sum_i \frac{1}{2} m \dot{x}_i^{\,2}$ (kinetic energy)



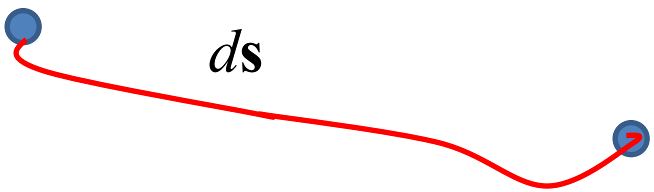
$$x_i = x_i\left(\{q_\sigma(t)\}, t\right)$$

Claim: $\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$

Details: $\dot{x}_i \equiv \dfrac{dx_i}{dt} = \sum_\sigma \dfrac{\partial x_i}{\partial q_\sigma}\dot{q}_\sigma + \dfrac{\partial x_i}{\partial t}$ Therefore: $\dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \dfrac{\partial x_i}{\partial q_\sigma}$

Claim: $\dfrac{d}{dt}\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial}{\partial q_\sigma}\dfrac{dx_i}{dt} \equiv \dfrac{\partial \dot{x}_i}{\partial q_\sigma}$

$$\sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_{\sigma'}\partial q_\sigma}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial t \partial q_\sigma} \qquad \sum_{\sigma'} \frac{\partial^2 x_i}{\partial q_\sigma \partial q_{\sigma'}}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial q_\sigma \partial t}$$



Generalized coordinates:

$$q_\sigma\left(\{x_i\}\right)$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2} m \dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial\left(\frac{1}{2} m \dot{x}_i^{\,2}\right)}{\partial q_\sigma} \right)\delta q_\sigma$$

Define -- kinetic energy : $\quad T \equiv \sum_i \frac{1}{2} m \dot{x}_i^{\,2}$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma$$

Recall:

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

$$\left(\mathbf{F} - m\mathbf{a}\right)\cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$m\mathbf{a}\cdot d\mathbf{s}=\sum_{\sigma}\sum_{i}\left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\ 2}\right)}{\partial\dot{q}_\sigma}\right)-\frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\ 2}\right)}{\partial q_\sigma}\right)\delta q_\sigma$$

Define -- kinetic energy: $T\equiv\sum_i \frac{1}{2}m\dot{x}_i^{\ 2}$

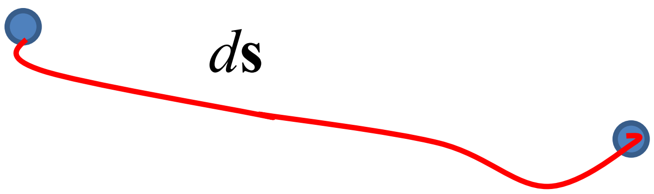
$$m\mathbf{a}\cdot d\mathbf{s}=\sum_{\sigma}\left(\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_\sigma}-\frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma$$

When do we need this term?

Single particle in 2 dimensions:

Cartesian coordinates: $T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)$

Polar coordinates: $T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$



Generalized coordinates :

$q_\sigma(\{x_i\})$

$$(\mathbf{F} - m\mathbf{a})\cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

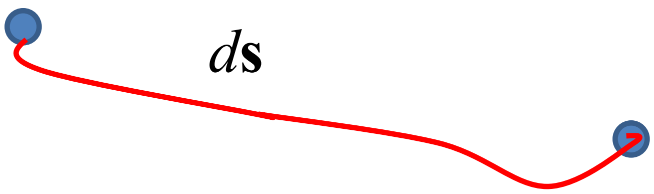
$$= \quad -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$$= \quad -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note : This is only true if

$$\frac{\partial U}{\partial \dot{q}_\sigma} = 0$$



Generalized coordinates :

$$q_\sigma(\{x_i\})$$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t)$$

$$(\mathbf{F} - m\mathbf{a})\cdot d\mathbf{s} = -\sum_\sigma\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$\Rightarrow$ Minimization integral: $S = \int_{t_i}^{t_f} L(\{q_\sigma\},\{\dot{q}_\sigma\},t)\,dt$

