PHY 337/637 Analytical Mechanics
12:30-1:45 PM MWF in Olin 103

Discussion of Lecture 5 – Chap. 7 & 8 in Cline

Lagrangian and Hamiltonian analysis

1. Review of Lagrangian mechanics
2. Legendre transformations and construction of the Hamiltonian
3. Canonical equations of motion

WFU Physics Department

Physics Career Coffee Chat

Join our casual coffee chat with WFU Physics alum and Duke Univeristy faculty member Dr. Salman Azhar

Thursday

1:30 - 2:30 PM

Olin 105

Questions? jurchescu@wfu.edu

Physics Colloquium

Thursday

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September 14th, 2023

Fantastic Frameworks From Physics

Dr. Salman Azhar is a co-founder or a co-investor in over 125 startups. He has scaled organizations by forming teams and solving enigmatic problems. Dr. Azhar has 35 years of experience in industry and academia, during which he has crafted and led talented teams in developing and launching innovative technical solutions with a wide range of applications. Dr. Azhar is a Faculty member and Executive in Residence at Duke University's Fuqua School of Business. He is a Charter Life Member of OPEN Global and a venture partner at SAP.io and the University of Minnesota. Dr. Azhar is an advisor to several companies, including Regiment Securities. His former business partners and clients include Toyota, Sony, SAP, and others. He is currently developing innovative technology initiatives and mentoring leaders. Dr. Azhar earned his MS and PhD in Computer Science from Duke as a James B. Duke Fellow and a BS in Math and Physics from Wake Forest University as a Carswell Scholar. In Thursday's talk, Dr. Salman plans to go over how he develops frameworks from physics to lead a more fulfilling life by making better decisions and solving real-world problems. He will share his thinking process and invite

Salman Azhar, PhD
WFU Alum, Faculty Member and Executive in Residence
Fuqua School of Business
Duke University

4 pm - Olin 101

Refreshments will be served in Olin Lobby beginning at 3:30pm.

Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	Reading	Topic	PP	Assign
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 6	Lagrangian mechanics	6E	#4
5	Tu, 9/12/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	#5
6	Th, 9/14/2023				
7	Tu, 9/19/2023				
8	Th, 9/21/2023				

PHY 337/637 – Assignment # 5

Assigned: 09/12/2023 Due: 09/19/2023

This exercise uses the Lagangian and Hamiltonian formalisms.

1. Suppose that the motion of a point particle of mass m can be described in cartesian coordinates by the Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + Cx\dot{z},$$

where C is a positive constant having the units of mass/time. At $t = 0$, the initial coordinates of the particle are $x(0) = y(0) = z(0) = 0$ and the initial velocities are $\dot{x}(0) = \dot{y}(0) = 0$ and $\dot{z}(0) = V_0$.

   (a) Write the Euler-Lagrange equations for this system and solve them to find the trajectories of the particle $x(t), y(t), z(t)$.

   (b) Evaluate the Hamiltonian for this system using the Legendre transformation and put it into Canonical form.

   (c) Evaluate and solve the Canonical equations of motion for this system and compare your answer with part (a).

7: Symmetries, Invariance and the Hamiltonian

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}$$

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = +\frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}$$

8: Hamiltonian Mechanics

Main focus of today's lecture

Review of Lagrangian mechanics --

Particle dynamics using the Lagrangian formalism, thanks to Hamilton's principle –

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$S = \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt$$

for a system with a single generalized coordinate *q(t)*

$$S = \int_{t_1}^{t_2} L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) dt$$

for a system with multiple dimensions and/or particles with generalized coordinates $q_\sigma(t)$

$$L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) = T\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) - U\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right)$$

Kinetic energy **Potential energy**

Digression: Is the Lagrangian function unique?

a. Yes
b. No

Consider

$$L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \text{ and } L_B\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)$$

where $L_B\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) = L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) + \dfrac{df\left(\{q_\sigma\},t\right)}{dt}$

$$S = \int_{t_1}^{t_2} L_B\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt = \int_{t_1}^{t_2} L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt + \int_{t_1}^{t_2} \frac{df\left(\{q_\sigma\},t\right)}{dt}dt$$

$$= \int_{t_1}^{t_2} L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt + f\left(\{q_\sigma\},t\right)\Big|_{t_1}^{t_2}$$

Uniqueness of the Lagrangian – continued --

$$S=\int_{t_1}^{t_2} L_B\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt=\int_{t_1}^{t_2} L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt+\int_{t_1}^{t_2}\frac{df\left(\{q_\sigma\},t\right)}{dt}dt$$

$$=\int_{t_1}^{t_2} L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt+f\left(\{q_\sigma\},t\right)\Big|_{t_1}^{t_2}$$

$$\delta S=\delta\int_{t_1}^{t_2} L_B\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt=\delta\int_{t_1}^{t_2} L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt+\delta f\left(\{q_\sigma\},t\right)\Big|_{t_1}^{t_2}$$

=0

➔It is possible to find two different Lagrangians for the same system that both satisfy Hamilton's according to:

$$L_B\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)=L_A\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)+\frac{df\left(\{q_\sigma\},t\right)}{dt}$$

