



PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

Notes for Lecture 12 – Chap. 5 (F &W)

Rotational motion of rigid bodies

1. Comment about HW #7
2. Review of rigid body motion
3. Euler angles
4. Symmetric top motion

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space	
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	#8
10	Mon, 9/18/2023	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/20/2023	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/22/2023	Chap. 5	Dyanmics of ridgid bodies	#11
13	Mon, 9/25/2023			
14	Wed, 9/27/2023			
15	Fri, 9/29/2023			

PHY 711 -- Assignment #11

Assigned: 9/22/2023 Due: 9/25/2023

This is an opportunity to re-examine HW 7 and earn more points for your corrected solution (up to 90 percent of the 2 point total).

PHY 711 – Assignment #7

Assigned: 09/11/2023 Due: 09/18/2023

The material for this exercise is covered in the lecture notes and in Chapters 3 and 6 of Fetter and Walecka.

1. A particle of mass m and charge q is subjected to a vector potential $\mathbf{A}(\mathbf{r}, t) = -(E_0ct + B_0x)\hat{\mathbf{z}}$. (Note that we are using the cgs Gaussian units of your text book.) Here E_0 denotes a constant electric field amplitude and B_0 denotes a constant magnetic field amplitude. The initial particle position is $\mathbf{r}(0) = 0$ and the initial particle velocity is $\dot{\mathbf{r}}(0) = 0$.
 - (a) Determine the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ which describes the particle's motion.
 - (b) Write the Euler-Lagrange equations for this system.
 - (c) Find the particle trajectories $x(t)$, $y(t)$, $z(t)$ by solving the equations and imposing the given initial conditions.
 - (d) Determine the Hamiltonian for this system and evaluate dH/dt .

PHY 711 – Assignment #7

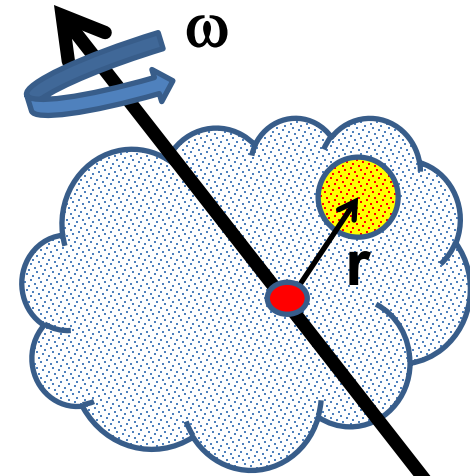
Assigned: 09/11/2023 Due: 09/18/2023

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Summary of previous results
describing rigid bodies rotating
about a fixed origin ●

$$\left(\frac{d\mathbf{r}}{dt} \right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$




Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left| \boldsymbol{\omega} \times \mathbf{r}_p \right| \right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p \right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p \right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega} \right) \left(\mathbf{r}_p \cdot \mathbf{r}_p \right) - \left(\mathbf{r}_p \cdot \boldsymbol{\omega} \right)^2 \right]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \vec{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \vec{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$

 Moment of inertia tensor
Matrix notation:

$$\vec{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \quad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

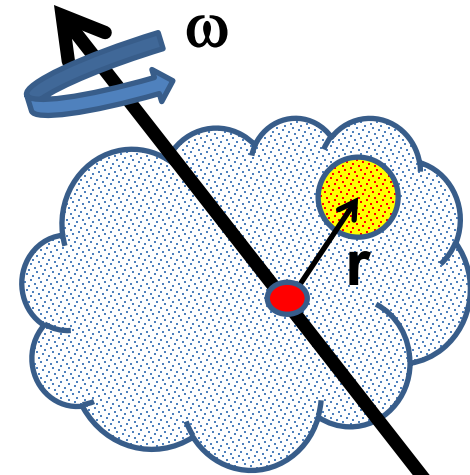
For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes

moment of inertia tensor: $\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad i = 1, 2, 3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \quad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

Continued -- summary of previous results describing rigid bodies rotating about a fixed origin ●



$$\left(\frac{d\mathbf{r}}{dt} \right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\text{Angular momentum: } \mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times (\boldsymbol{\omega} \times \mathbf{r}_p)$$

$$\mathbf{L} = \sum_p m_p \left[\boldsymbol{\omega} (\mathbf{r}_p \cdot \mathbf{r}_p) - \mathbf{r}_p (\mathbf{r}_p \cdot \boldsymbol{\omega}) \right]$$

$$\mathbf{L} = \vec{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \vec{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right)$$

Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Time derivative:
$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

$$\begin{aligned} \frac{d\mathbf{L}}{dt} = & I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \\ & \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3 \end{aligned}$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$
$$\tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3$$

