PHY 337/637 Analytical Mechanics
12:30-1:45 PM TR in Olin 103

Notes for Lecture 9: Scattering analysis – Chap. 11 (Cline; especially 11.12)

1. Some comments on rigid body motion; especially the dynamics of a rotation symmetric top
2. Review of particle interactions
3. Two particles interacting with a central potential
4. Conservation of energy and angular momentum
5. Definition of differential scattering cross section

Physics Colloquium

Thursday

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September 28th, 2023

Culture-Based Approaches to Physics Instruction

Physics faculty and secondary teachers have a variety of methods at their disposal to embrace a culturally based approach to teaching. In this context, we want to spotlight the endeavors of professional learning communities composed of physics instructors. These educators have made deliberate efforts to infuse culture-based pedagogical elements into their classrooms, focusing on three key areas: (1) cultivating a strong physics teacher identity, (2) designing effective curricula, and (3) assessing the impact of curriculum materials on students' comprehension of physics concepts. The outcomes of this study reveal that these instructors exhibit notable signs of employing culturally relevant pedagogy. They exhibit a willingness to tackle sensitive subjects, consistently motivate students to strive for academic excellence, and adapt their curricula to incorporate students' strengths. Despite these positive aspects, instructors still grapple with certain challenges. These challenges include encouraging students to adopt a critical stance towards physics knowledge and effectively managing students' ability to take charge of their own learning process. The implications of the research findings are significant for physics instructors aiming to implement culturally relevant pedagogy in their teaching methodologies.

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Department of Teacher Education
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4 pm - Olin 101

Refreshments will be served in Olin Lobby beginning at 3:30pm.

Course schedule

In the table below, **Reading** refers to the chapters in the Cline textbook , **PP** refers to textbook section listing practice problems to be discussed at the course tutorials, and **Assign** is a link to the graded homework for the lecture. The graded homeworks are due each Tuesday following the associated lecture.

(Preliminary schedule -- subject to frequent adjustment.)

	Date	Reading	Topic	PP	Assign
1	Tu, 8/29/2023	Ch. 1 & 2	Introduction, history, and motivation	2E	#1
2	Th, 8/31/2023	Ch. 5	Introduction to Calculus of variation	5E	#2
3	Tu, 9/05/2023	Ch. 5	More examples of the calculus of variation	5E	#3
4	Th, 9/07/2023	Ch. 6	Lagrangian mechanics	6E	#4
5	Tu, 9/12/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	#5
6	Th, 9/14/2023	Ch. 7 & 8	Hamiltonian mechanics	8E	
7	Tu, 9/19/2023	Ch. 13	Dynamics of rigid bodies	13E	#6
8	Th, 9/21/2023	Ch. 13	Dynamics of rigid bodies	13E	#7
9	Tu, 9/26/2023	Ch. 13 & 11	Review of rigid bodies and intro to scattering	11E	#8
10	Th, 9/28/2023	Ch. 11	Scattering theory	11E	
11	Tu, 10/3/2023	Ch. 11	Scattering theory	11E	
12	Th, 10/5/2023				
13	Tu, 10/10/2023				
	Th, 10/12/2023	Fall Break			
14	Tu, 10/17/2023				

PHY 337/637 – Assignment #8

Assigned: 09/26/2023 Due: 10/03/2023

1. Consider a particle of mass m moving in the vicinity of another particle of mass M, initially at rest, where $m << M$. The particles interact with a conservative central potential of the form

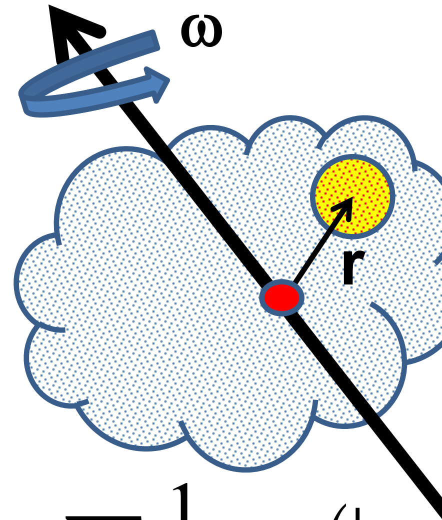
$$V(r) = V_0\left(\left(\frac{r_0}{r}\right)^2 - \left(\frac{r_0}{r}\right)\right),$$

where r denotes the magnitude of the particle separation and V_0 and r_0 denote energy and length constants, respectively. The total energy of the system E is constant and $E = V_0$.

(a) First consider the case where the impact parameter $b = 0$. Find the distance of closest approach of the particles.

(b) Now consider the case where the impact parameter $b = r_0$. Find the distance of closest approach of the particles.

Summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega} \times \mathbf{r}_p\right|\right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right)\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)^2\right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$$

Moment of inertia tensor

Matrix notation:

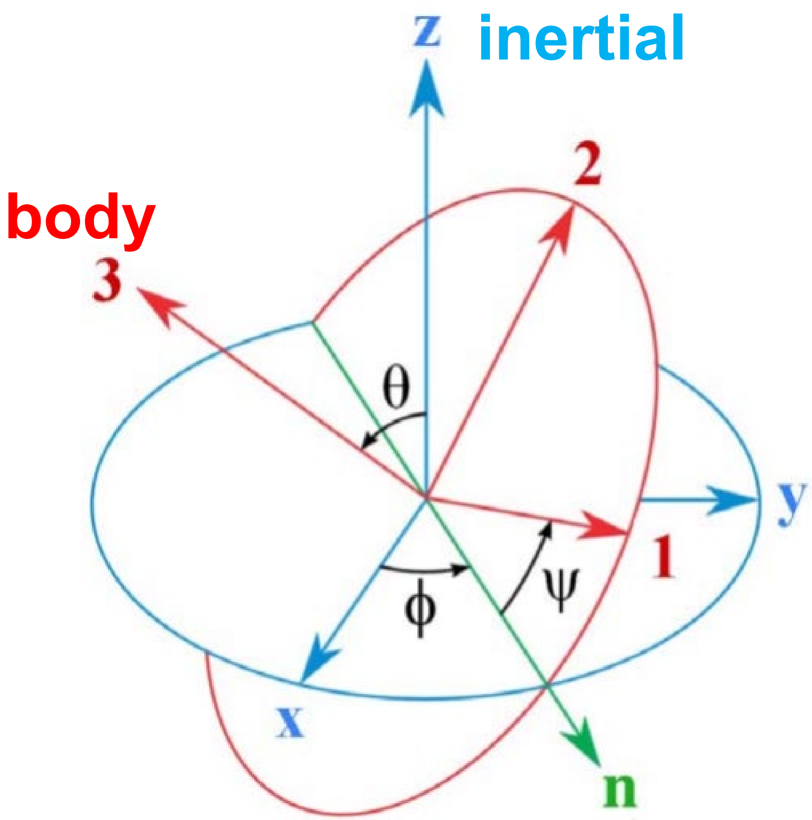
$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1, 2, 3$

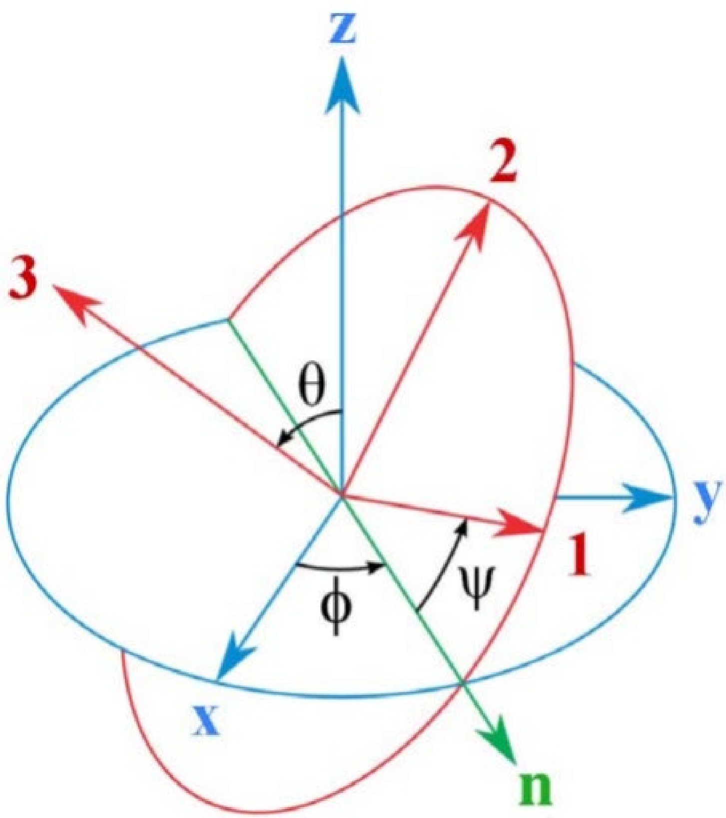
$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

Transformation between body-fixed and inertial coordinate systems – Euler angles



Comment – Since this is an old and intriguing subject, there are a lot of terminologies and conventions, not all of which are compatible. We are following the convention found in your textbook mechanics texts and NOT the convention found in most quantum mechanics texts. Euler's main point is that any rotation can be described by 3 successive rotations about 3 different (not necessarily orthogonal) axes. In this case, one is along the inertial z axis and another is along the body fixed 3 axis. The middle rotation is along an intermediate n axis.