Idea of Legendre transformations

Adrien-Marie Legendre

1752-1833   Paris, France

Mathematical transformations for continuous functions of several variables & Legendre transforms --

$$z(x,y) \quad \Rightarrow \quad dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$\text{Let} \quad u \equiv \left(\frac{\partial z}{\partial x}\right)_y \quad \text{and} \quad v \equiv \left(\frac{\partial z}{\partial y}\right)_x \quad \Rightarrow \quad dz = u\,dz + v\,dy$$

Define new function

$$w(u,y) \quad \Rightarrow \quad dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

$$\text{For} \quad w = z - ux, \qquad dw = dz - u\,dx - x\,du = u\,dx + v\,dy - u\,dx - x\,du$$

$$dw = -x\,du + v\,dy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \qquad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

Now that we see that these transformations are possible, we should ask the question why we might want to do this?

An example comes from thermodynamics where we have various interdependent variables such as temperature T, pressure P, volume V, etc. etc. Often a measurable property can be specified as a function of two of those, while the other variables are also dependent on those two. For example we might specify T and P while the volume will be V(T,P). Or we might specify T and V while the pressure will be P(T,V).

Other examples from thermo --
For thermodynamic functions:

$$\text{Internal energy:} \quad U = U(S,V)$$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$$

$$\text{Enthalpy:} \quad H = H(S,P) = U + PV$$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \qquad V = \left(\frac{\partial H}{\partial P}\right)_S$$

Name	Potential	Differential Form
Internal energy	$E(S,V,N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E,V,N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S,P,N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T,V,N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T,P,N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T,V,\mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Application of the Legendre transformation for the Lagrangian and Hamiltonian

$L(q,\dot{q},t)$ and $H(q,p,t)$

suppose $H(q,p,t) = \dot{q}p - L(q,\dot{q},t)$

$$dH = \dot{q}dp + pd\dot{q} - \left(\frac{\partial L}{\partial q}\right)dq - \left(\frac{\partial L}{\partial \dot{q}}\right)d\dot{q} - \left(\frac{\partial L}{\partial t}\right)dt = \left(\frac{\partial H}{\partial q}\right)dq + \left(\frac{\partial H}{\partial p}\right)dp + \left(\frac{\partial H}{\partial t}\right)dt$$

Note that these two terms cancel if $p = \frac{\partial L}{\partial \dot{q}}$

$$\Rightarrow dH = \dot{q}dp - \left(\frac{\partial L}{\partial q}\right)dq - \left(\frac{\partial L}{\partial t}\right)dt = \left(\frac{\partial H}{\partial q}\right)dq + \left(\frac{\partial H}{\partial p}\right)dp + \left(\frac{\partial H}{\partial t}\right)dt$$

Generalization to multiple dimensions q_σ and p_σ is straightforward …

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

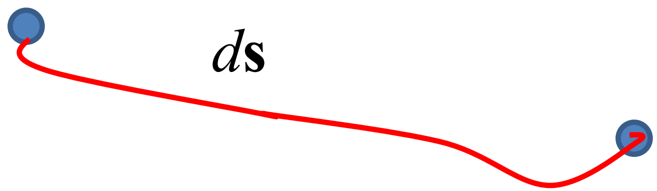
$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$dH = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Direct application of Hamiltonian's principle using the Hamiltonian function --



Generalized coordinates :

$q_\sigma(\{x_i\})$

Define -- Lagrangian: $L \equiv T - U$

$$L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)$$

$\Rightarrow$ Minimization integral: $S = \int_{t_i}^{t_f} L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Expressed in terms of Hamiltonian:

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

Hamilton's principle continued:

Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_\sigma \dot{q}_\sigma p_\sigma - H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + \delta \dot{q}_\sigma p_\sigma - \frac{\partial H}{\partial q_\sigma} \delta q_\sigma - \frac{\partial H}{\partial p_\sigma} \delta p_\sigma \right) \right) dt = 0$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Canonical equations

Detail:

0

$$\int_{t_i}^{t_f} \left(\sum_\sigma \left(\delta \dot{q}_\sigma p_\sigma \right) \right) dt = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d(\delta q_\sigma p_\sigma)}{dt} - \delta q_\sigma \dot{p}_\sigma \right) \right) dt = \sum_\sigma \delta q_\sigma p_\sigma \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \left(\sum_\sigma \left(\delta q_\sigma \dot{p}_\sigma \right) \right) dt$$

More comments about "details"

Detail :

$$\int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta\dot{q}_\sigma p_\sigma\right)\right)dt = \int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d\left(\delta q_\sigma p_\sigma\right)}{dt} - \delta q_\sigma \dot{p}_\sigma\right)\right)dt = \sum_\sigma \delta q_\sigma p_\sigma\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta q_\sigma \dot{p}_\sigma\right)\right)dt$$

Vanishes because $\delta q(t_f)=\delta q(t_i)$ due to the premise of Hamilton's principle.

