PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 11: Rigid bodies – Chap. 5 (F &W)

1. More about moment of inertia tensor
2. Torque free motion
3. Euler angles

COLLOQUIUM

SEPTEMBER 21ST, 2023

Phase Transformations via Surface Defects in Halide Perovskites

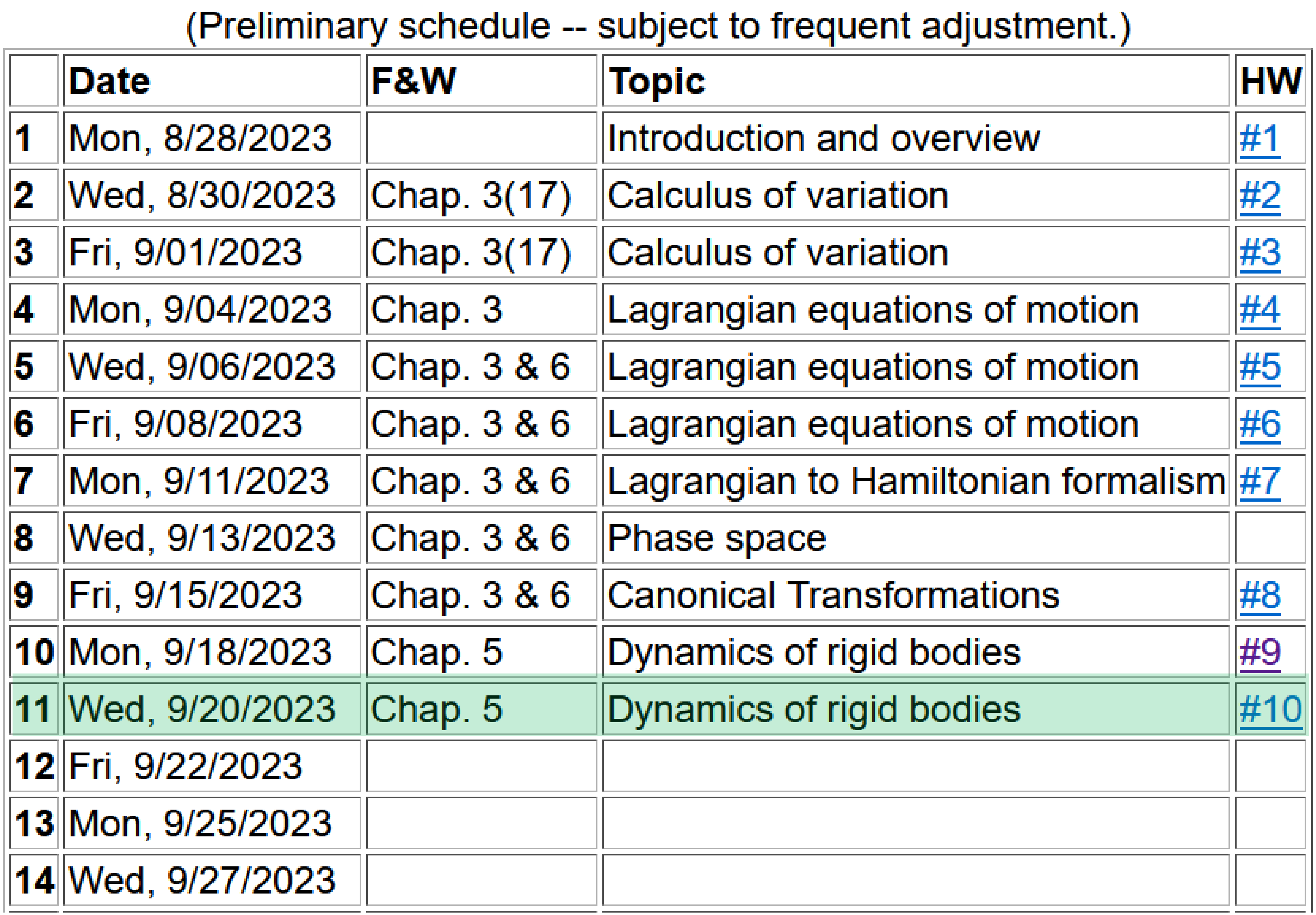
Perovskite solar cells promise to yield efficiencies beyond 30% by further improving the quality of the materials and devices. Electronic defect passivation and suppression of detrimental charge-carrier recombination at the different device interfaces has been used as a strategy to achieve high performance perovskite solar cells. However, the mechanisms that allow for carriers to be transferred across these interfaces are still unknown. Through the contributions to better understand 2D and 3D defects the perovskite solar cell field has been able to improve device performance. Albeit the rapid improvements in performance, there is still a need to understand how these defects affect long term structural stability and thus optoelectronic performance over the long term. In this presentation, I will discuss the role of crystal surface structural defects on optoelectronic properties of lead halide perovskites through synchrotron-based techniques. The importance of interfaces and their contribution to detrimental recombination will also be discussed. Finally, a discussion on the current state-of-the-art of performance and stability will be presented.

Juan-Pablo Correa-Baena, PhD

Assistant Professor
School of Materials Science and Engineering
Georgia Institute of Technology

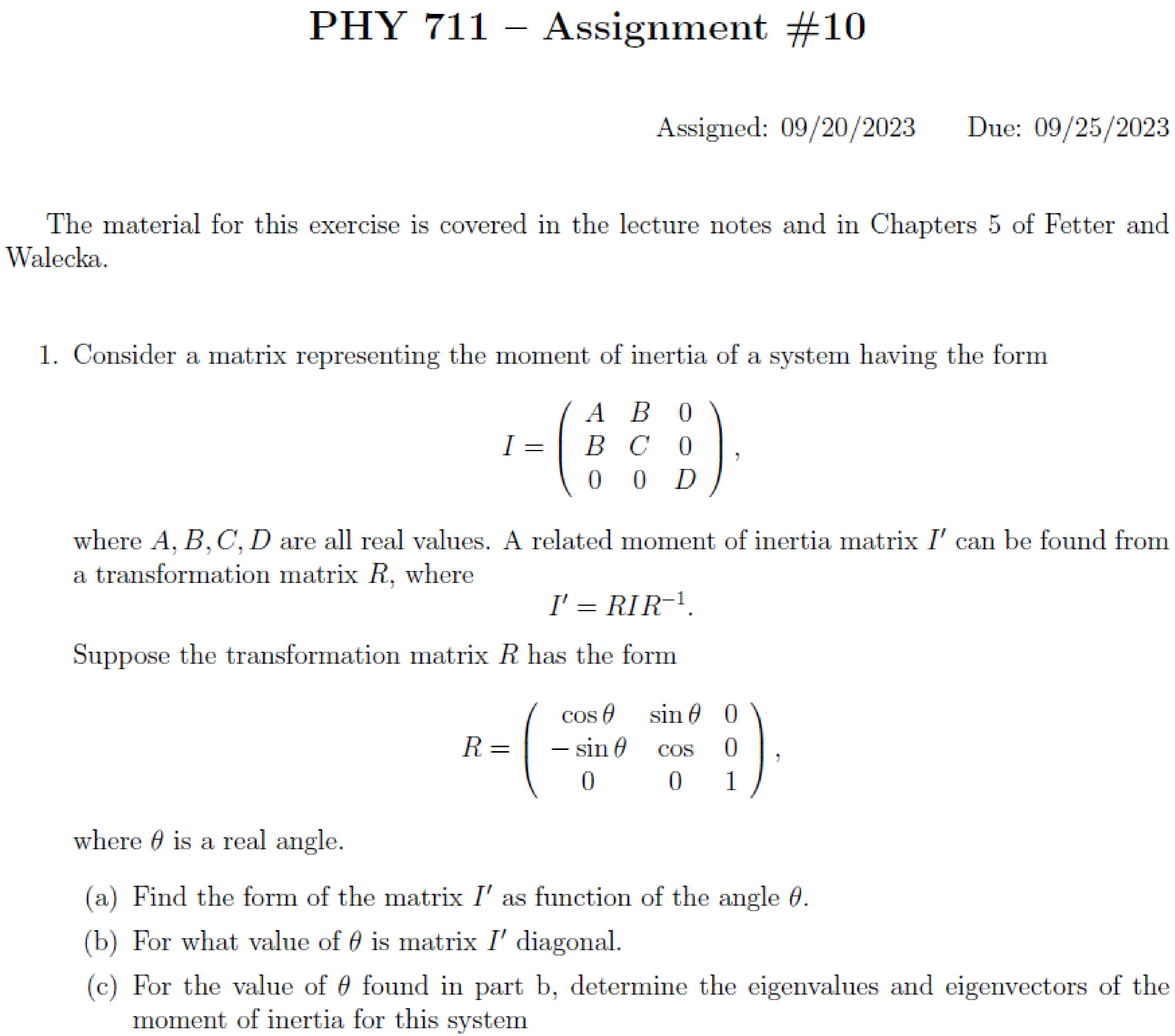
4 pm - Olin 101
Refreshments will be served in Olin Lobby beginning at 3:30pm.