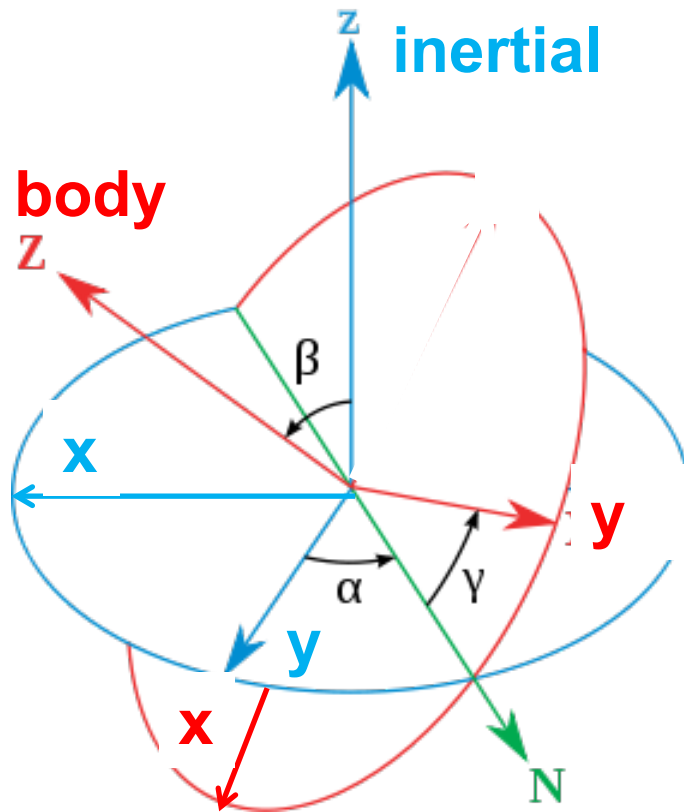
$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Want to determine
angular velocities $\omega_i(t)$

Transformation between body-fixed and inertial coordinate systems – Euler angles

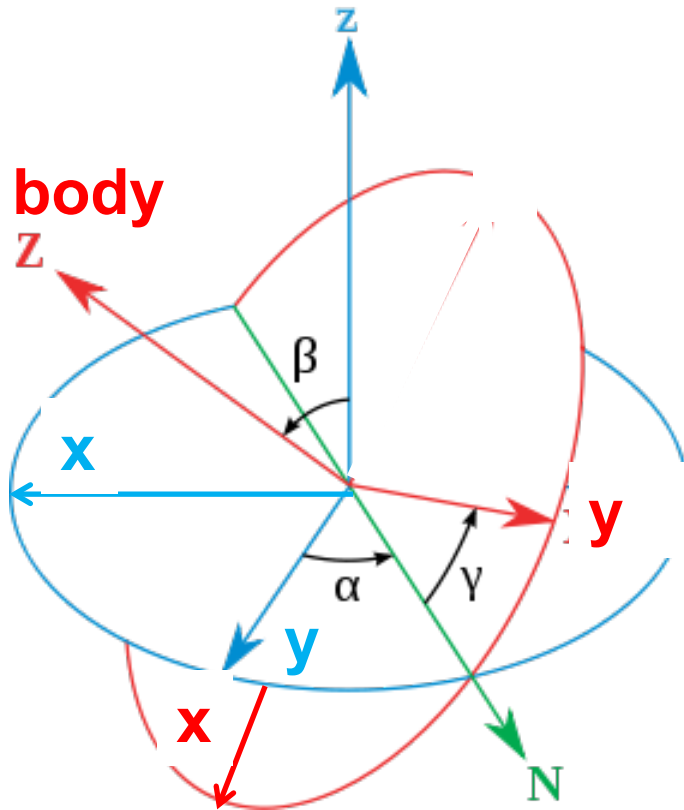


Comment – Since this is an old and intriguing subject, there are a lot of terminologies and conventions, not all of which are compatible. We are following the convention found in most quantum mechanics texts and NOT the convention found in most classical mechanics texts. Euler's main point is that any rotation can be described by 3 successive rotations about 3 different (not necessarily orthogonal) axes. In this case, one is along the inertial z axis and another is along the body fixed Z axis. The middle rotation is along an intermediate N axis.