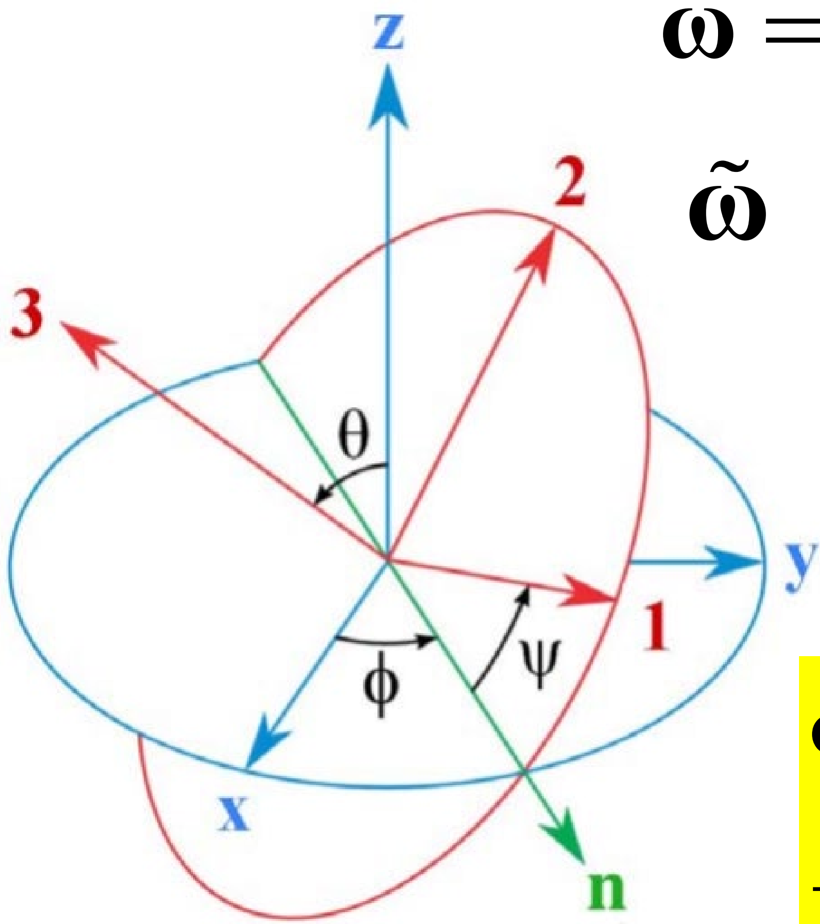
From section 13.13 of Cline



$$\tilde{\boldsymbol{\omega}} = \dot{\phi}\ \hat{\mathbf{z}} + \dot{\theta}\ \hat{\mathbf{n}} + \dot{\psi}\ \hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$



$$\tilde{\boldsymbol{\omega}} = \dot{\phi}\ \hat{\mathbf{z}} + \dot{\theta}\ \hat{\mathbf{n}} + \dot{\psi}\ \hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\phi}\left(\sin\theta\sin\psi\right) + \dot{\theta}\cos\psi\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\phi}\left(\sin\theta\cos\psi\right) - \dot{\theta}\sin\psi\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\phi}\cos\theta + \dot{\psi}\right]\hat{\mathbf{e}}_3\end{aligned}$$

Rotational kinetic energy

$$T\left(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi}\right)=\frac{1}{2}I_1\tilde{\omega}_1^2+\frac{1}{2}I_2\tilde{\omega}_2^2+\frac{1}{2}I_3\tilde{\omega}_3^2$$

$$=\frac{1}{2}I_1\left[\dot{\phi}\left(\sin\theta\sin\psi\right)+\dot{\theta}\cos\psi\right]^2$$

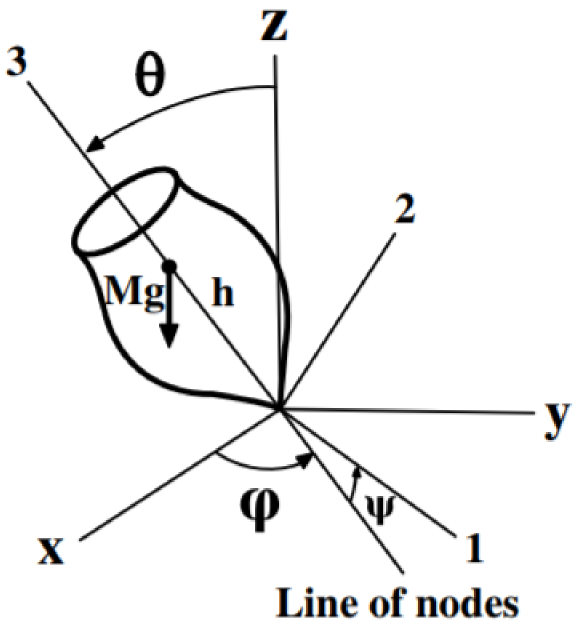
$$+\frac{1}{2}I_2\left[\dot{\phi}\left(\sin\theta\cos\psi\right)-\dot{\theta}\sin\psi\right]^2$$

$$+\frac{1}{2}I_3\left[\dot{\phi}\cos\theta+\dot{\psi}\right]^2$$

If $I_1 = I_2$:

$$T\left(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi}\right)=\frac{1}{2}I_1\left[\dot{\phi}^2\sin^2\theta+\dot{\theta}^2\right]+\frac{1}{2}I_3\left[\dot{\phi}\cos\theta+\dot{\psi}\right]^2$$

Motion of a symmetric top under the influence of the torque of gravity:



Lagrangian

$$L\left(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi}\right)=\frac{1}{2}I_1\left[\dot{\phi}^2\sin^2\theta+\dot{\theta}^2\right]+\frac{1}{2}I_3\left[\dot{\phi}\cos\theta+\dot{\psi}\right]^2 - Mgh\cos\theta$$

Some analysis --

$$L\left(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi}\right)=\frac{1}{2}I_1\left[\dot{\phi}^2\sin^2\theta+\dot{\theta}^2\right]+\frac{1}{2}I_3\left[\dot{\phi}\cos\theta+\dot{\psi}\right]^2-Mgh\cos\theta$$

Constants of the motion:

$$p_\phi=\frac{\partial L}{\partial\dot{\phi}}=I_1\dot{\phi}\sin^2\theta+I_3\left[\dot{\phi}\cos\theta+\dot{\psi}\right]\cos\theta$$

$$p_\psi=\frac{\partial L}{\partial\dot{\psi}}=I_3\left[\dot{\phi}\cos\theta+\dot{\psi}\right]$$