Course schedule



(Preliminary schedule -- subject to frequent adjustment.)

	Date	**F&W**	**Topic**	**HW**
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space	
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	#8
10	Mon, 9/18/2023	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/20/2023	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/22/2023			
13	Mon, 9/25/2023			
14	Wed, 9/27/2023			



PHY 711 – Assignment #10

Assigned: 09/20/2023 Due: 09/25/2023

The material for this exercise is covered in the lecture notes and in Chapters 5 of Fetter and Walecka.

1. Consider a matrix representing the moment of inertia of a system having the form

$$I = \begin{pmatrix} A & B & 0 \\ B & C & 0 \\ 0 & 0 & D \end{pmatrix},$$

where A, B, C, D are all real values. A related moment of inertia matrix I' can be found from a transformation matrix R, where

$$I' = RIR^{-1}.$$

Suppose the transformation matrix R has the form

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where θ is a real angle.

(a) Find the form of the matrix I' as function of the angle θ.

(b) For what value of θ is matrix I' diagonal.

(c) For the value of θ found in part b, determine the eigenvalues and eigenvectors of the moment of inertia for this system

Moment of inertia tensor:

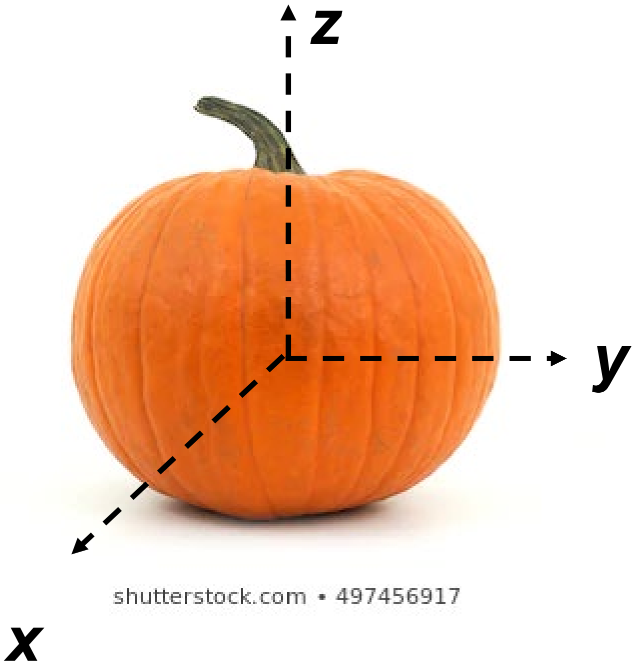
$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

Note: For a given object and a given coordinate system, one can find the moment of inertia matrix

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$



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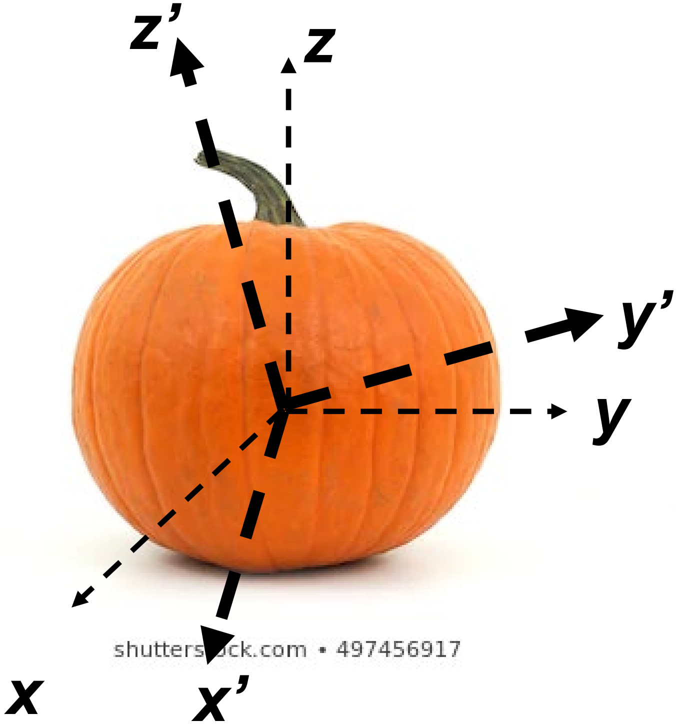


Moment of inertia in original coordinates

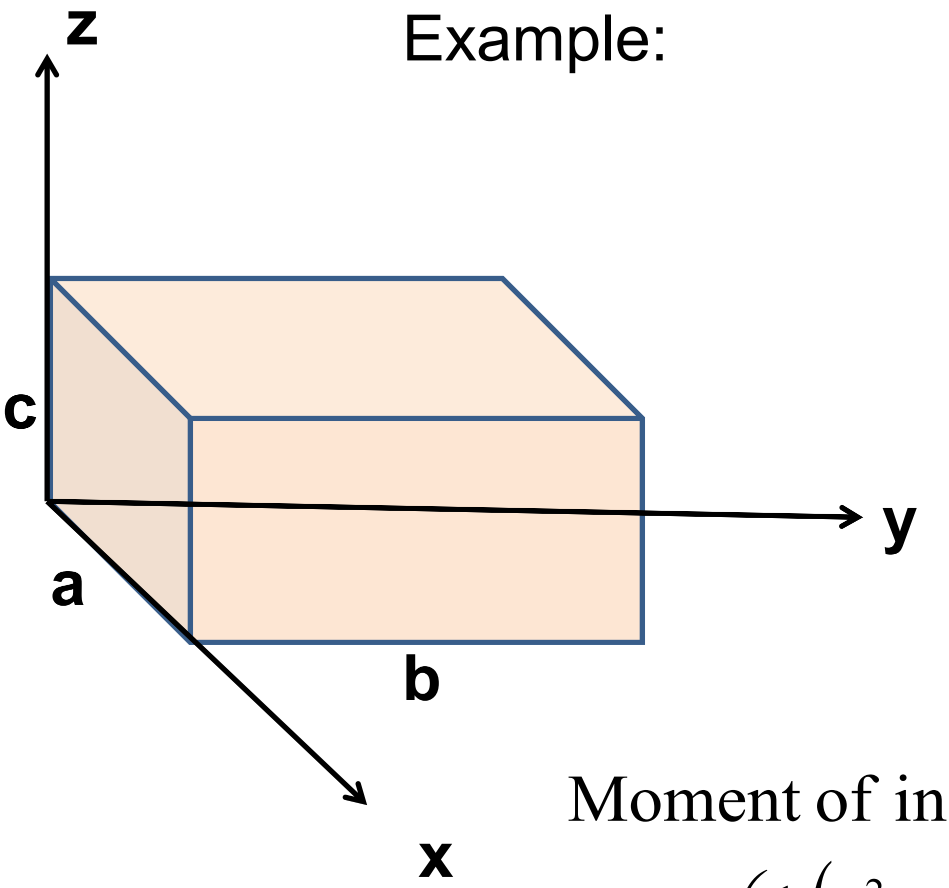
$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Moment of inertia in principal axes ***(x',y',z')***

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Example:



Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

Properties of moment of inertia tensor:

- Symmetric matrix ➔real eigenvalues I_1,I_2,I_3
- ➔orthogonal eigenvectors

$$\overleftrightarrow{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad i = 1,2,3$$