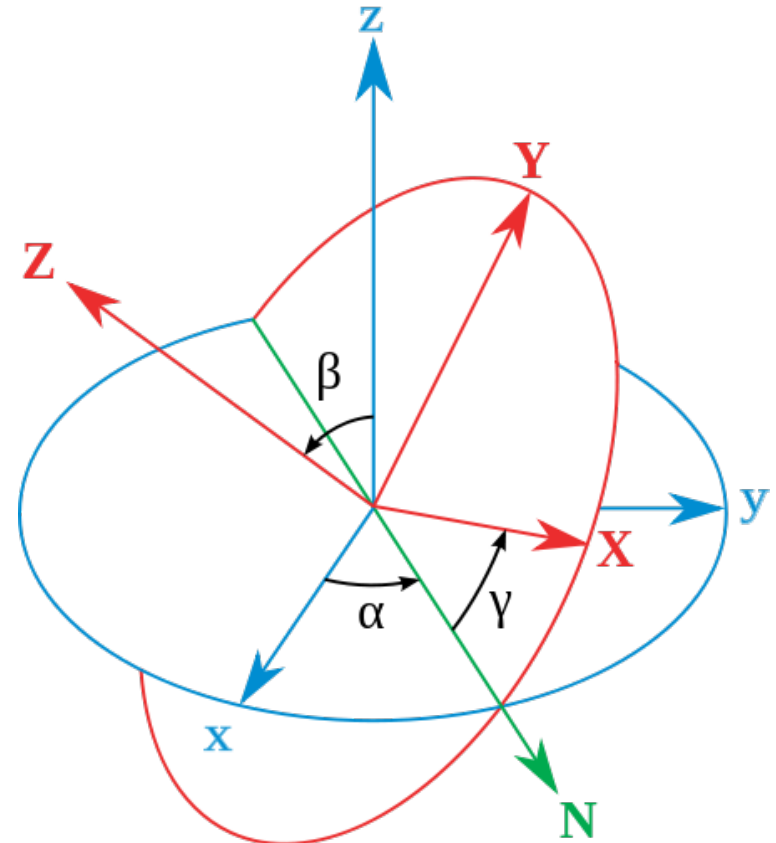
http://en.wikipedia.org/wiki/Euler_angles

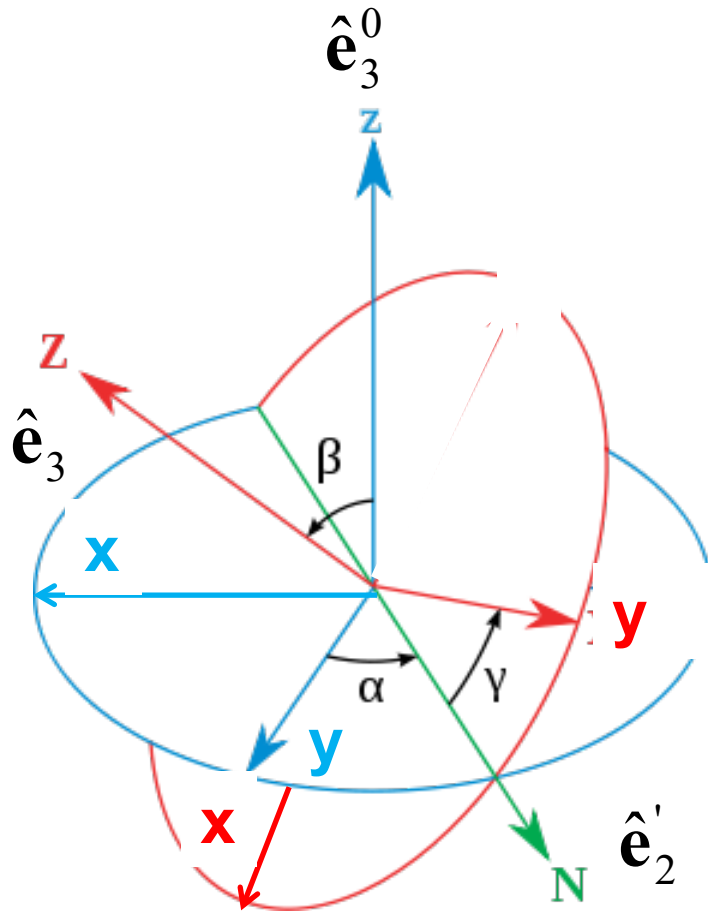
Comment on conventions

Our diagram



On web (for CM)





$$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2' + \dot{\gamma} \hat{e}_3$$

Need to express all components in body-fixed frame:

$$\tilde{\omega} = \tilde{\omega}_1 \hat{e}_1 + \tilde{\omega}_2 \hat{e}_2 + \tilde{\omega}_3 \hat{e}_3$$