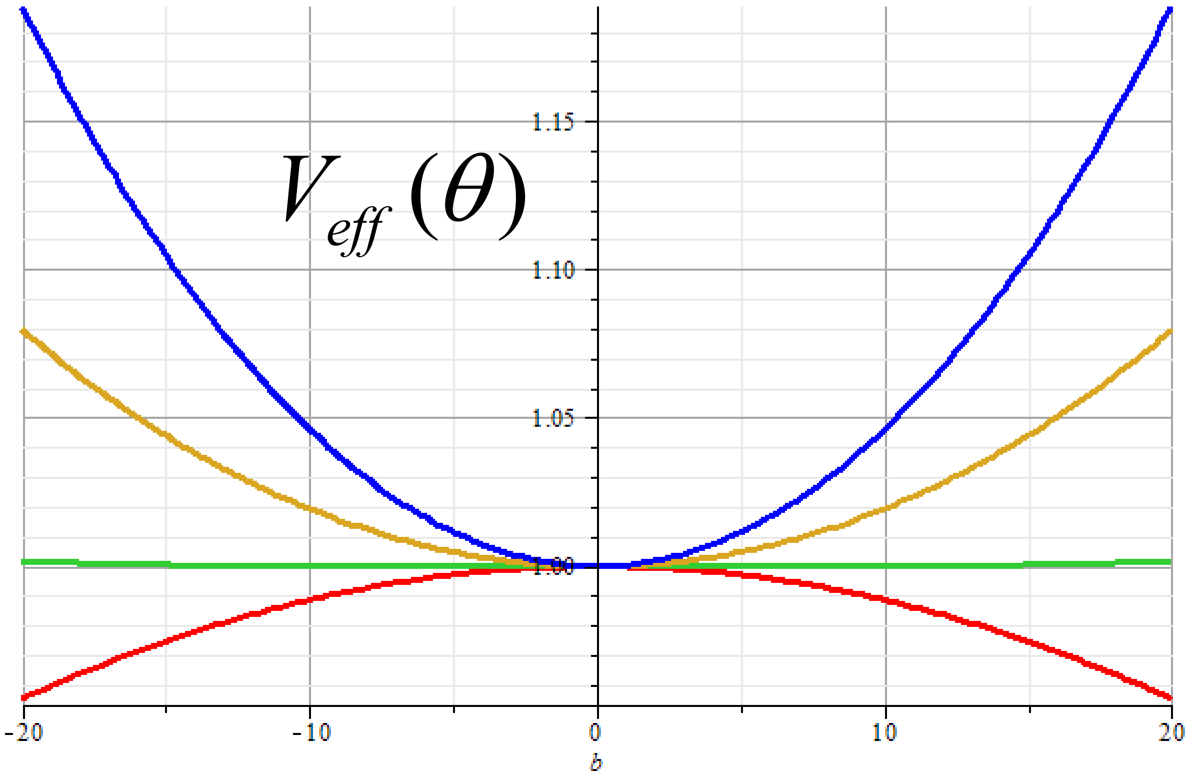
$$L\left(\theta,\dot{\theta}\right)=\frac{1}{2}I_1\dot{\theta}^2+\frac{\left(p_\phi-p_\psi\cos\theta\right)^2}{2I_1\sin^2\theta}+\frac{p_\psi^2}{2I_3}-Mgl\cos\theta$$

Note that energy is also conserved:

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{p_\psi^2}{2I_3} + \frac{\left(p_\phi - p_\psi \cos\theta\right)^2}{2I_1 \sin^2\theta} + Mgh\cos\theta$$

$$E' = E - \frac{p_\psi^2}{2I_3} = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{\left(p_\phi - p_\psi \cos\theta\right)^2}{2I_1 \sin^2\theta} + Mgh\cos\theta$$

Stable/unstable solutions near θ=0



$$E' = E - \frac{p_\psi^2}{2I_3} = \frac{1}{2}I_1\dot{\theta}^2 + \frac{\left(p_\phi - p_\psi\cos\theta\right)^2}{2I_1\sin^2\theta} + Mgh\cos\theta \equiv \frac{1}{2}I_1\dot{\theta}^2 + V_{eff}(\theta)$$

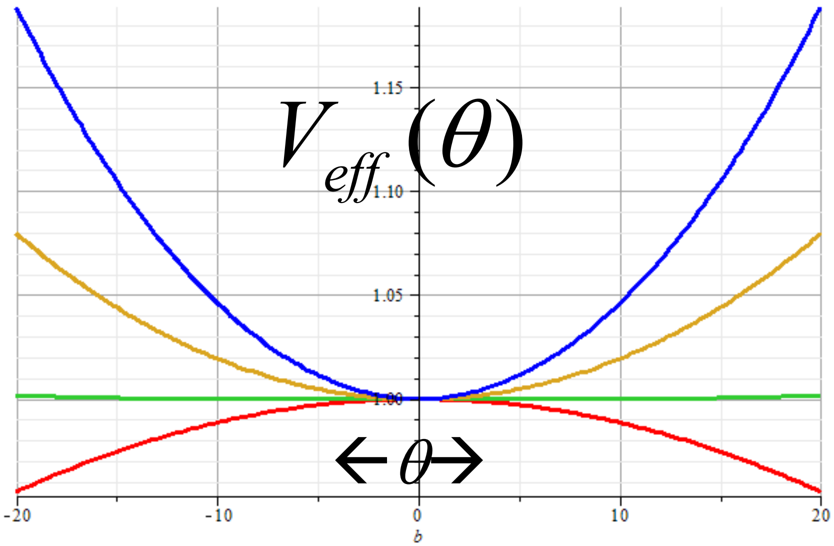
Note that for $\theta \approx 0$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \ldots$

$\sin\theta \approx \theta$ and $p_\phi \approx p_\psi$

$$\Rightarrow \quad V_{eff}(\theta) \approx \frac{p_\psi^2}{8I_1}\theta^2 + Mgh\left(1 - \frac{1}{2}\theta^2\right) + \ldots$$

$$E' \approx \frac{1}{2}I_1\dot{\theta}^2 + \frac{p_\psi^2}{8I_1}\theta^2 + Mgh\left(1 - \frac{1}{2}\theta^2\right) + \ldots$$

$$E' \approx \frac{1}{2}I_1\dot{\theta}^2 + Mgh + \left(\frac{p_\psi^2}{8I_1} - \frac{Mgh}{2}\right)\theta^2 + \ldots$$



Note that

$p_\psi = I_3\omega_3$

$\Rightarrow \omega_3$ must be sufficiently large for the top to maintain vertical orientation $(\theta \approx 0)$.

$\Rightarrow$ Stable solution if

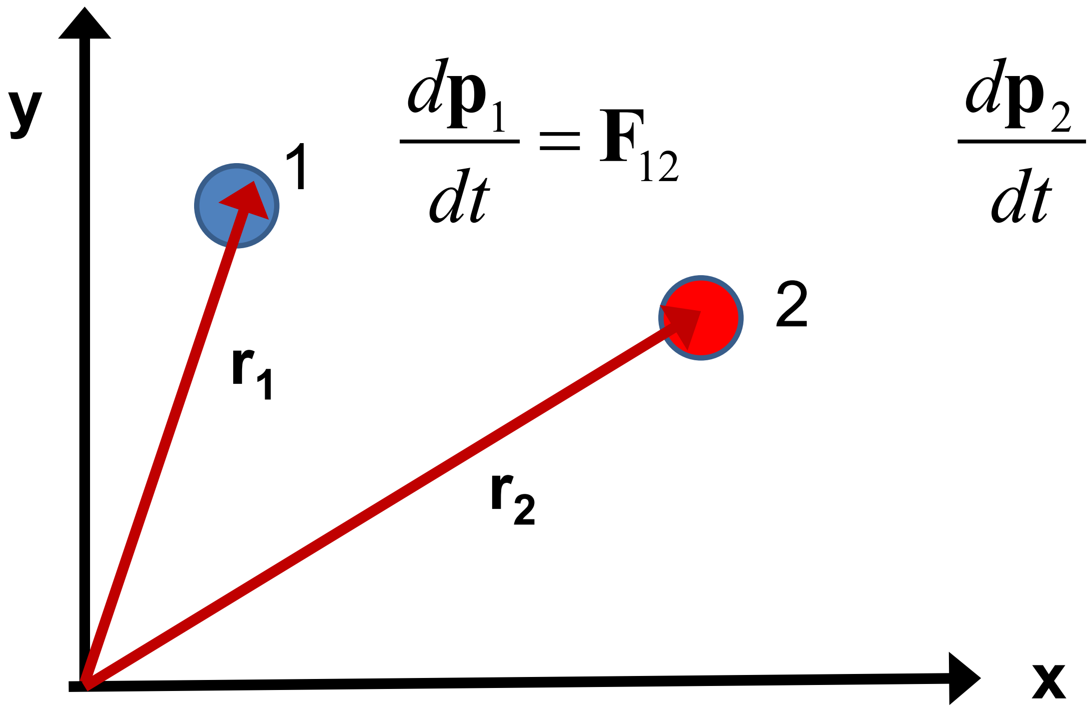
$$p_\psi \geq \sqrt{4MghI_1}$$

Introduction to the analysis of the energy and forces between two particles –

This treatment can be formulated with Lagrangians and Hamiltonians, but we will directly use the Newtonian approach for now..

First consider fundamental picture of particle interactions

Classical mechanics of a conservative 2-particle system.