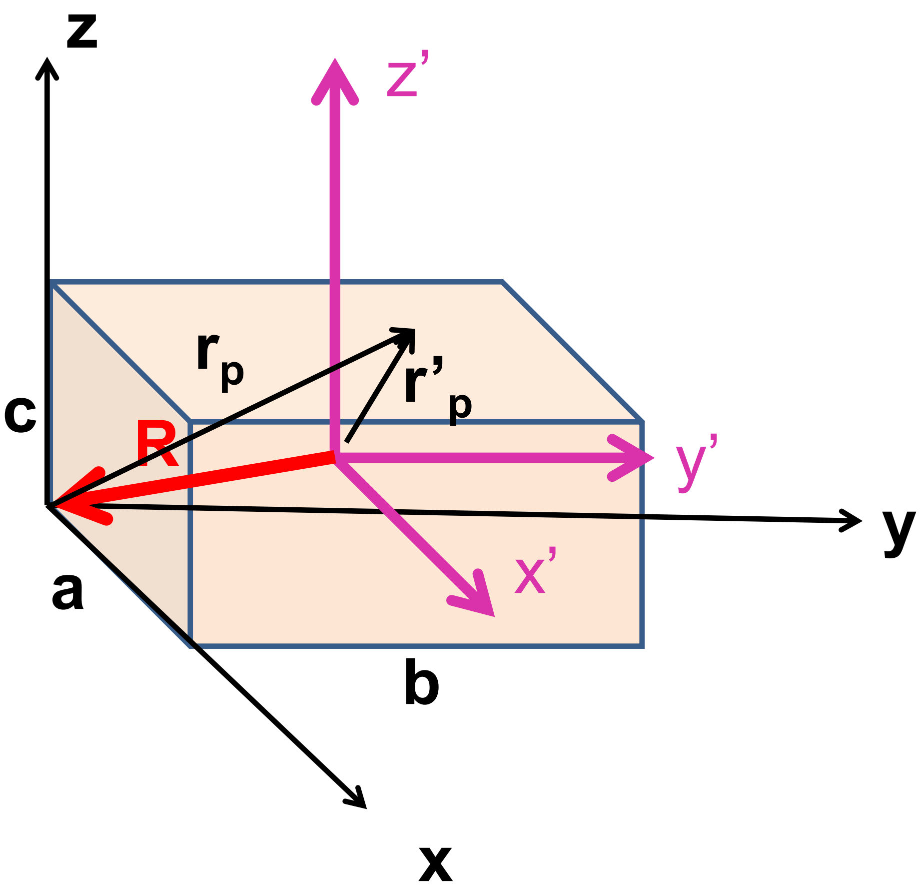
Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

For $a = b = c$:

$$I_1 = \frac{1}{6}Ma^2 \quad I_2 = \frac{11}{12}Ma^2 \quad I_3 = \frac{11}{12}Ma^2$$

Changing origin of rotation



$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj}\right)$$

$$I'_{ij} \equiv \sum_p m_p \left(\delta_{ij} r'^2_p - r'_{pi}\, r'_{pj}\right)$$

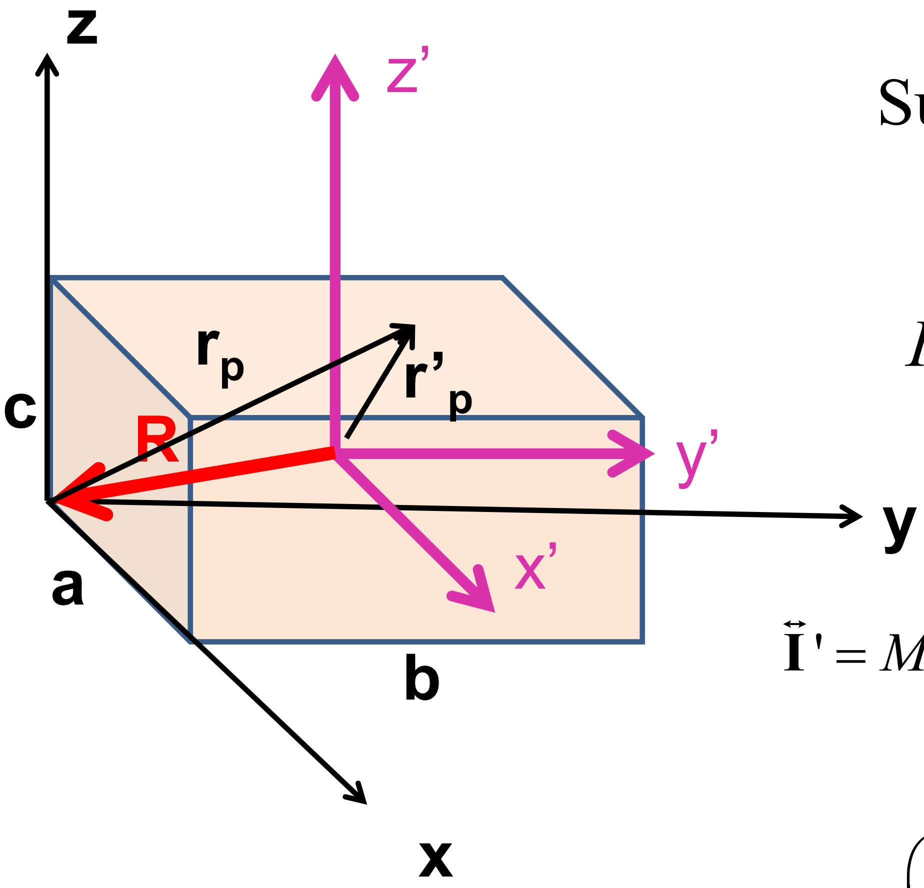
$$\mathbf{r}'_p = \mathbf{r}_p + \mathbf{R}$$

Define the center of mass:

$$\mathbf{r}_{CM} = \frac{\sum_p m_p \mathbf{r}_p}{\sum_p m_p} \equiv \frac{\sum_p m_p \mathbf{r}_p}{M}$$

$$I'_{ij} = I_{ij} + M\left(R^2\delta_{ij} - R_i R_j\right) + M\left(2\mathbf{r}_{CM}\cdot\mathbf{R}\,\delta_{ij} - r_{CMi} R_j - R_i r_{CMj}\right)$$

$$I'_{ij} = I_{ij} + M\left(R^2\delta_{ij} - R_iR_j\right) + M\left(2\mathbf{r}_{CM}\cdot\mathbf{R}\delta_{ij} - r_{CMi}R_j - R_ir_{CMj}\right)$$



Suppose that $\mathbf{R} = -\frac{a}{2}\hat{\mathbf{x}} - \frac{b}{2}\hat{\mathbf{y}} - \frac{c}{2}\hat{\mathbf{z}}$

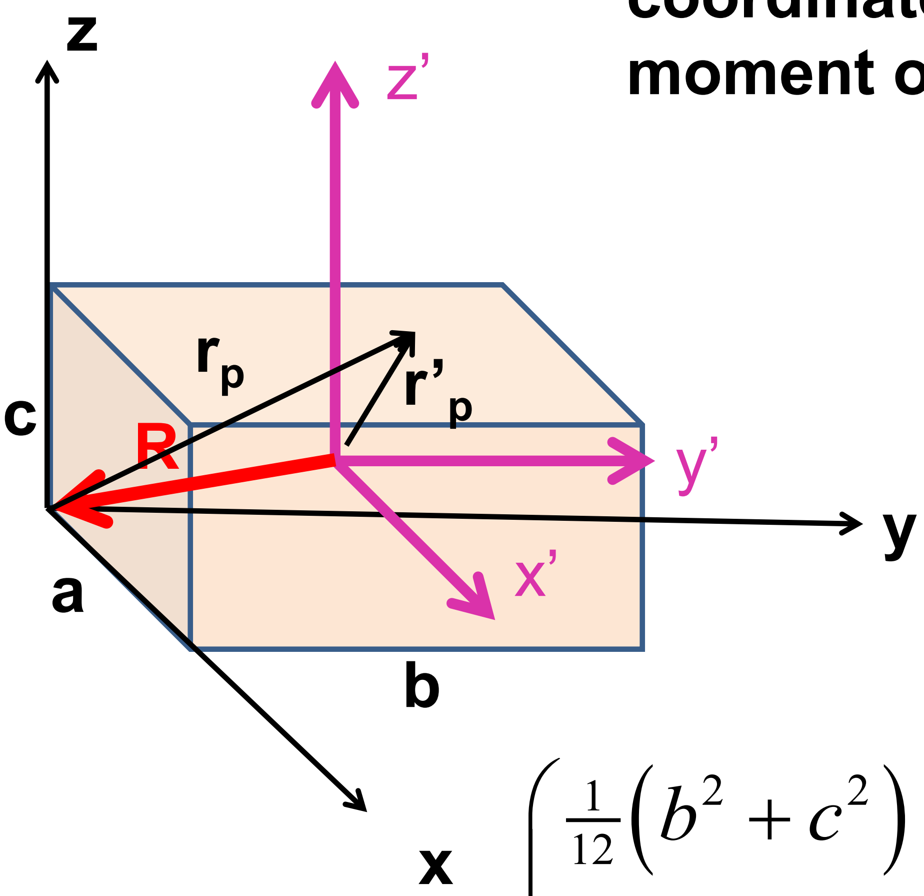
and $\mathbf{r}_{\mathrm{CM}} = -\mathbf{R}$

$$I'_{ij} = I_{ij} - M\left(R^2\delta_{ij} - R_iR_j\right)$$

$$\overleftrightarrow{\mathbf{I}}' = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

$$- \; M\begin{pmatrix} \frac{1}{4}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{4}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{4}\left(a^2+b^2\right) \end{pmatrix}$$

Note that changing origin of coordinate system changes moment of inertia tensor.