When the dust clears --

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

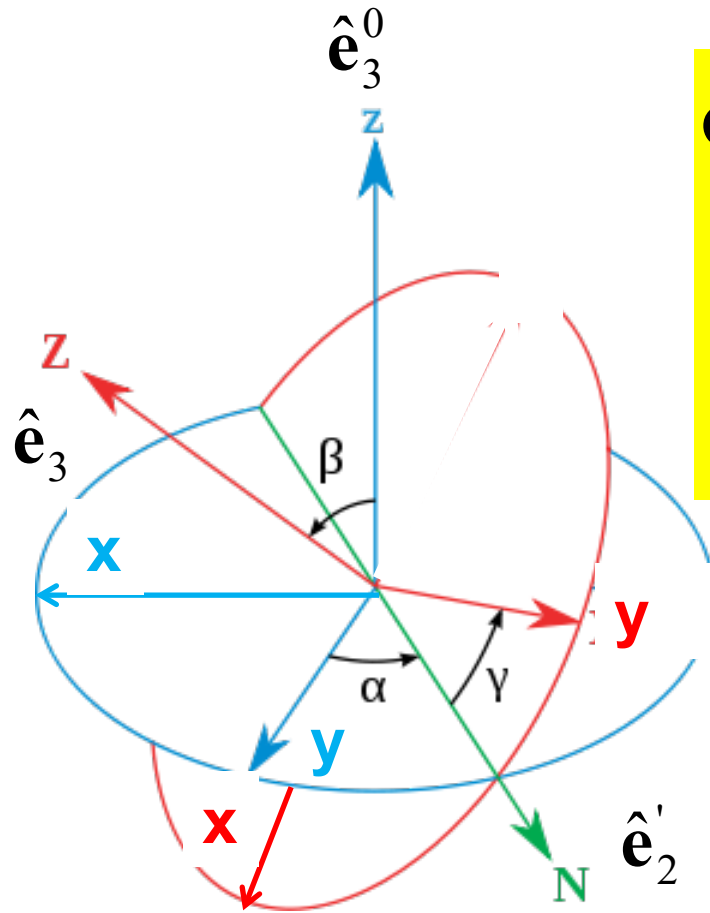
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma$$

$$\tilde{\omega}_3 = \dot{\alpha} \cos \beta + \dot{\gamma}$$

$$\tilde{\omega} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$



$$\begin{aligned} \tilde{\omega} = & \left[\dot{\alpha} (-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma \right] \hat{\mathbf{e}}_1 \\ & + \left[\dot{\alpha} (\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma \right] \hat{\mathbf{e}}_2 \\ & + \left[\dot{\alpha} \cos \beta + \dot{\gamma} \right] \hat{\mathbf{e}}_3 \end{aligned}$$

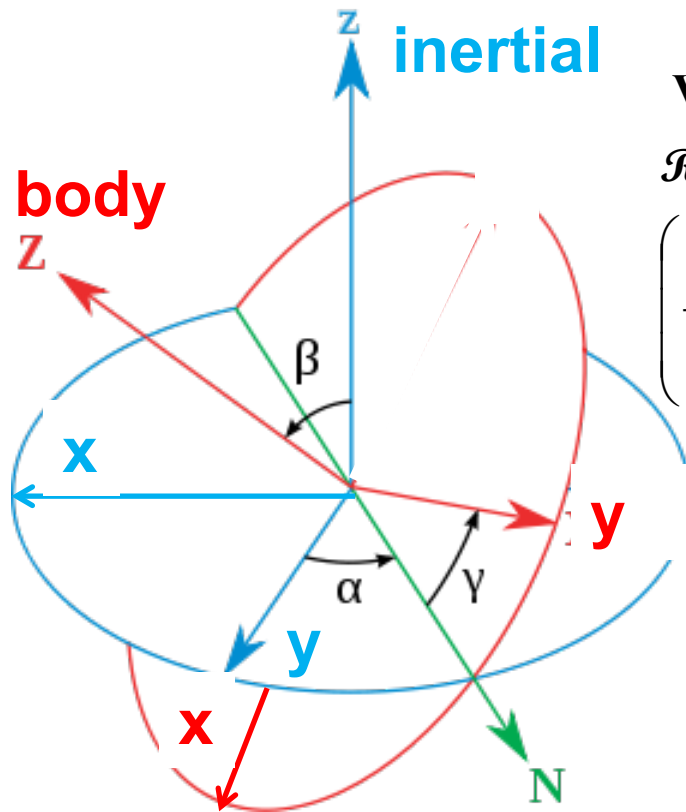
Rotational kinetic energy

$$\begin{aligned} T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2 \\ &= \frac{1}{2} I_1 \left[\dot{\alpha} (-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma \right]^2 \\ &\quad + \frac{1}{2} I_2 \left[\dot{\alpha} (\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma \right]^2 \\ &\quad + \frac{1}{2} I_3 \left[\dot{\alpha} \cos \beta + \dot{\gamma} \right]^2 \end{aligned}$$