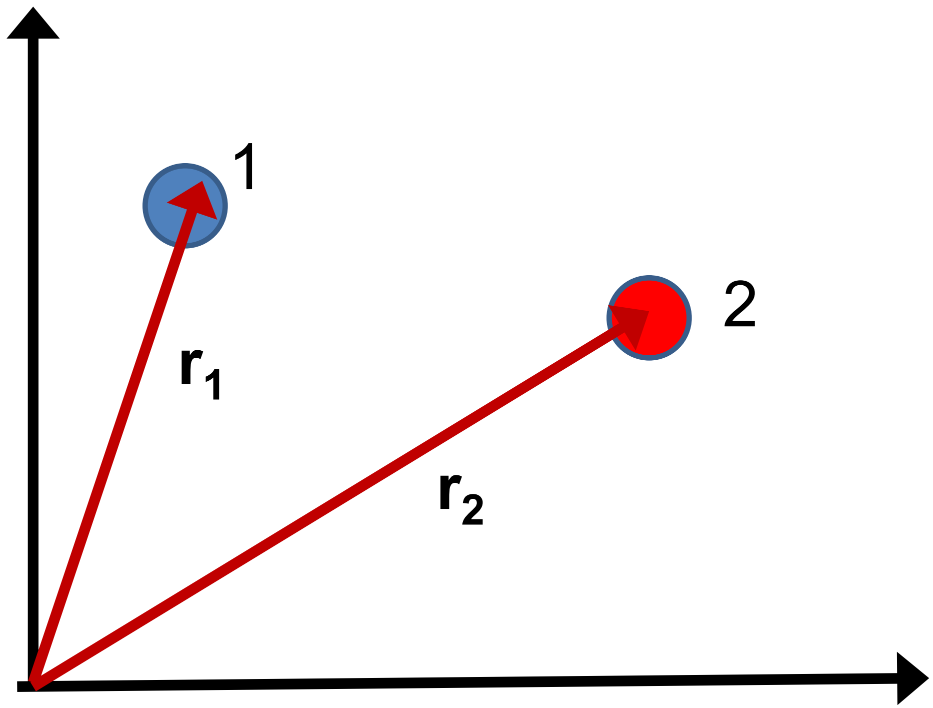


$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) \qquad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

For this discussion, we will assume that $V(\mathbf{r})=V(r)$ (a central potential).

Energy is conserved: $$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$



For a central potential $V(\mathbf{r})=V(r)$, angular momentum is conserved. For the moment we also make the simplifying assumption that $m_2 >> m_1$ so that particle 1 dominates the motion.

Typical two-particle interactions –

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

Hard sphere: $V(r) = \begin{cases} \infty & r \leq a \\ 0 & r > a \end{cases}$

Coulomb or gravitational: $V(r) = \dfrac{K}{r}$

Lennard-Jones: $V(r) = \dfrac{A}{r^{12}} - \dfrac{B}{r^6}$

More details of two particle interaction potentials

Central potential: $$V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$$

This means that the interaction only depends on the distance between the particles and not on the angle between them. This would typically be true of the particles are infinitesimal points without any internal structure such as two infinitesimal charged particles or two infinitesimal masses separated by a distance *r*:

$$V(r) = \frac{K}{r}$$

Example – Interaction between a proton and an electron. Note we are treating the interactions with classical mechanics; in some cases, quantum effects are non-trivial.

Other examples of central potentials --

Example

Hard sphere: $V(r) = \begin{cases} \infty & r \leq a \\ 0 & r > a \end{cases}$ Two marbles

Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^{6}}$ Two Ar atoms

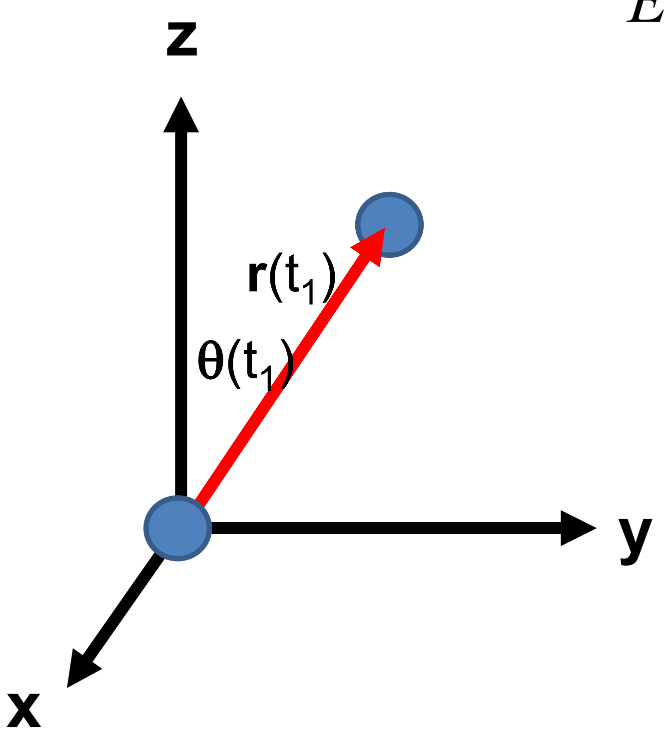
Note – not all systems are described by this form. Some counter examples:

1. Molecules (internal degrees of freedom)
2. Systems with more than two particles such as crystals

Some more details --

Here we are assuming that the target particle is stationary and $m_1 \equiv m$.

The origin of our coordinate system is taken at the position of the target particle.



Conservation of energy:

$$E = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V(r)$$

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + V(r)$$

Conservation of angular momentum:

$$|\mathbf{L}| = L \;\; = mr^2 \frac{d\theta}{dt}$$

Comments continued --

Conservation of energy:

$$E = \frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V(r)$$

$$= \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) + V(r)$$

$$= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \boxed{\frac{L^2}{2mr^2} + V(r)}$$

⇐ $V_{eff}(r)$

Conservation of angular momentum:

$$L = mr^2\frac{d\theta}{dt}$$

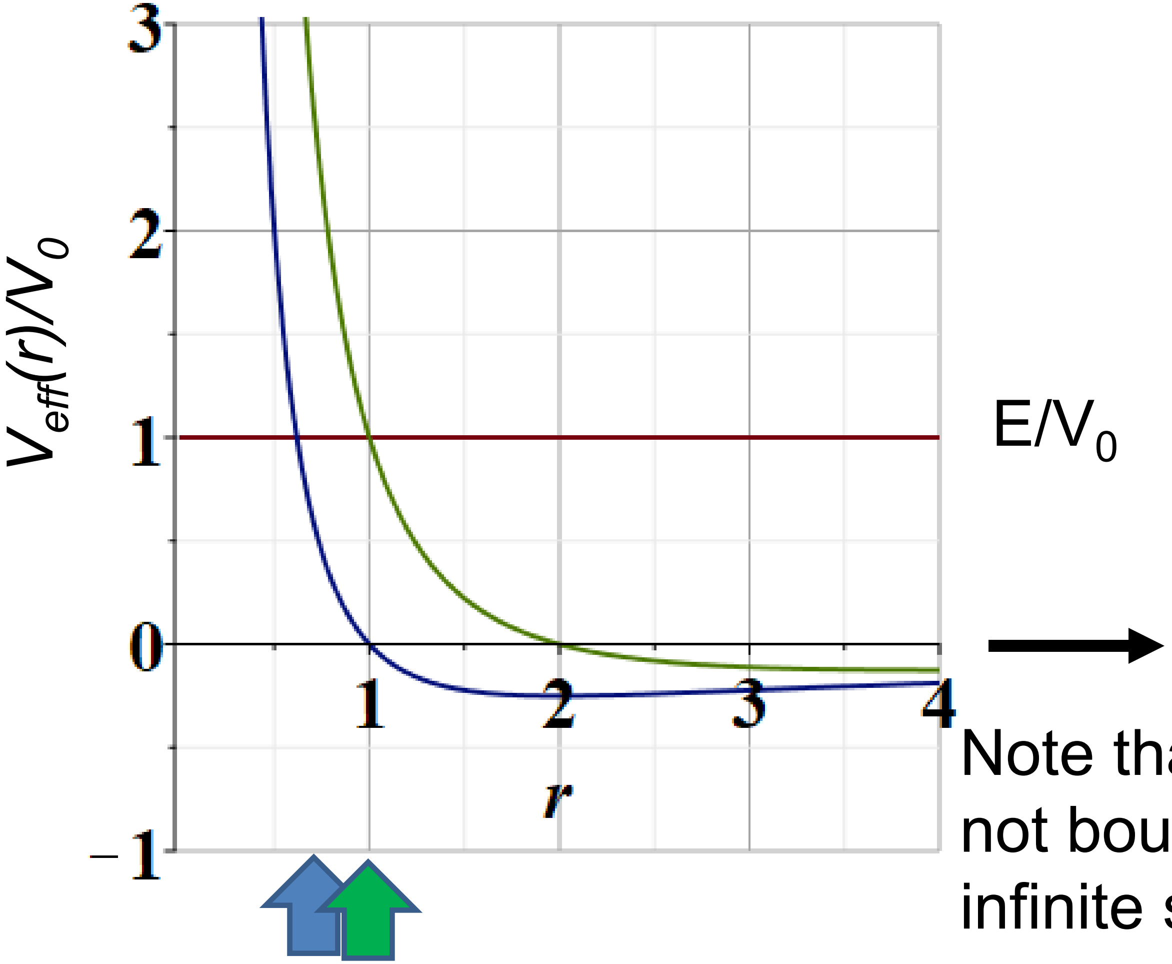
Also note that when $r \rightarrow \infty, \quad V(r) \rightarrow 0$

$$\mathbf{L} \equiv \mathbf{r} \times m\frac{d\mathbf{r}}{dt} \qquad L = bmv_\infty = b\sqrt{2mE}$$