Note: This is a special case; changing the center of rotation does not necessarily result in a diagonal **I'**

$$\overleftrightarrow{\mathbf{I}}' = M\begin{pmatrix} \frac{1}{12}\left(b^2+c^2\right) & 0 & 0 \\ 0 & \frac{1}{12}\left(a^2+c^2\right) & 0 \\ 0 & 0 & \frac{1}{12}\left(a^2+b^2\right) \end{pmatrix}$$

Descriptions of rotation about a given origin

For general coordinate system

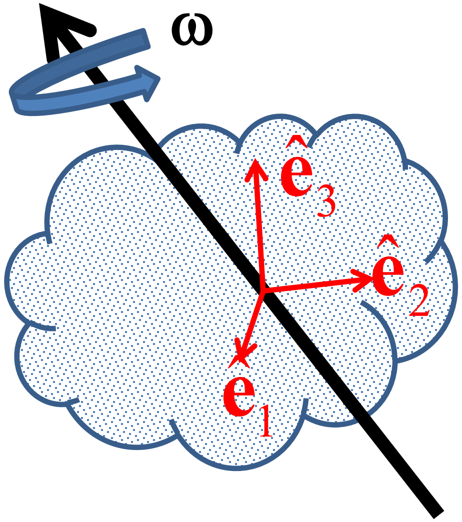
$$T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad\qquad i = 1,2,3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2$$



Descriptions of rotation about a given origin -- continued

Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L}$$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\overleftrightarrow{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad\qquad \boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1$$

$$+ \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations :

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1$$
$$+ \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3 = 0$$

Torqueless Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1+\tilde{\omega}_2\tilde{\omega}_3\left(I_3-I_2\right)=0$$

$$I_2\dot{\tilde{\omega}}_2+\tilde{\omega}_3\tilde{\omega}_1\left(I_1-I_3\right)=0$$

$$I_3\dot{\tilde{\omega}}_3+\tilde{\omega}_1\tilde{\omega}_2\left(I_2-I_1\right)=0$$

➔ Solution for symmetric object with $I_2=I_1$:

$$I_1\dot{\tilde{\omega}}_1+\tilde{\omega}_2\tilde{\omega}_3\left(I_3-I_1\right)=0$$

$$I_1\dot{\tilde{\omega}}_2+\tilde{\omega}_3\tilde{\omega}_1\left(I_1-I_3\right)=0$$

$$I_3\dot{\tilde{\omega}}_3=0 \qquad \Rightarrow \tilde{\omega}_3=(\text{constant})$$

Define: $\Omega\equiv\tilde{\omega}_3\dfrac{I_3-I_1}{I_1}$

$$\dot{\tilde{\omega}}_1=-\tilde{\omega}_2\Omega$$

$$\dot{\tilde{\omega}}_2=\tilde{\omega}_1\Omega$$

Solution of Euler equations for symmetric object continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

where $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

Solution: $\tilde{\omega}_1(t) = A\cos(\Omega t + \phi)$

$$\tilde{\omega}_2(t) = A\sin(\Omega t + \phi)$$

$$\tilde{\omega}_3(t) = \tilde{\omega}_3 \quad \text{(constant)}$$

$$T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2 = \frac{1}{2}I_1A^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$= I_1A\left(\cos\left(\Omega t + \varphi\right)\hat{\mathbf{e}}_1 + \sin\left(\Omega t + \varphi\right)\hat{\mathbf{e}}_2\right) + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

Torqueless Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

→ Solution for asymmetric object: $I_3 \neq I_2 \neq I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$

Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Solution for asymmetric object $I_3 \neq I_2 \neq I_1$:

Approximate solution --

Suppose: $\dot{\tilde{\omega}}_3 \approx 0$

Define: $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define: $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

Euler equations for asymmetric object continued

$$I_1\dot{\tilde{\omega}}_1+\tilde{\omega}_2\tilde{\omega}_3\left(I_3-I_2\right)=0$$

$$I_2\dot{\tilde{\omega}}_2+\tilde{\omega}_3\tilde{\omega}_1\left(I_1-I_3\right)=0$$

$$I_3\dot{\tilde{\omega}}_3+\tilde{\omega}_1\tilde{\omega}_2\left(I_2-I_1\right)=0$$

If $\dot{\tilde{\omega}}_3 \approx 0,$ Define: $\mathit{\Omega}_1 \equiv \tilde{\omega}_3 \frac{I_3-I_2}{I_1}$ $\mathit{\Omega}_2 \equiv \tilde{\omega}_3 \frac{I_3-I_1}{I_2}$

$$\dot{\tilde{\omega}}_1=-\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2=\Omega_2\tilde{\omega}_1$$

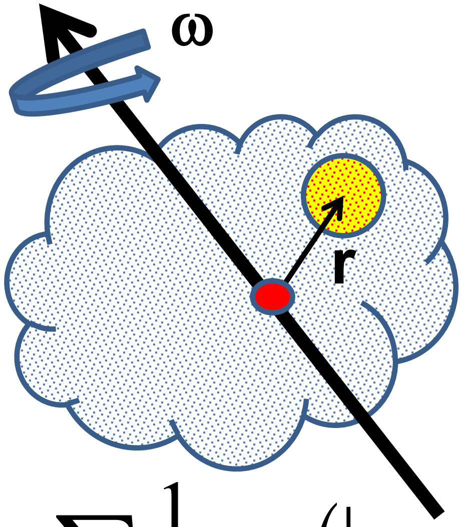
If Ω_1 and Ω_2 are both positive or both negative:

$$\tilde{\omega}_1(t)\approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t+\varphi\right)$$

$$\tilde{\omega}_2(t)\approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t+\varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

Summary of previous results describing rigid bodies rotating about a fixed origin ●



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

Kinetic energy: $T=\sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_p\cdot\mathbf{r}_p\right)-\left(\mathbf{r}_p\cdot\boldsymbol{\omega}\right)^2\right]$$

$$= \frac{1}{2}\boldsymbol{\omega}\cdot\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega} \qquad \overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p\left(\mathbf{1}r_p^2-\mathbf{r}_p\mathbf{r}_p\right)$$

Moment of inertia tensor

Matrix notation:

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\overleftrightarrow{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

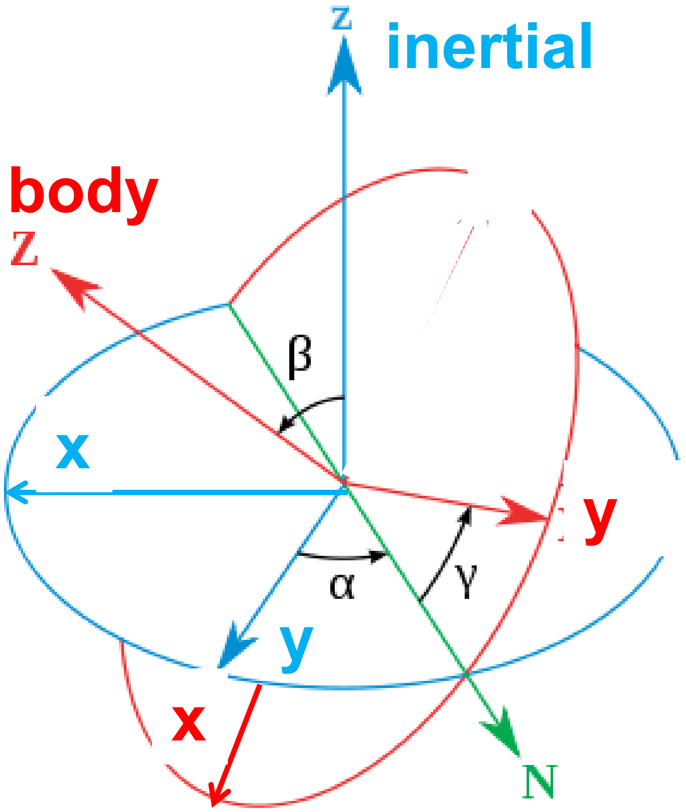
$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