If $I_1 = I_2$:

$$T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2$$

General transformation between rotated coordinates – Euler angles



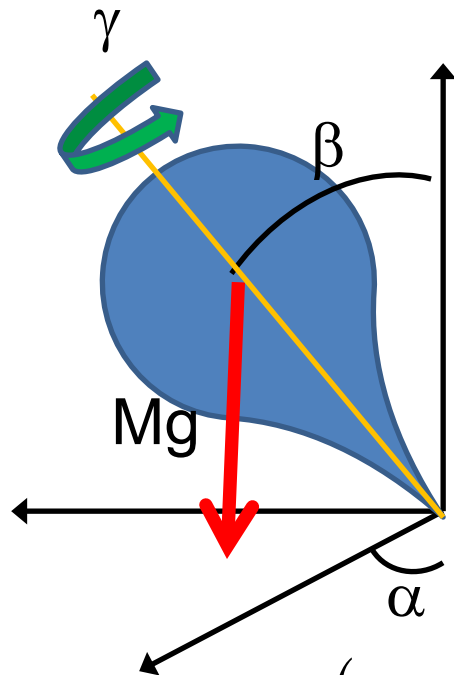
$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} =$$

$$\begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta$$

$$L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$

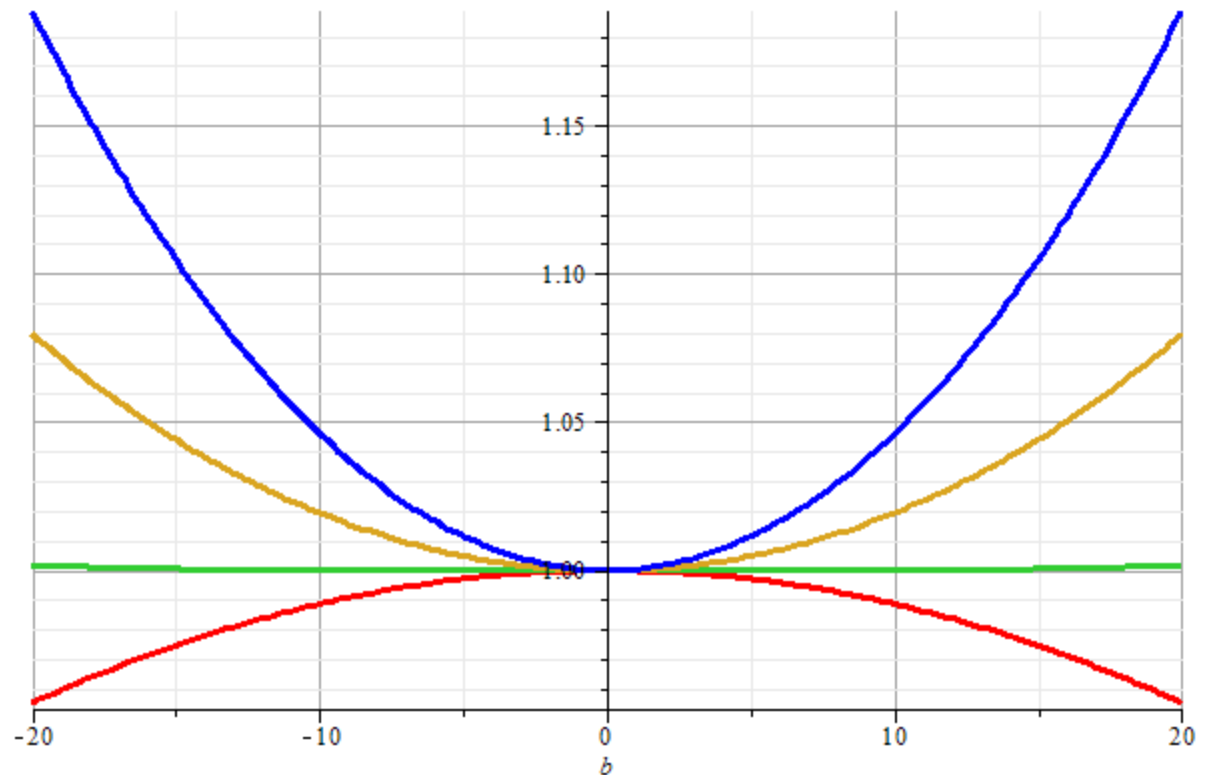
$$L(\beta, \dot{\beta}) = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - Mgl \cos \beta$$

$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

Stable/unstable
solutions near
 $\beta=0$



What happens when $\beta \rightarrow 0$?

Constants of the motion:

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{\text{eff}}(\beta)$$

$$L(\beta, \dot{\beta}) = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - Mgl \cos \beta$$

$$V_{\text{eff}}(\beta) = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$

$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

Note that for $\beta \approx 0$, $\cos \beta \approx 1 - \frac{1}{2} \beta^2 + \dots$