For $r \rightarrow \infty, \quad \frac{d\mathbf{r}}{dt} \rightarrow v_\infty = \sqrt{\frac{2E}{m}}$

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{b^2E}{r^2} + V(r)$$

Representative plots of $V_{eff}\ (r)$



Note that particles are not bound; can reach infinite separation

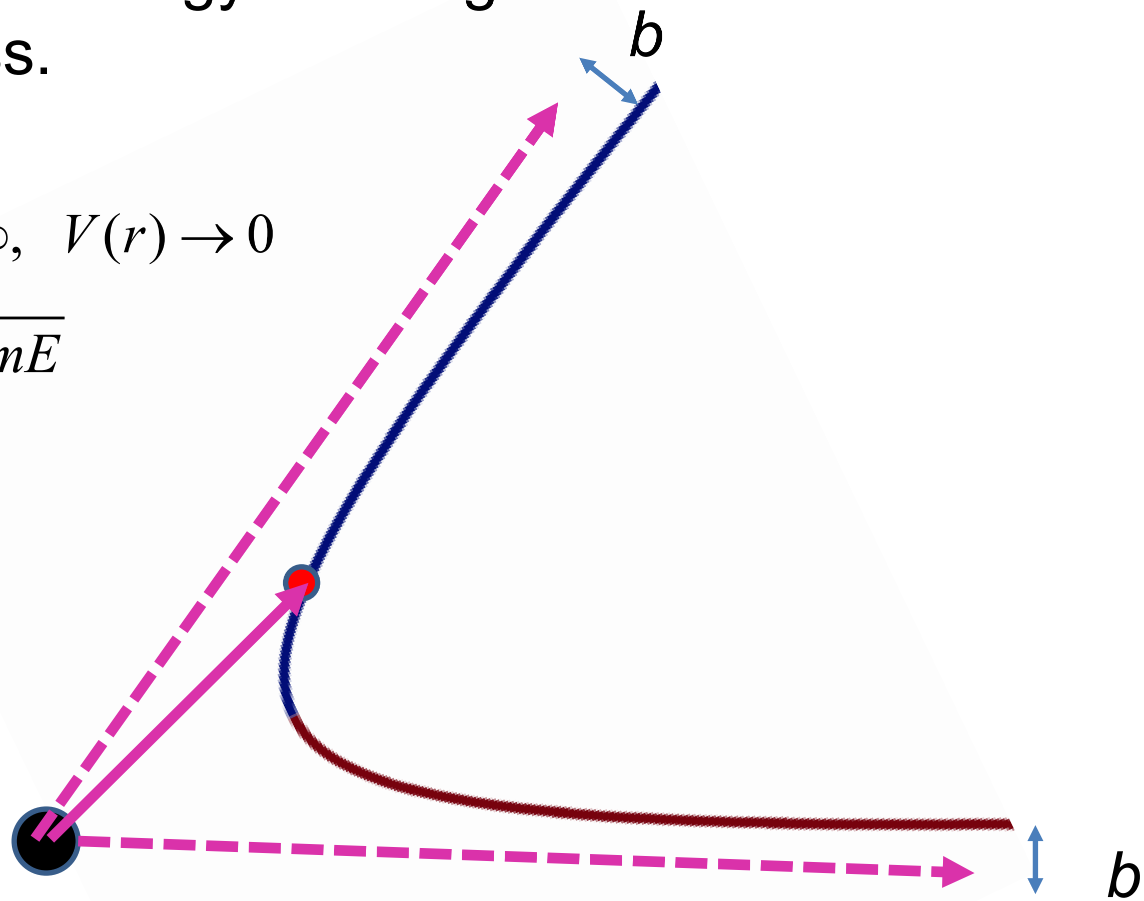
Distances of closest approach

What is the impact parameter?

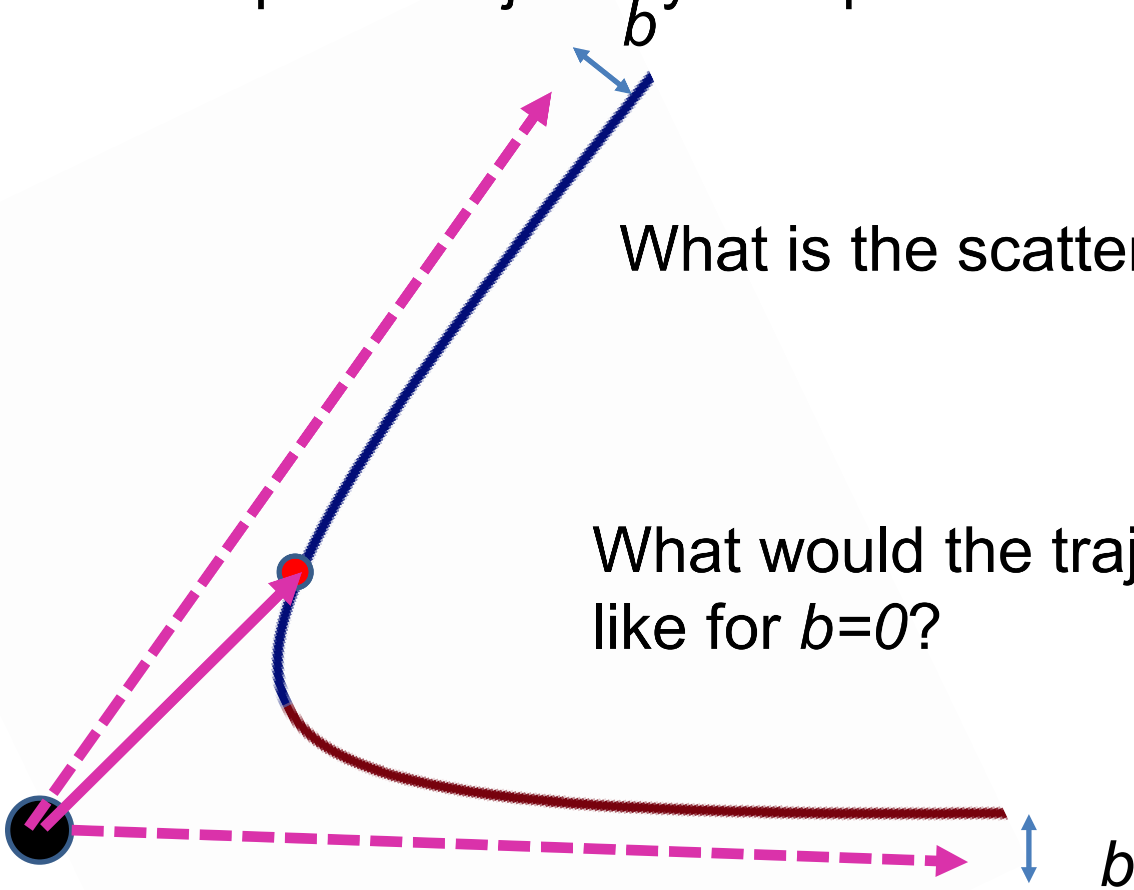
Briefly, a convenient distance that depends on the conserved energy and angular momentum of the process.

Also note that when $r \to \infty, \quad V(r) \to 0$

$$\mathbf{L} \equiv \mathbf{r} \times m \frac{d\mathbf{r}}{dt} \qquad L = b\sqrt{2mE}$$



Example of trajectory with particular *b*



What is the scattering angle?

What would the trajectory look like for *b=0*?

Which of the following are true for a particle moving in a central potential field:

a. The particle moves in a plane.
b. For any interparticle, potential the trajectory can be determined/calculated.
c. Only for a few special interparticle potential forms can the trajectory be determined.

Why should we care about this?

a. We shouldn’t really care.
b. It is only of academic interest
c. It is of academic interest but can be measured.
d. Many experiments can be analyzed in terms of the particle trajectory.