Want to determine angular velocities $\omega_i(t)$

Transformation between body-fixed and inertial coordinate systems – Euler angles

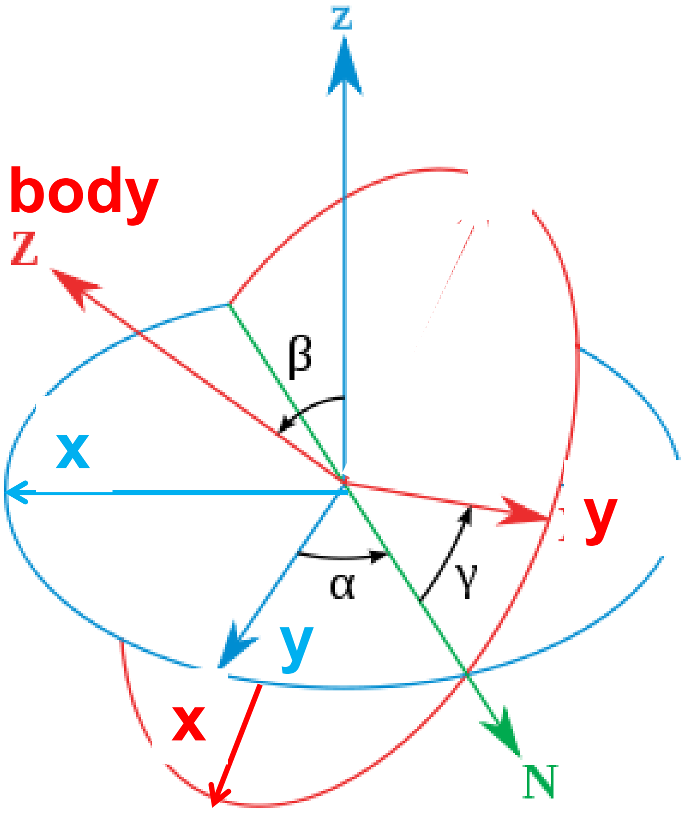


Comment – Since this is an old and intriguing subject, there are a lot of terminologies and conventions, not all of which are compatible. We are following the convention found in most quantum mechanics texts and NOT the convention found in most classical mechanics texts. Euler's main point is that any rotation can be described by 3 successive rotations about 3 different (not necessarily orthogonal) axes. In this case, one is along the inertial z axis and another is along the body fixed Z axis. The middle rotation is along an intermediate N axis.

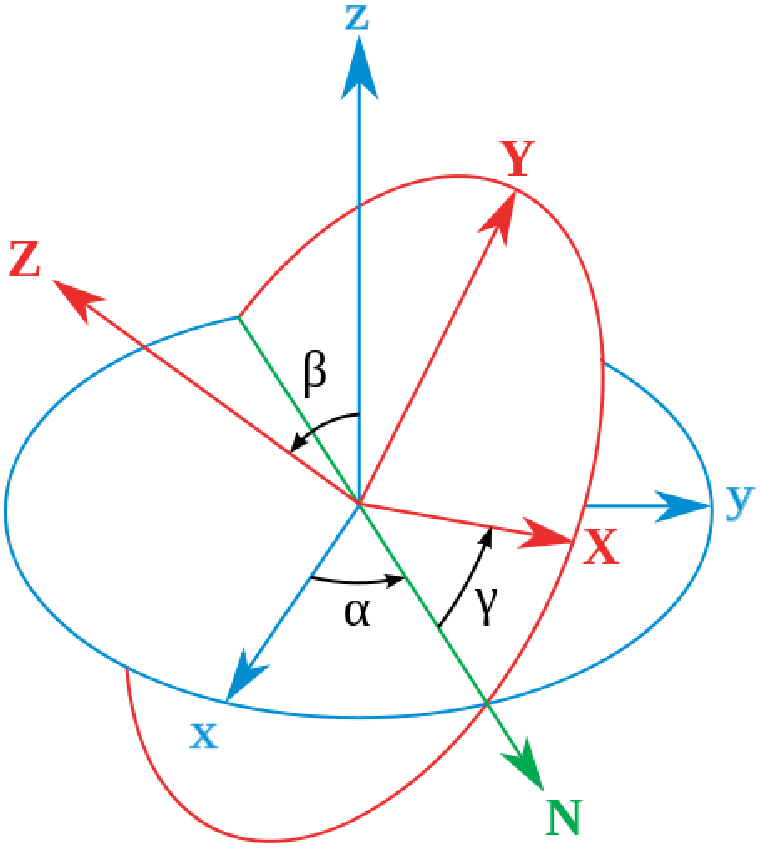
http://en.wikipedia.org/wiki/Euler_angles

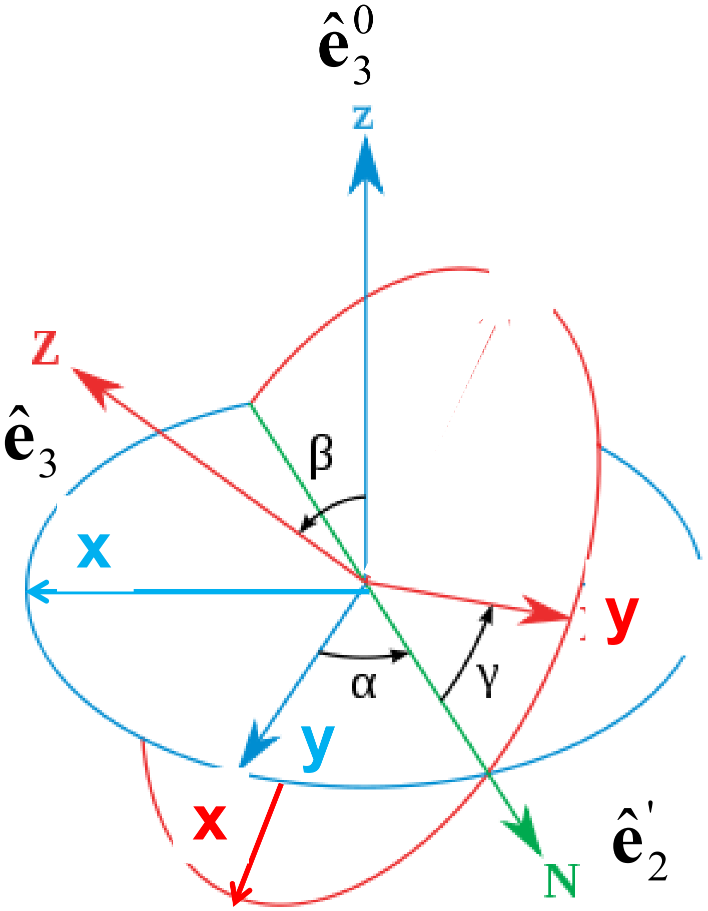
Comment on conventions

Our diagram



On web (for CM)