$$\sin \beta \approx \beta$$

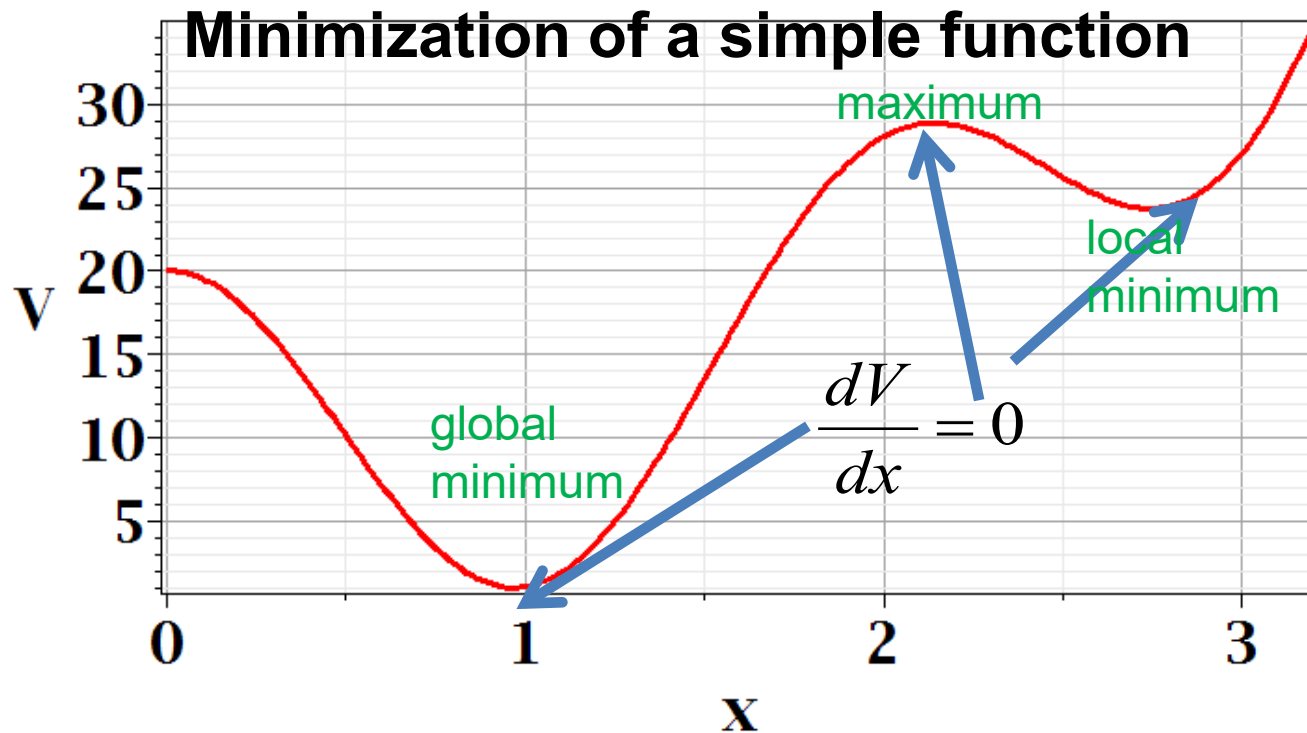
If $p_\alpha = p_\gamma$: $V_{eff}(\beta) \approx \frac{p_\gamma^2}{8I_1} \beta^2 + Mgl \left(1 - \frac{1}{2} \beta^2 \right) + \dots$

$$E - \frac{p_\gamma^2}{2I_3} \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{8I_1} \beta^2 + Mgl \left(1 - \frac{1}{2} \beta^2 \right) + \dots$$

$$E - \frac{p_\gamma^2}{2I_3} \approx \frac{1}{2} I_1 \dot{\beta}^2 + Mgl + \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2} \right) \beta^2 + \dots$$

Question: How do we decide stable/unstable solutions for the symmetric top motion?

Comment – When we discussed one dimensional motion, we discussed stable and unstable equilibrium points. At equilibrium $dV/dx=0$, but only when $V(x)$ has a minimum at that point, is the system stable in the sense that for small displacements from equilibrium, there are restoring forces to move the system back to the equilibrium point.



Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_1} \frac{(1 - 1 + \frac{1}{2} \beta^2)^2}{\beta^2} + Mgl(1 - \frac{1}{2} \beta^2)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2} \right) \beta^2 + Mgl$$