Scattering theory:

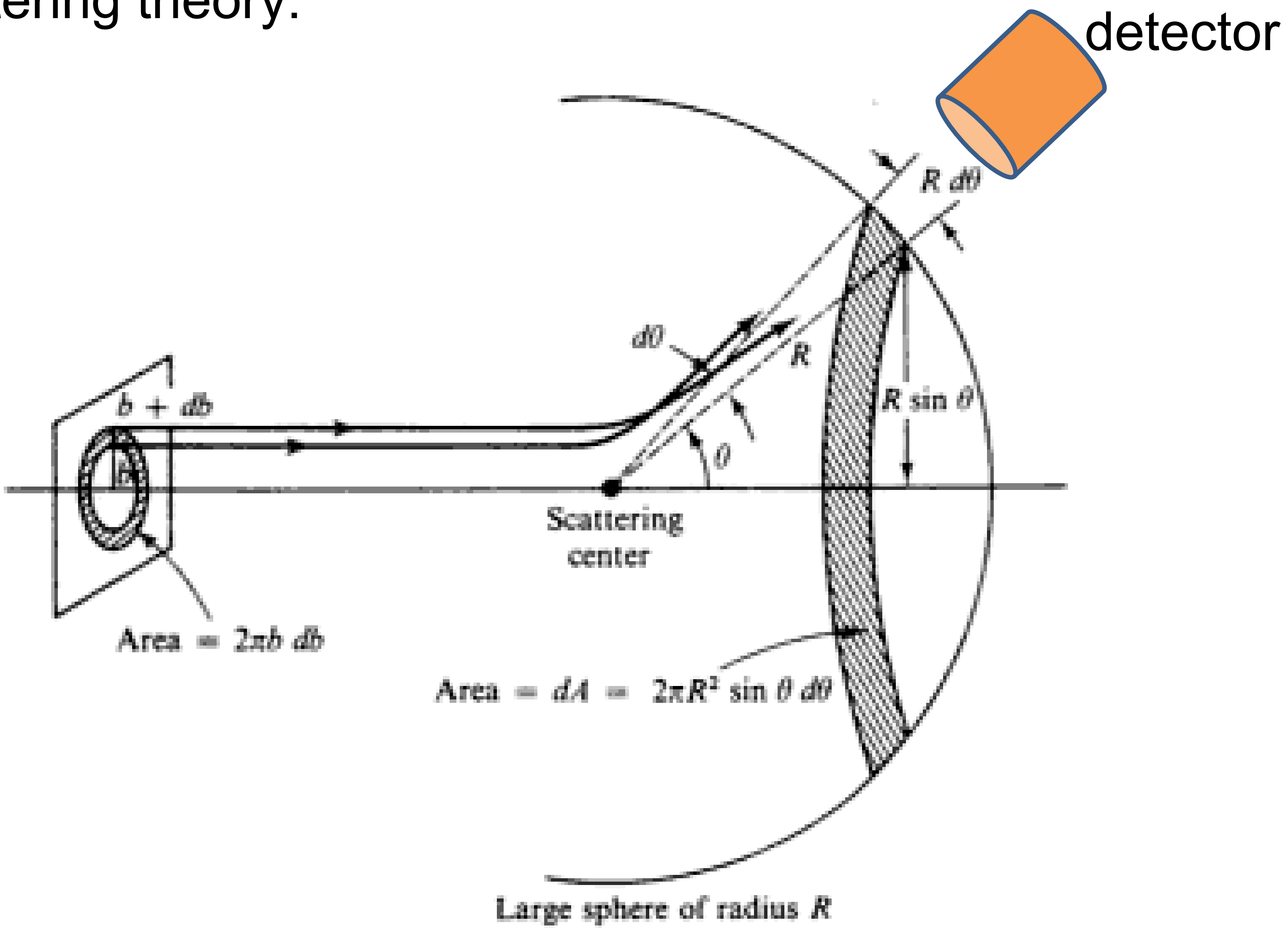


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Scattering theory:

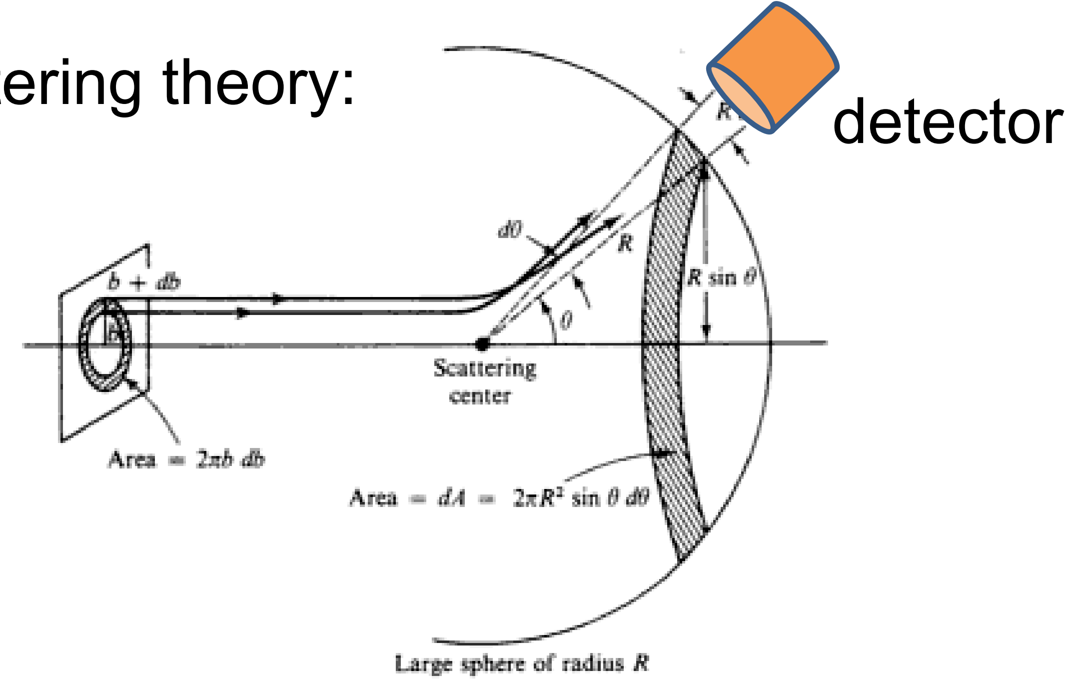


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Some reasons that scattering theory is useful:

1. It allows comparison between measurement and theory
2. The analysis depends on knowledge of the scattering particles when they are far apart
3. The scattering results depend on the interparticle interactions

Standard measure of differential cross section

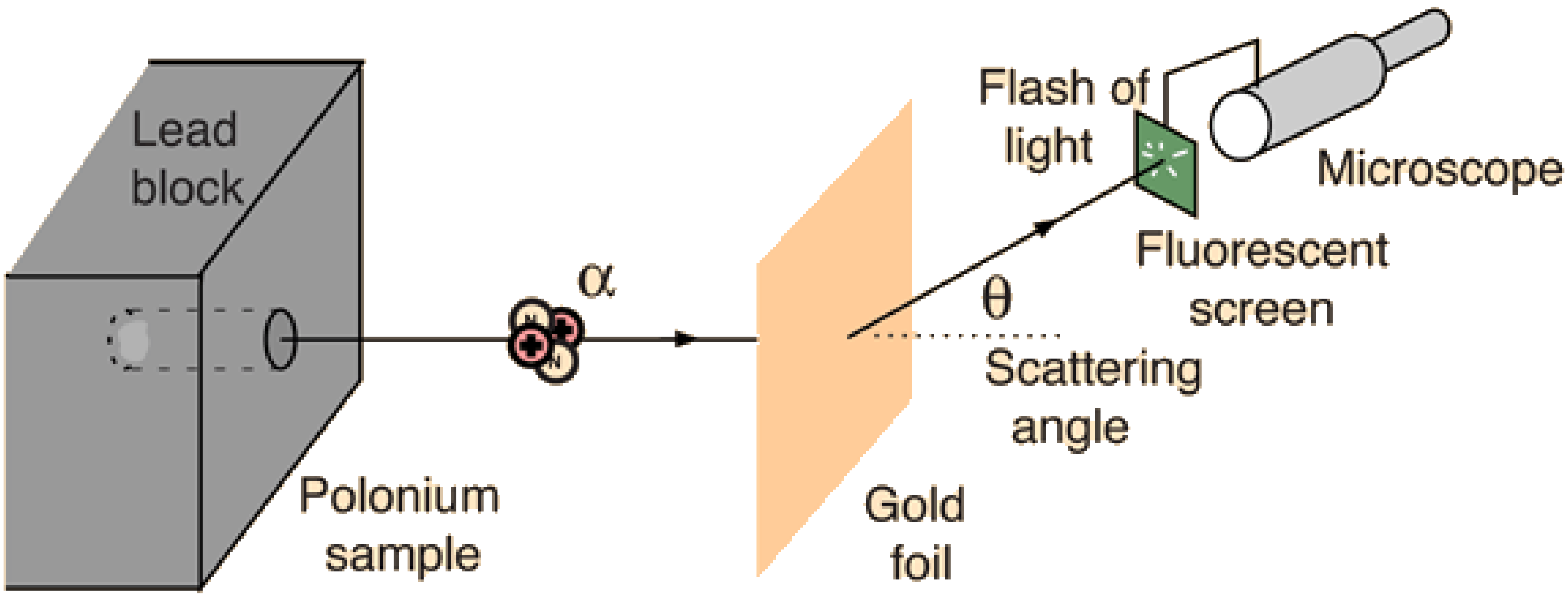
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle per unit time}}{\text{Number of incident particles per unit area per unit time}}$$