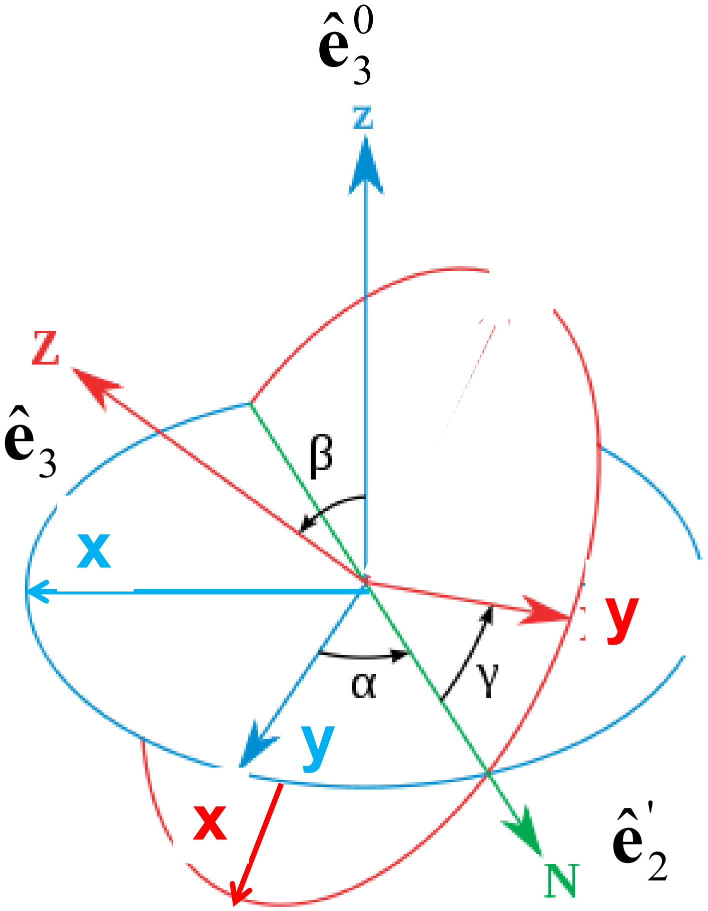


$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

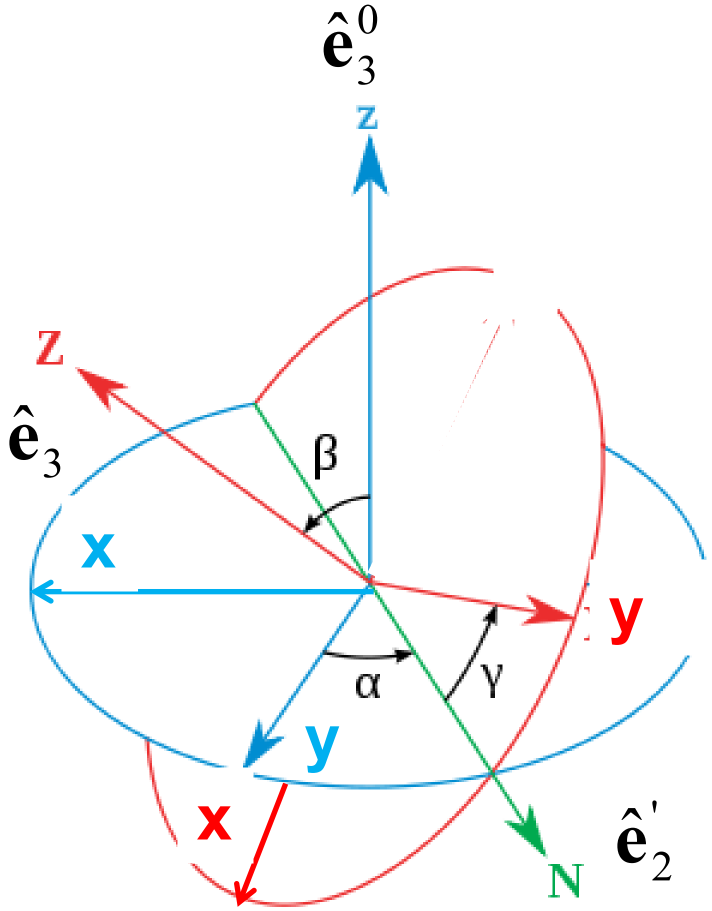


$$\hat{\mathbf{e}}_2' = \sin\gamma\;\hat{\mathbf{e}}_1 + \cos\gamma\;\hat{\mathbf{e}}_2$$

Matrix representation:

$$\hat{\mathbf{e}}_2' = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\hat{\mathbf{e}}_3^0 = -\sin\beta\ \hat{\mathbf{e}}'_1 + \cos\beta\ \hat{\mathbf{e}}'_3$$

$$= -\cos\gamma\sin\beta\ \hat{\mathbf{e}}_1 + \sin\gamma\sin\beta\ \hat{\mathbf{e}}_2 + \cos\beta\ \hat{\mathbf{e}}_3$$

Matrix representation:

$$\hat{\mathbf{e}}_3^0 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

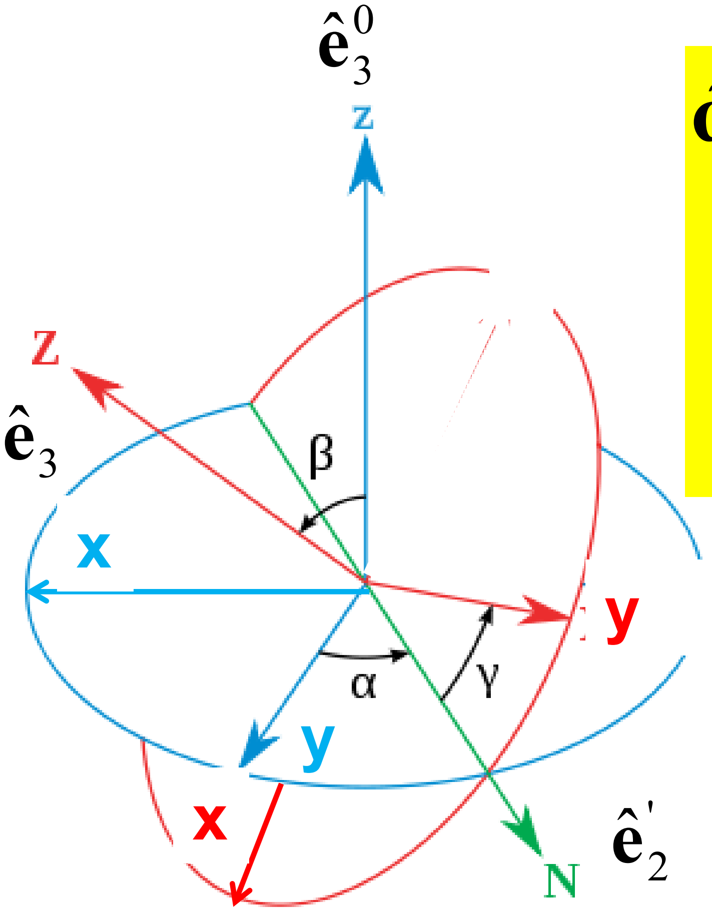
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma$$

$$\tilde{\omega}_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

Rotational kinetic energy

$$T\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\tilde{\omega}_1^2+\frac{1}{2}I_2\tilde{\omega}_2^2+\frac{1}{2}I_3\tilde{\omega}_3^2$$

$$=\frac{1}{2}I_1\left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right)+\dot{\beta}\sin\gamma\right]^2$$

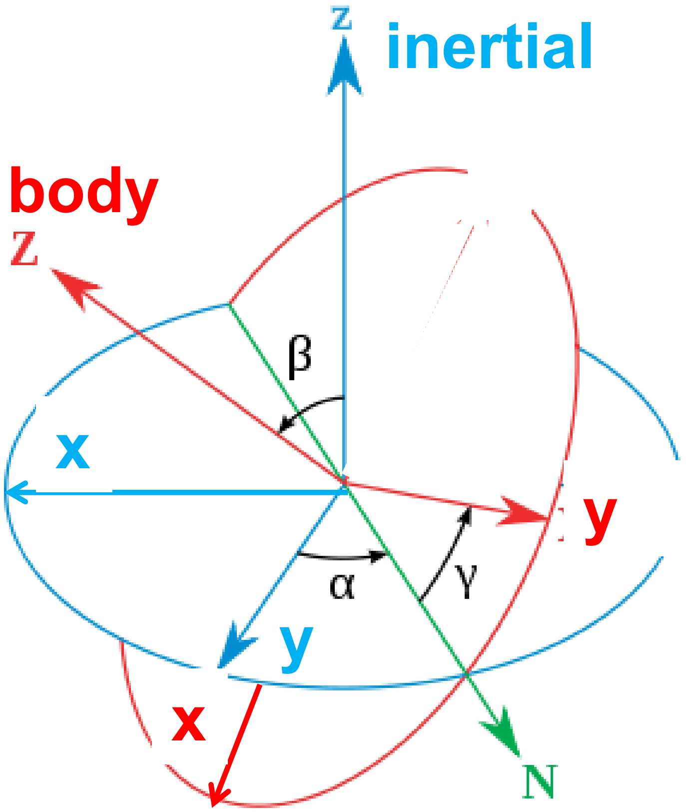
$$+\frac{1}{2}I_2\left[\dot{\alpha}\left(\sin\beta\sin\gamma\right)+\dot{\beta}\cos\gamma\right]^2$$

$$+\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2$$

If $I_1 = I_2$:

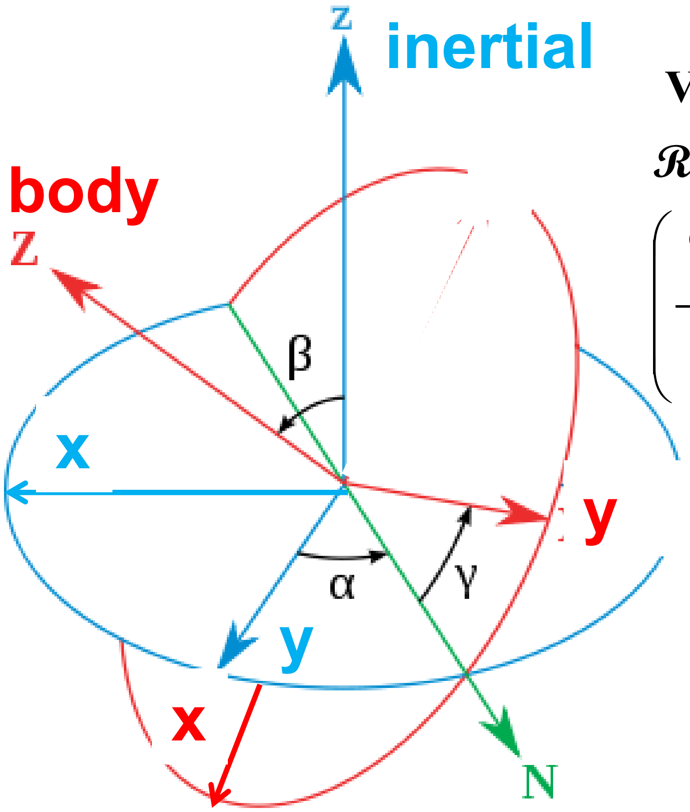
$$T\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)\ +\frac{1}{2}I_3\left(\dot{\alpha}\cos\beta+\dot{\gamma}\right)^2$$

Recap --
Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

General transformation between rotated coordinates – Euler angles



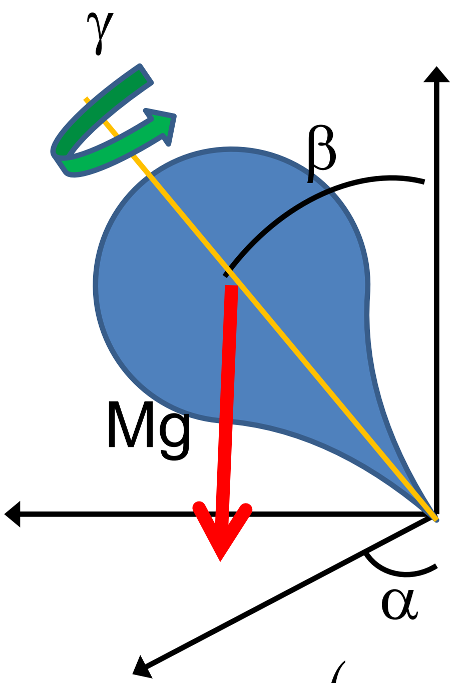
$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} =$$

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma})=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+$$

$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2-Mgl\cos\beta$$

$$L\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+$$

$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2-Mgl\cos\beta$$

Constants of the motion :

$$p_\alpha=\frac{\partial L}{\partial\dot{\alpha}}=I_1\dot{\alpha}\sin^2\beta+I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]\cos\beta$$

$$p_\gamma=\frac{\partial L}{\partial\dot{\gamma}}=I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]$$

$$E=\frac{1}{2}I_1\dot{\beta}^2+\frac{p_\gamma^2}{2I_3}+V_{eff}\left(\beta\right)$$

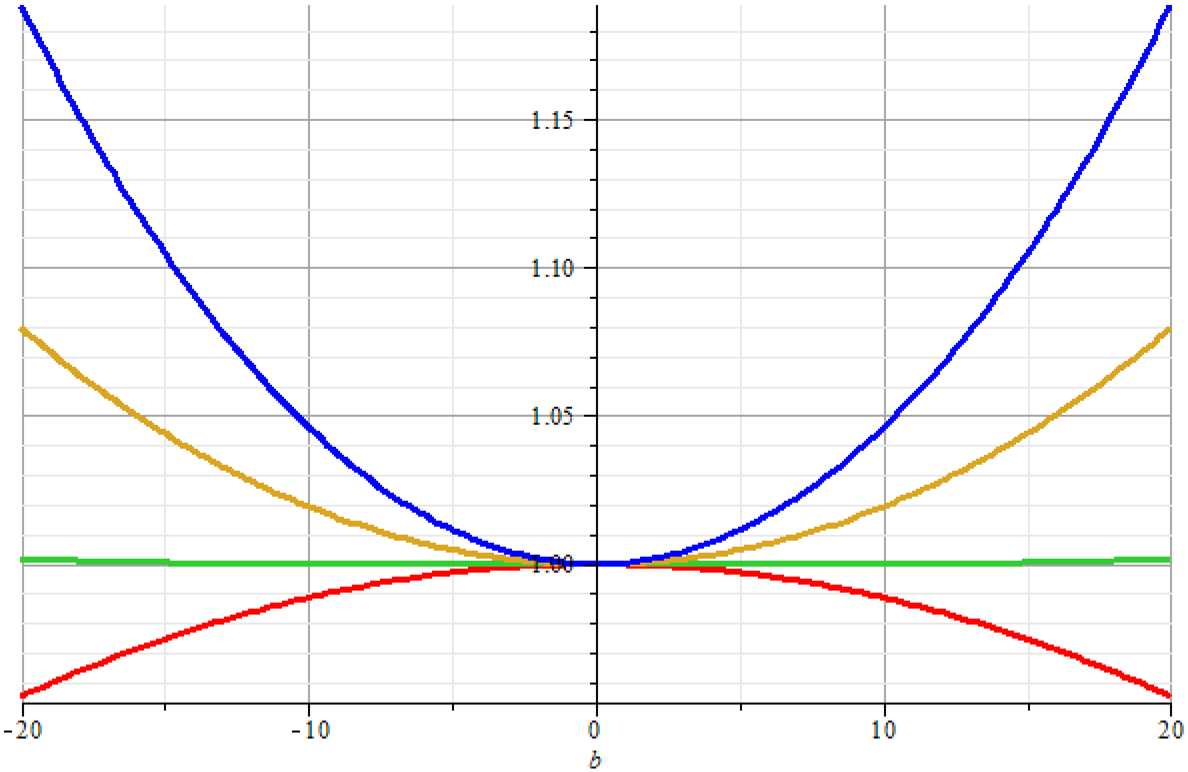
$$L\left(\beta,\dot{\beta}\right)=\frac{1}{2}I_1\dot{\beta}^2+\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+\frac{p_\gamma^2}{2I_3}-Mgl\cos\beta$$

$$V_{eff}\left(\beta\right)=\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{\left(p_\alpha - p_\gamma \cos\beta\right)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

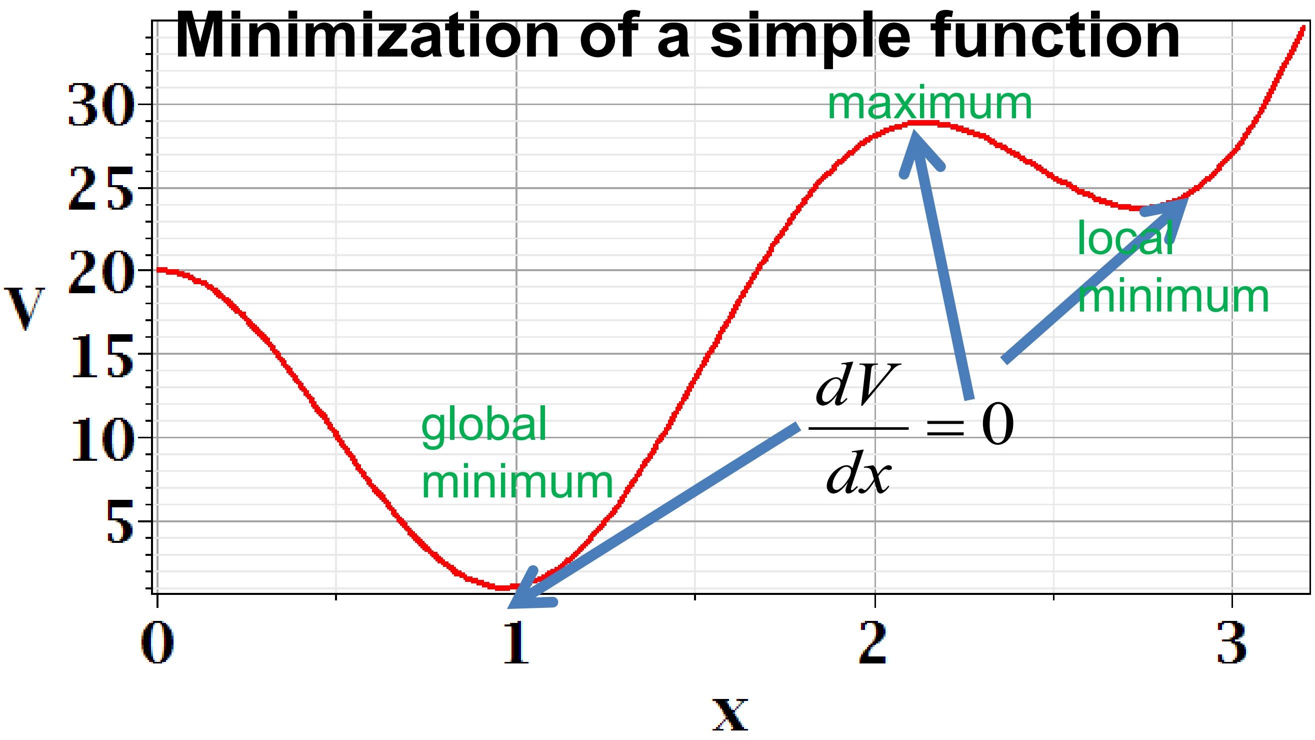
$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma \cos\beta\right)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

Stable/unstable solutions near β=0



Question: How do we decide stable/unstable solutions for the symmetric top motion?