\Rightarrow Stable solution if

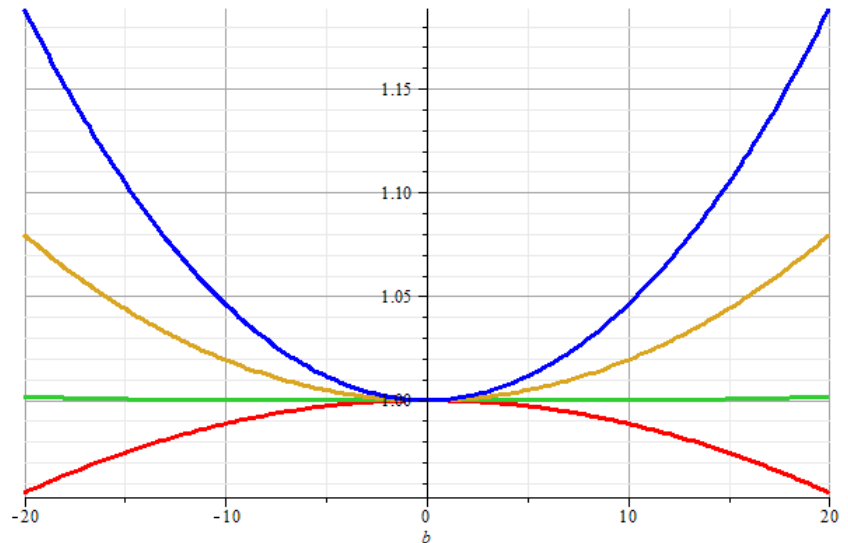
$$p_\gamma \geq \sqrt{4MglI_1}$$

Note that

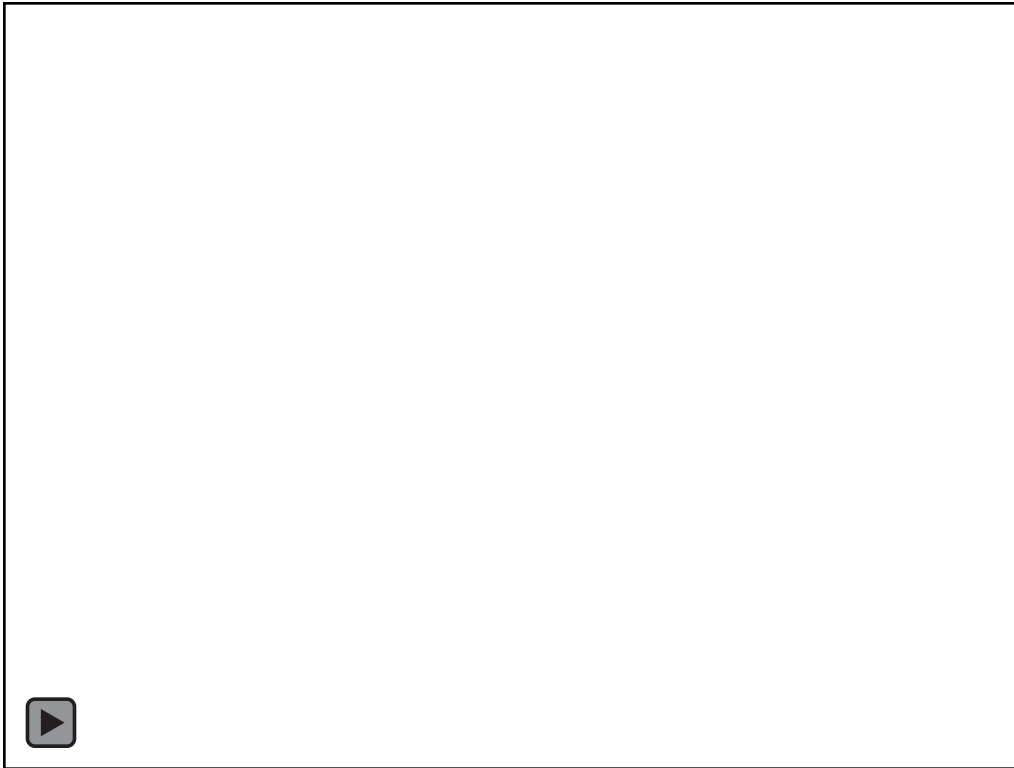
$$p_\gamma = I_3 \omega_3$$

$\Rightarrow \omega_3$ must be sufficiently large


for the top to maintain vertical orientation ($\beta \approx 0$).



<http://www.physics.usyd.edu.au/~cross/SPINNING%20TOPS.htm>



[Home](#) > [American Journal of Physics](#) > [Volume 81, Issue 4](#) > [10.1119/1.4776195](https://doi.org/10.1119/1.4776195)