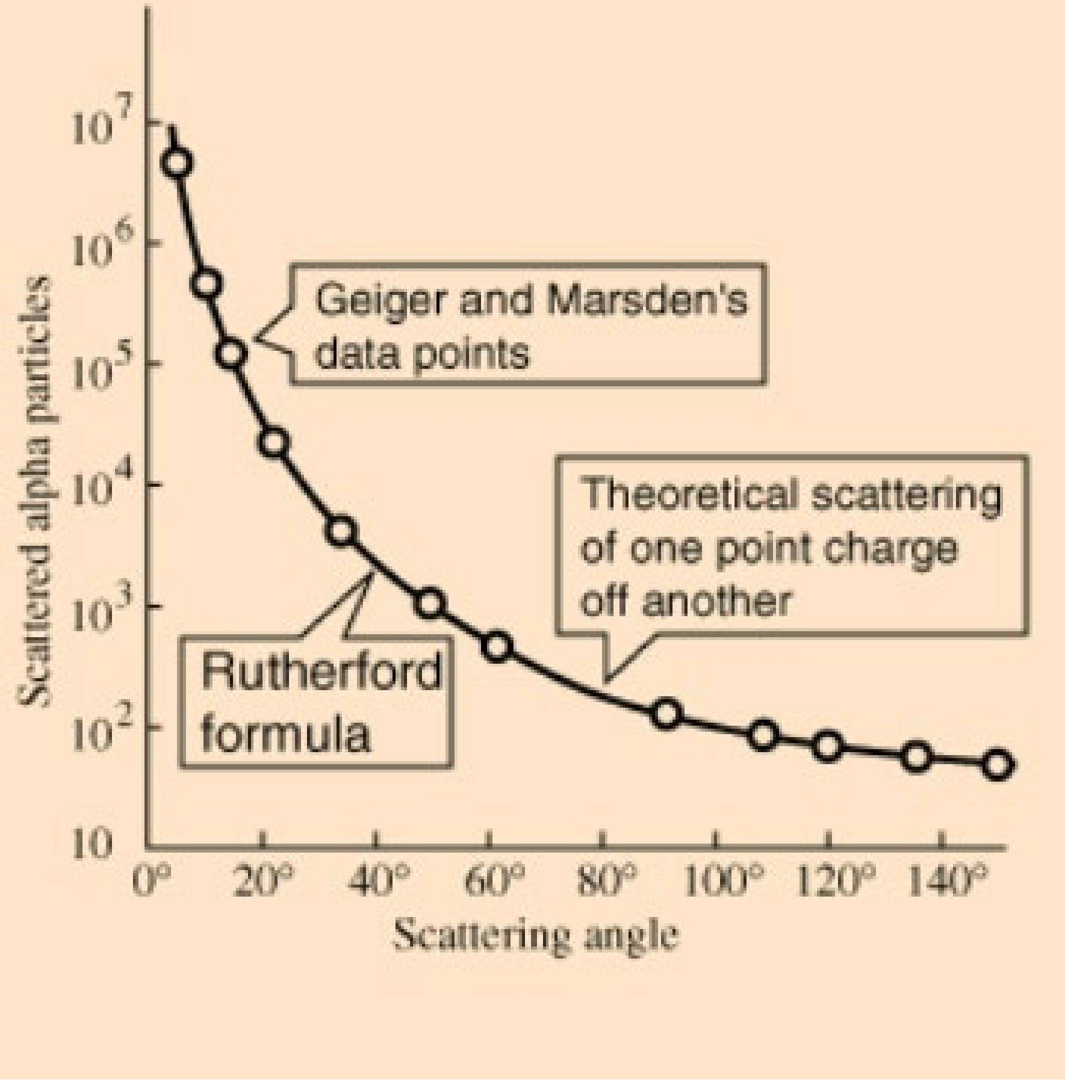
$$= \text{ Area of incident beam that is scattered into detector}$$

at angle θ

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Graph of data from scattering experiment



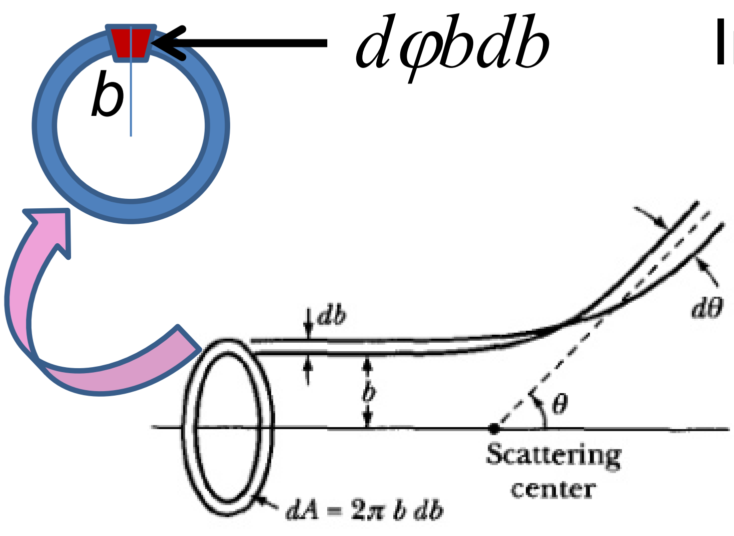
From website: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/rutsca2.html

Standardization of scattering experiments --

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle per unit time}}{\text{Number of incident particles per unit area per unit time}}$$

$$= \text{ Area of incident beam that is scattered into detector at angle } \theta$$

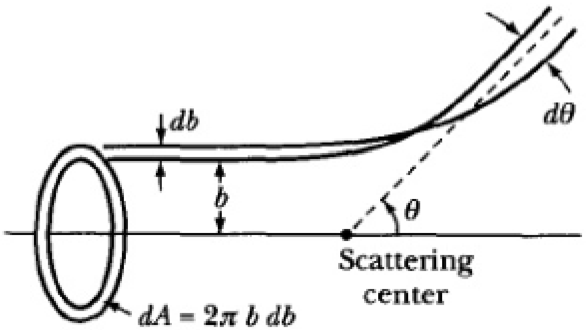


Impact parameter: *b*

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi\ b\ db}{d\phi\ \sin\theta\ d\theta} = \left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



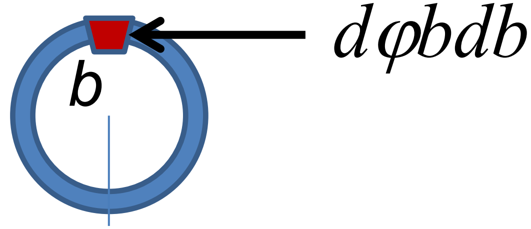
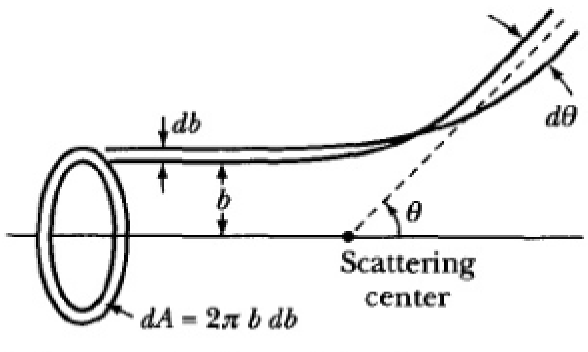


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi\ b\ db}{d\phi\ \sin\theta\ d\theta} = \left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right|$$

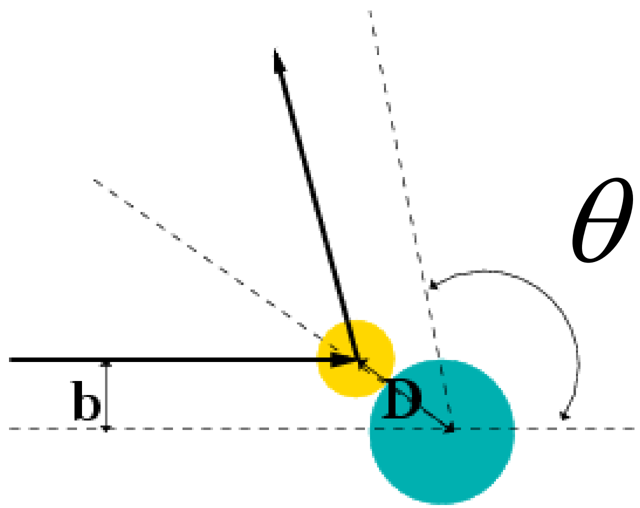
Note: We are assuming that the process is isotropic in φ