In the Hamiltonian formulation --

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Why are these equations known as the "canonical equations"?

a. Because they are beautiful.
b. The term is meant to elevate their importance to the level of the music of J. S. Bach
c. To help you remember them
d. No good reason; it is just a name

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

2. Compute generalized momenta: $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$

3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$

4. Form Hamiltonian function: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

5. Analyze canonical equations of motion:

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

What happens when you miss a step in the recipe?

a. No big deal
b. Big deal – can lead to shame and humiliation (or at least wrong analysis)

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \Rightarrow \quad \text{Second order differential equations for } q_\sigma(t)$$

Hamiltonian picture

For independent generalized coordinates $q_\sigma(t)$ and momenta $p_\sigma(t)$:

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \quad \text{Two first order differential equations}$$

Constants of the motion in Hamiltonian formalism

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma \left(-\dot{p}_\sigma\dot{q}_\sigma + \dot{q}_\sigma\dot{p}_\sigma \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } H \text{ if } \frac{\partial H}{\partial t} = 0$$

What is the physical meaning of a constant H?

Comment -- Whenever you find a constant of the motion, it is helpful for analyzing the trajectory. In this case, H often represents the mechanical energy of the system so that constant H implies that energy is conserved.

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$p_x = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)\right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants: $\bar{p}_x, \bar{p}_y, \bar{H}$ (using bar to indicate constant)

Equations of motion: $\dfrac{dz}{dt} = \dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{m} \qquad \dfrac{dp_z}{dt} = -\dfrac{dV}{dz}$

Example 2: Motion in a central potential

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

$$p_r = m\dot{r} \qquad\qquad p_\varphi = mr^2\dot{\varphi}$$

$$H = m\dot{r}^2 + mr^2\dot{\varphi}^2 - \left(\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)\right)$$

$$= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r)$$

Constants: $\overline{p}_\varphi, \overline{H}$

Equations of motion:

$$\frac{dr}{dt} = \frac{p_r}{m} \qquad\qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{\overline{p}_\varphi^2}{mr^3} - \frac{\partial V}{\partial r}$$

Other examples

Lagrangian for symmetric top with Euler angles α,β,γ :

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2$$
$$- Mgh\cos\beta$$

$$p_\alpha = I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta$$

$$p_\beta = I_1\dot{\beta}$$

$$p_\gamma = I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)$$

$$H = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 + Mgh\cos\beta$$

$$H = \frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} + Mgh\cos\beta$$

Constants: $\overline{p}_\alpha, \overline{p}_\gamma, \overline{H}$

Other examples

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c}B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$p_x = m\dot{x} - \frac{q}{2c}B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c}B_0 x$$

$$p_z = m\dot{z}$$

$$H = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Canonical form

$$H = \frac{\left(p_x + \frac{q}{2c}B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c}B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants: $\overline{p}_z, \overline{H}$

Canonical equations of motion for constant magnetic field:

$$H=\frac{\left(p_x+\frac{q}{2c}B_0y\right)^2}{2m}+\frac{\left(p_y-\frac{q}{2c}B_0x\right)^2}{2m}+\frac{p_z^2}{2m}$$

Constants: $\bar{p}_z, \bar{H}$

$$\frac{dx}{dt}=\frac{p_x+\frac{q}{2c}B_0y}{m} \qquad \frac{dy}{dt}=\frac{p_y-\frac{q}{2c}B_0x}{m}$$

$$\frac{dp_x}{dt}=-\frac{\partial H}{\partial x}=\frac{qB_0}{2mc}\left(p_y-\frac{q}{2c}B_0x\right)$$

$$\frac{dp_y}{dt}=-\frac{\partial H}{\partial y}=-\frac{qB_0}{2mc}\left(p_x+\frac{q}{2c}B_0y\right)$$

Canonical equations of motion for constant magnetic field -- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c} B_0 x\right) = \frac{qB_0}{2c}\frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c} B_0 y\right) = -\frac{qB_0}{2c}\frac{dx}{dt}$$

$$\frac{d^2 x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc} B_0 \dot{y} = \frac{qB_0}{mc}\frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc} B_0 \dot{x} = -\frac{qB_0}{mc}\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{qB_0}{mc}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = -\frac{qB_0}{mc}\frac{dx}{dt}$$

Are these results equivalent to the results of the Lagrangian analysis?

a. Yes
b. No

General treatment of particle of mass m and charge q moving in 3 dimensions in an potential $U(\mathbf{r})$ as well as electromagnetic scalar and vector potentials $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

Lagrangian: $$L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Hamiltonian: $$\mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{r}}}=m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)$$

$$H(\mathbf{r},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{r}}-L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$=\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

Some details: $L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

Hamiltonian: $\mathbf{p}=\frac{\partial L}{\partial \dot{\mathbf{r}}}=m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)$

$$H(\mathbf{r},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{r}}-L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$=\left(m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)\cdot\dot{\mathbf{r}}-\left(\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\right)$$

$$=\frac{1}{2}m\dot{\mathbf{r}}^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

$$H(\mathbf{r},\mathbf{p},t)=\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

Canonical form