➔Hamilton’s principle from the “backwards” application of the Euler-Lagrange equations --

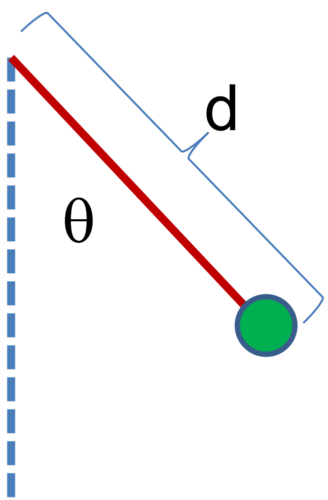
Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t)$$

Euler – Lagrange equations: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta, \dot{\theta}) = \tfrac{1}{2} m d^2 \dot{\theta}^2 - mg(d - d\cos\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \qquad \Rightarrow \frac{d}{dt} m d^2 \dot{\theta} + mgd\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{d}\sin\theta$$

Another example: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2}I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}\left(I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}\left(I_1\dot{\beta}\right) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}\left(I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\right) = 0$$

Example – simple harmonic oscillator

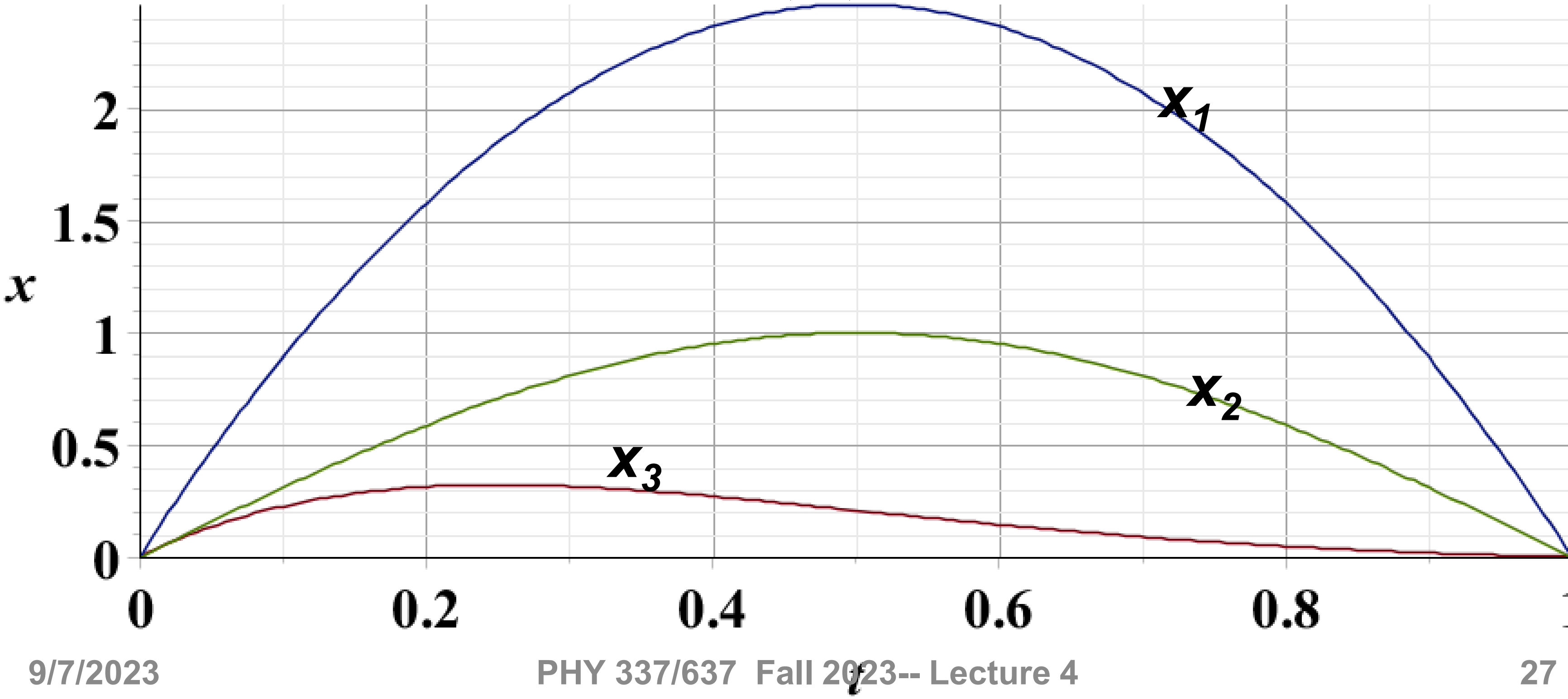
$T = \frac{1}{2} m\dot{x}^2 \qquad U = \frac{1}{2} m\omega^2 x^2$