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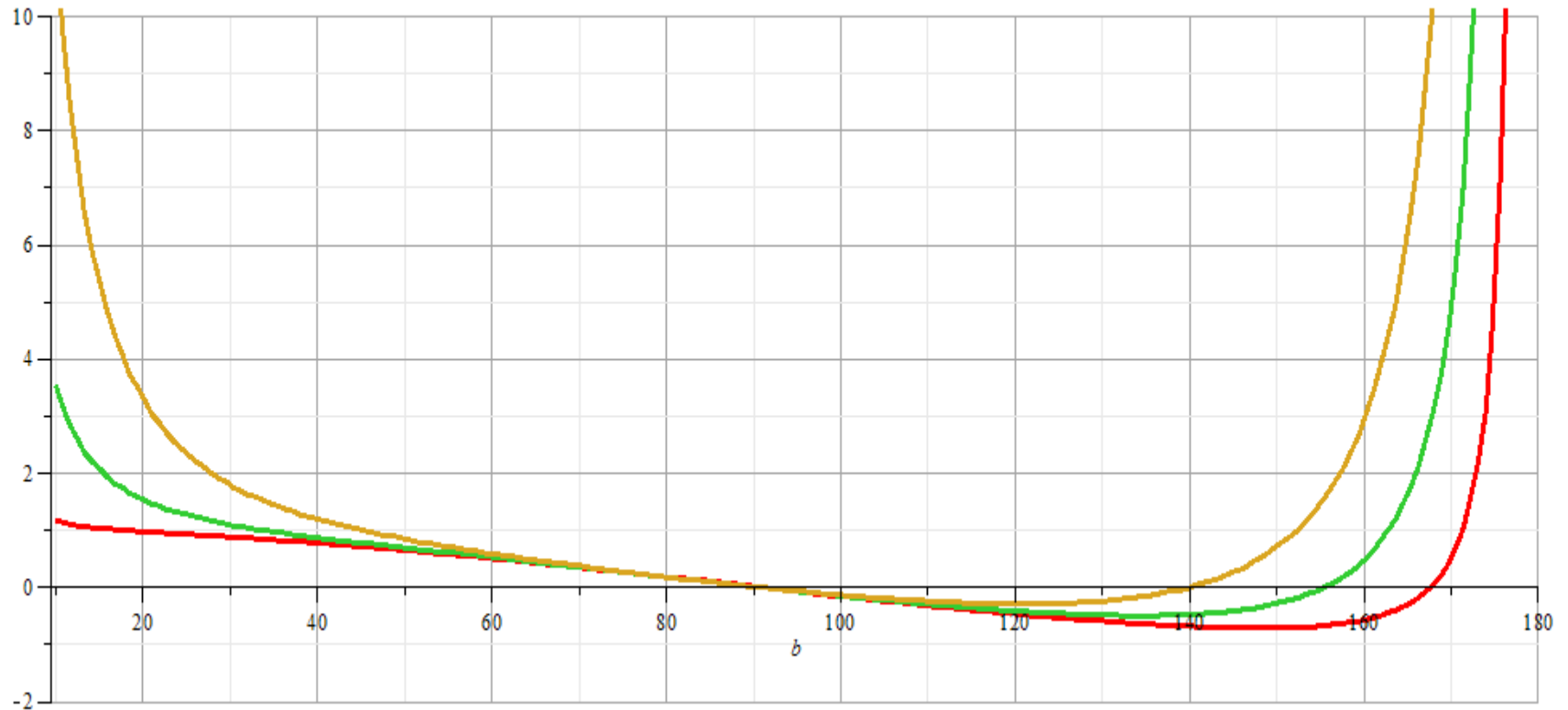
See also --

The rise and fall of spinning tops

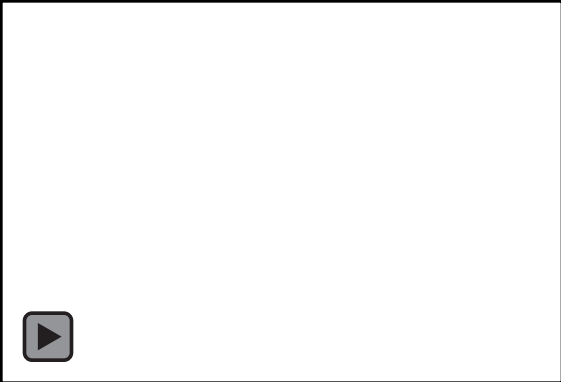
American Journal of Physics **81**, 280 (2013); <https://doi.org/10.1119/1.4776195>

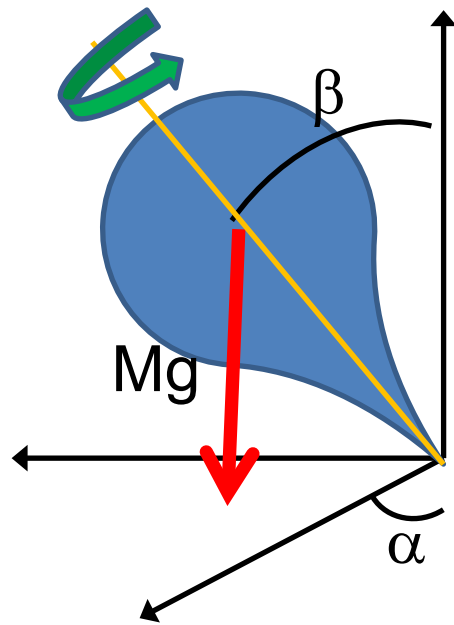
More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$



<https://drive.google.com/file/d/0B14RyYwpwSDNcXdxTWI3OExHX1k/view>





Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$= I_1 \dot{\alpha} \sin^2 \beta + p_\gamma \cos \beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$