Comment – When we discussed one dimensional motion, we discussed stable and unstable equilibrium points. At equilibrium dV/dx=0, but only when V(x) has a minimum at that point, is the system stable in the sense that for small displacements from equilibrium, there are restoring forces to move the system back to the equilibrium point.



Minimization of a simple function

maximum

local minimum

global minimum

$$\frac{dV}{dx} = 0$$

Suppose $\quad p_\alpha = p_\gamma \quad$ and $\quad \beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma{}^2}{2I_1} \frac{\left(1 - 1 + \frac{1}{2}\beta^2\right)^2}{\beta^2} + Mgl\left(1 - \frac{1}{2}\beta^2\right)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma{}^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

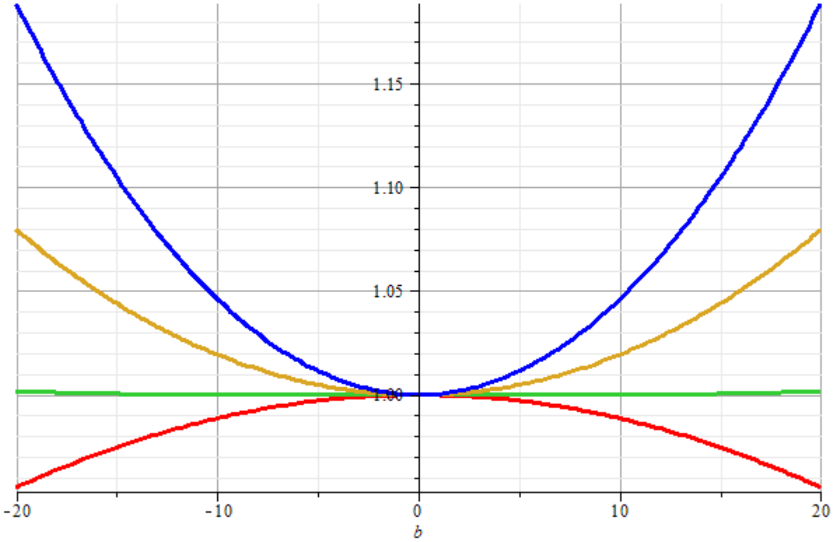
$\Rightarrow$ Stable solution if

$$p_\gamma \geq \sqrt{4MglI_1}$$

Note that

$p_\gamma = I_3\omega_3$

$\Rightarrow \omega_3$ must be sufficiently large for the top to maintain vertical orientation $\quad (\beta \approx 0)$.



http://www.physics.usyd.edu.au/~cross/SPINNING%20TOPS.htm

See also --

Home > American Journal of Physics > Volume 81, Issue 4 > 10.1119/1.4776195

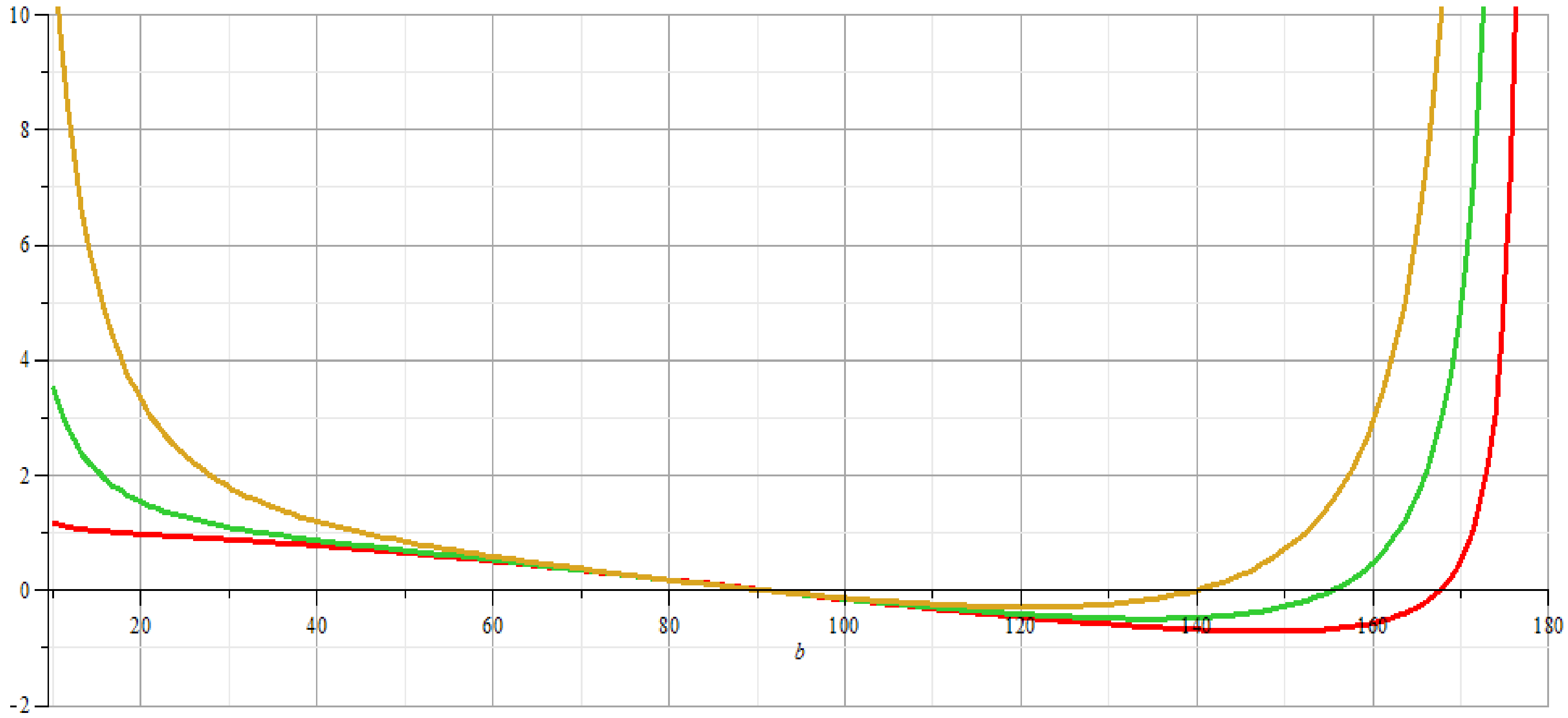
Full . Published Online: 18 March 2013 Accepted: December 2012

The rise and fall of spinning tops

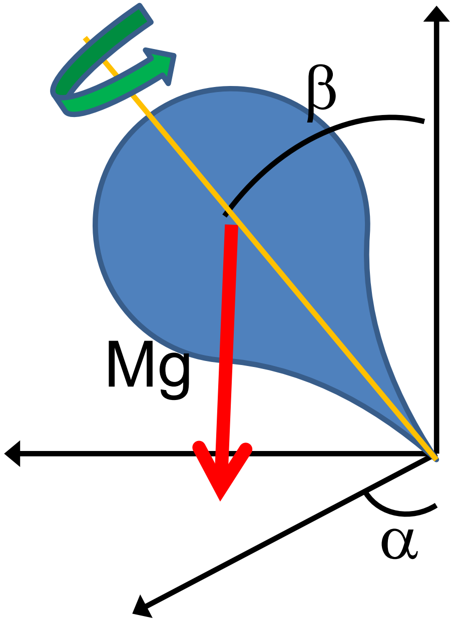
American Journal of Physics **81**, 280 (2013); https://doi.org/10.1119/1.4776195

More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$



https://drive.google.com/file/d/0B14RyYwpwSDNcXdxTWI3OExHX1k/view



Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\cos\beta$$

$$= I_1\dot{\alpha}\sin^2\beta + p_\gamma\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$