Simple example – collision of hard spheres having mutual radius D; very large target mass



$$\left(\frac{d\sigma}{d\Omega}\right)=\left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right|$$

Microscopic view:



$$b(\theta)=?$$

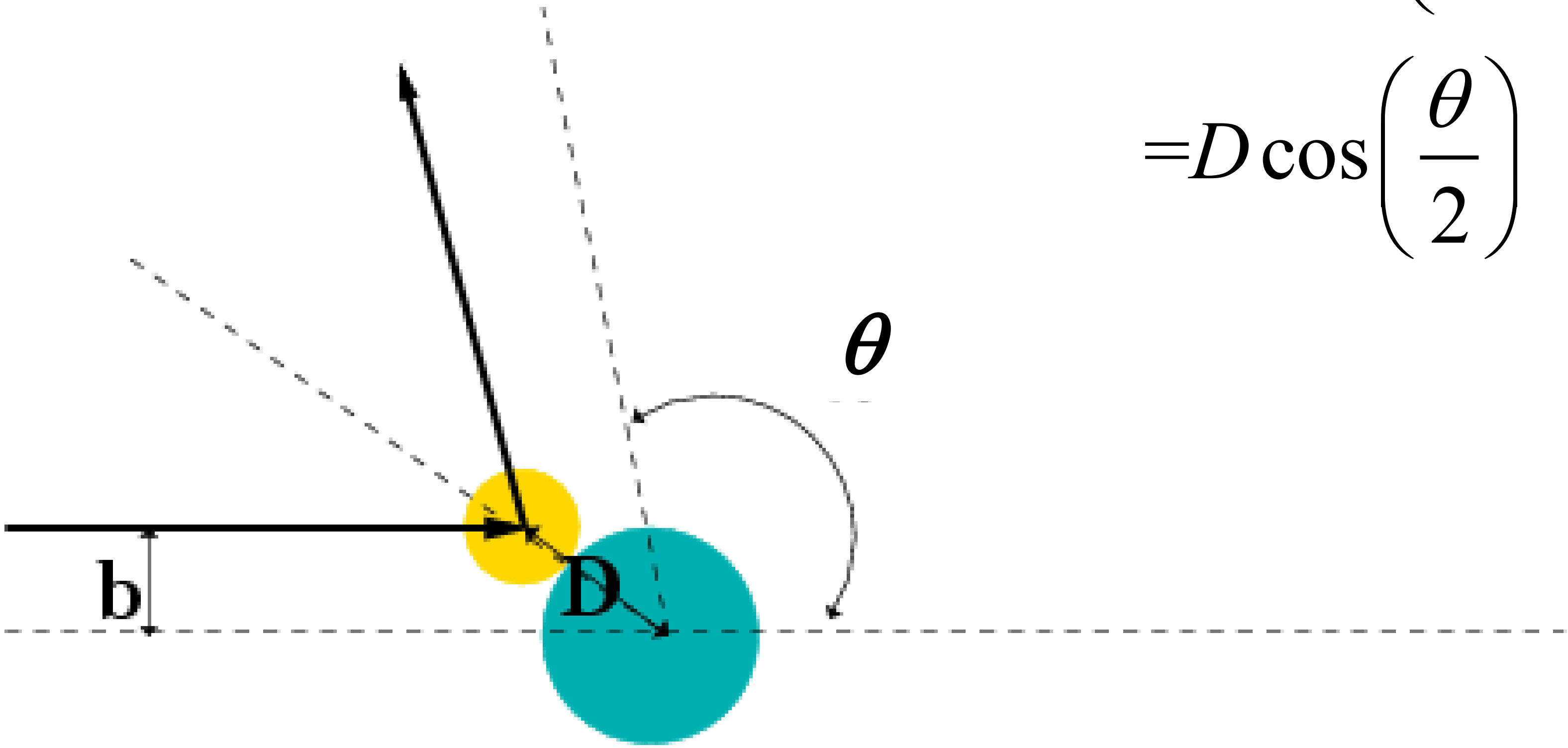
$$b(\theta)=D\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{D^2}{4}$$

Some details --

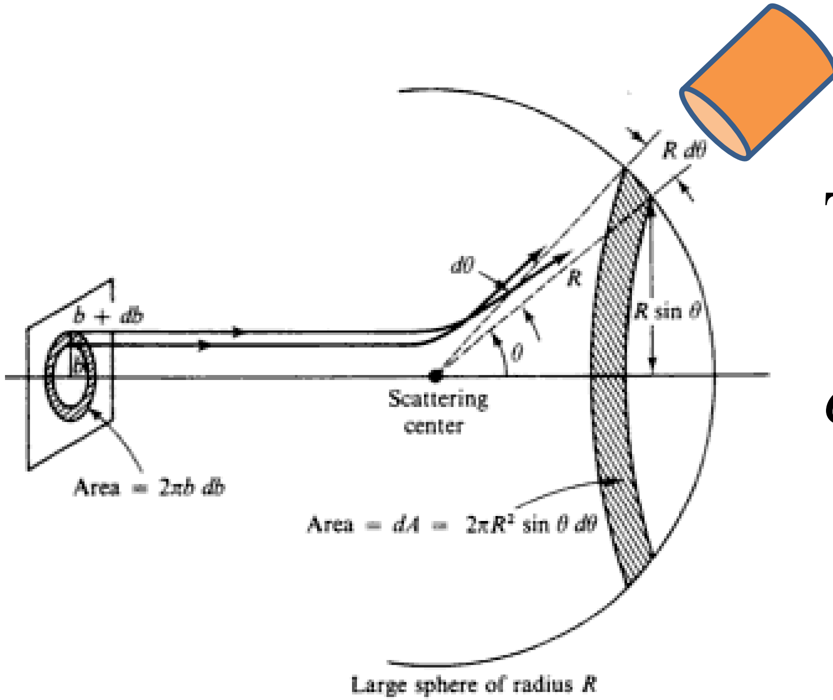
$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= D\cos\left(\frac{\theta}{2}\right)$$



$$\left(\frac{d\sigma}{d\Omega}\right) = \left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right| = \frac{D\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\frac{D\sin\left(\frac{\theta}{2}\right)}{2} = \frac{D^2}{4}$$

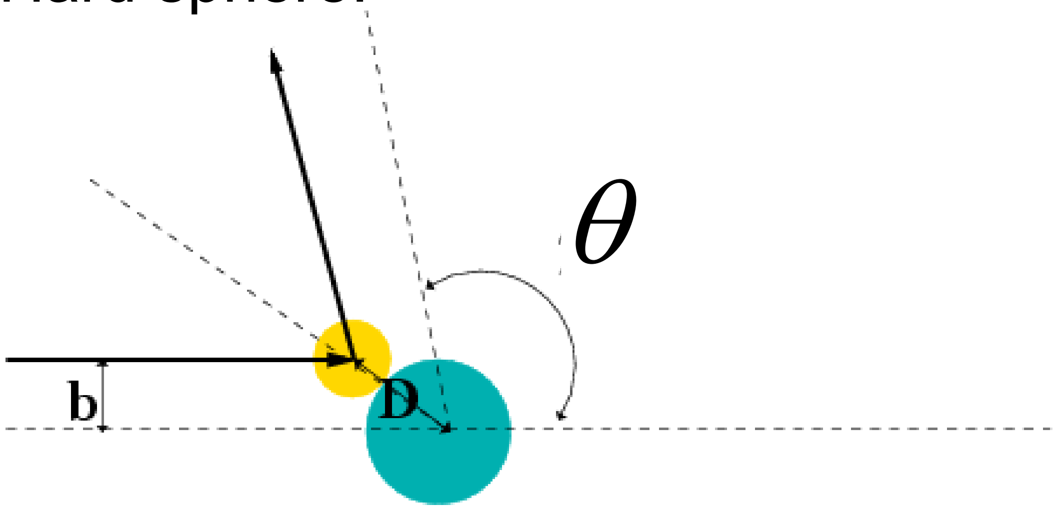
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega} \right) d\Omega$$

Hard sphere:



$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

More details of hard sphere scattering –

Hidden in the analysis are assumptions about the scattering process such as:

- No external forces ➔ linear momentum is conserved
- No dissipative phenomena ➔ energy is conserved
- No torque on the system ➔ angular momentum is conserved
- Target particle is much more massive than scattering particle
- Other assumptions??

Note that for quantum mechanical hard spheres at low energy the total cross section is 4 times as large.