Assume $x(0) = 0$ and $x(\frac{\pi}{\omega}) = 0 \qquad S = \frac{1}{2} m \int_0^{\pi/\omega} \left(\dot{x}^2 - \omega^2 x^2 \right) dt$

Trial functions $x_1(t) = A\sin(\omega t) \qquad S_1 = 0$

$x_2(t) = A\omega t \cdot (\pi - \omega t) \qquad S_2 = 0.067 A^2 m\omega^2$

$x_3(t) = Ae^{-\omega t} \sin(\omega t) \qquad S_3 = 0.062 A^2 m\omega^2$



Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

are those which minimize the action: $S = \int L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \qquad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

$\Rightarrow$ How can we represent velocity-dependent forces?

Why do we need velocity dependent forces?

a. Friction is sometimes represented as a velocity dependent force. (difficult to treat with Lagrangian mechanics.)
b. Lorentz force on a moving charged particle in the presence of a magnetic field.

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$ (in SI units)

$x-$component : $F_x = q\left(E_x + \left(\mathbf{v}\times\mathbf{B}\right)_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Note: Here we are using cartesian coordinates for convenience.

x-component: $\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0$

Apparently: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

Answer: $U = q\Phi\left(\mathbf{r},t\right) - q\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$

where $\mathbf{E}\left(\mathbf{r},t\right) = -\nabla\Phi\left(\mathbf{r},t\right) - \frac{\partial \mathbf{A}\left(\mathbf{r},t\right)}{\partial t}$ $\quad\mathbf{B}\left(\mathbf{r},t\right) = \nabla\times\mathbf{A}\left(\mathbf{r},t\right)$

More details --

Consider: $$0=-\sum_\sigma\left(\frac{d}{dt}\frac{\partial(T-U)}{\partial\dot{q}_\sigma}-\frac{\partial(T-U)}{\partial q_\sigma}\right)\delta q_\sigma$$

Suppose $T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$$\Rightarrow 0=\left(\frac{d}{dt}\frac{\partial(T-U)}{\partial\dot{x}}-\frac{\partial(T-U)}{\partial x}\right)=\frac{d}{dt}m\dot{x}-\frac{d}{dt}\left(\frac{\partial U}{\partial\dot{x}}\right)+\frac{\partial U}{\partial x}$$

$$\Rightarrow m\ddot{x}=F_x=\frac{d}{dt}\left(\frac{\partial U}{\partial\dot{x}}\right)-\frac{\partial U}{\partial x}$$

Units for electromagnetic fields and forces

cgs Gaussian units -- (used in older textbooks)

E and **B** fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

SI units --

E and **B** fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r},t) - q\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Lorentz forces, continued:

$$x\text{-component of Lorentz force:}\quad F_x = q\left(E_x + \left(\mathbf{v}\times\mathbf{B}\right)_x\right)$$

Suppose: $$U = q\Phi\left(\mathbf{r},t\right) - q\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$$

Consider: $$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi\left(\mathbf{r},t\right)}{\partial x} + q\left(\dot{x}\frac{\partial A_x\left(\mathbf{r},t\right)}{\partial x} + \dot{y}\frac{\partial A_y\left(\mathbf{r},t\right)}{\partial x} + \dot{z}\frac{\partial A_z\left(\mathbf{r},t\right)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -qA_x\left(\mathbf{r},t\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -q\frac{dA_x\left(\mathbf{r},t\right)}{dt} = -q\left(\frac{\partial A_x\left(\mathbf{r},t\right)}{\partial x}\dot{x} + \frac{\partial A_x\left(\mathbf{r},t\right)}{\partial y}\dot{y} + \frac{\partial A_x\left(\mathbf{r},t\right)}{\partial z}\dot{z} + \frac{\partial A_x\left(\mathbf{r},t\right)}{\partial t}\right)$$

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + q\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -q\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + q\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - q\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - q\frac{\partial A_x(\mathbf{r},t)}{\partial t} + q\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + q\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + q\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Some details on last step:

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + q\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - q\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - q\frac{\partial A_x(\mathbf{r},t)}{\partial t} + q\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

Note that: $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$ $\qquad$ $\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$

So that:

$$F_x(\mathbf{r},t) = qE_x(\mathbf{r},t) + q\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + q\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi(\mathbf{r},t) - q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \dfrac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$ $\qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Suppose $\mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2} B_0 y\right) - \frac{q}{2} B_0 \dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2} B_0 x\right) + \frac{q}{2} B_0 \dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt} m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2} B_0 y \right) - \frac{q}{2} B_0 \dot{y} = 0 \qquad \Rightarrow m\ddot{x} - qB_0 \dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2} B_0 x \right) + \frac{q}{2} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + qB_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +qB_0\dot{y}$$

$$m\ddot{y} = -qB_0\dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained from direct application of Newton's laws:

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times B_0\hat{\mathbf{z}}$$

Example Lorentz force -- continued

Evaluation of equations:

$$m\ddot{x} - qB_0\dot{y} = 0$$

$$m\ddot{y} + qB_0\dot{x} = 0$$

$$m\ddot{z} = 0$$

$$\dot{x}(t) = V_0 \sin\left(\frac{qB_0}{m}t + \phi\right)$$

$$\dot{y}(t) = V_0 \cos\left(\frac{qB_0}{m}t + \phi\right)$$

$$\dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \frac{m}{qB_0}V_0 \cos\left(\frac{qB_0}{m}t + \phi\right)$$

$$y(t) = y_0 + \frac{m}{qB_0}V_0 \sin\left(\frac{qB_0}{m}t + \phi\right)$$

$$z(t) = z_0 + V_{0z}t$$

Example Lorentz force -- continued

Consider formulation with different Gauge: $\mathbf{A}(\mathbf{r}) = -B_0 y\hat{\mathbf{x}}$

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - qB_0\dot{x}y$$

$$\frac{d}{dt}\left(m\dot{x} - qB_0 y\right) = 0 \qquad \Rightarrow m\ddot{x} - qB_0\dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y}\right) + qB_0\dot{x} = 0 \qquad \Rightarrow m\ddot{y} + qB_0\dot{x} = 0$$

$$\frac{d}{dt}m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Does it surprise you that the same equations of motion are obtained with a different Gauge?

How do these two different forms of **A** correspond to the same **B**?

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

Consider $\mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla f(\mathbf{r},t)$

Note that $\nabla \times \mathbf{A}(\mathbf{r},t) = \nabla \times \mathbf{A}'(\mathbf{r},t)$

In our case, $\mathbf{A}(\mathbf{r},t) = \frac{1}{2} B_0 \left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$

$$\mathbf{A}'(\mathbf{r},t) = -B_0 y\hat{\mathbf{x}}$$

What is $f(\mathbf{r},t)$?