PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 15: Scattering analysis – Chap. 1 (F & W)

1. Definition of differential scattering cross section
2. Calculation of particle trajectories for a central potential
3. Relation of particle trajectories to the differential scattering cross section
4. Example of Rutherford scattering

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space	
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	#8
10	Mon, 9/18/2023	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/20/2023	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/22/2023	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/25/2023	Chap. 1	Scattering analysis	#12
14	Wed, 9/27/2023	Chap. 1	Scattering analysis	#13
15	Fri, 9/29/2023	Chap. 1	Scattering analysis	#14
16	Mon, 10/2/2023			
17	Wed, 10/4/2023			
18	Fri, 10/6/2023			
22	Mon, 10/9/2023			

PHY 711 – Assignment #14

Assigned: 09/29/2023 Due: 10/02/2023

1. Suppose that a particle is scattered by a very massive target particle such that energy and angular momentum are conserved. The trajectory of the scattering particle is found to have an impact parameter b which depends on the scattering angle θ according to the formula

$$b(\theta) = K \left| \frac{1}{\sin(\theta/2)} \right|,$$

where K denotes a constant which depends on energy and other parameters. What is the differential cross section for this process?

Scattering theory:

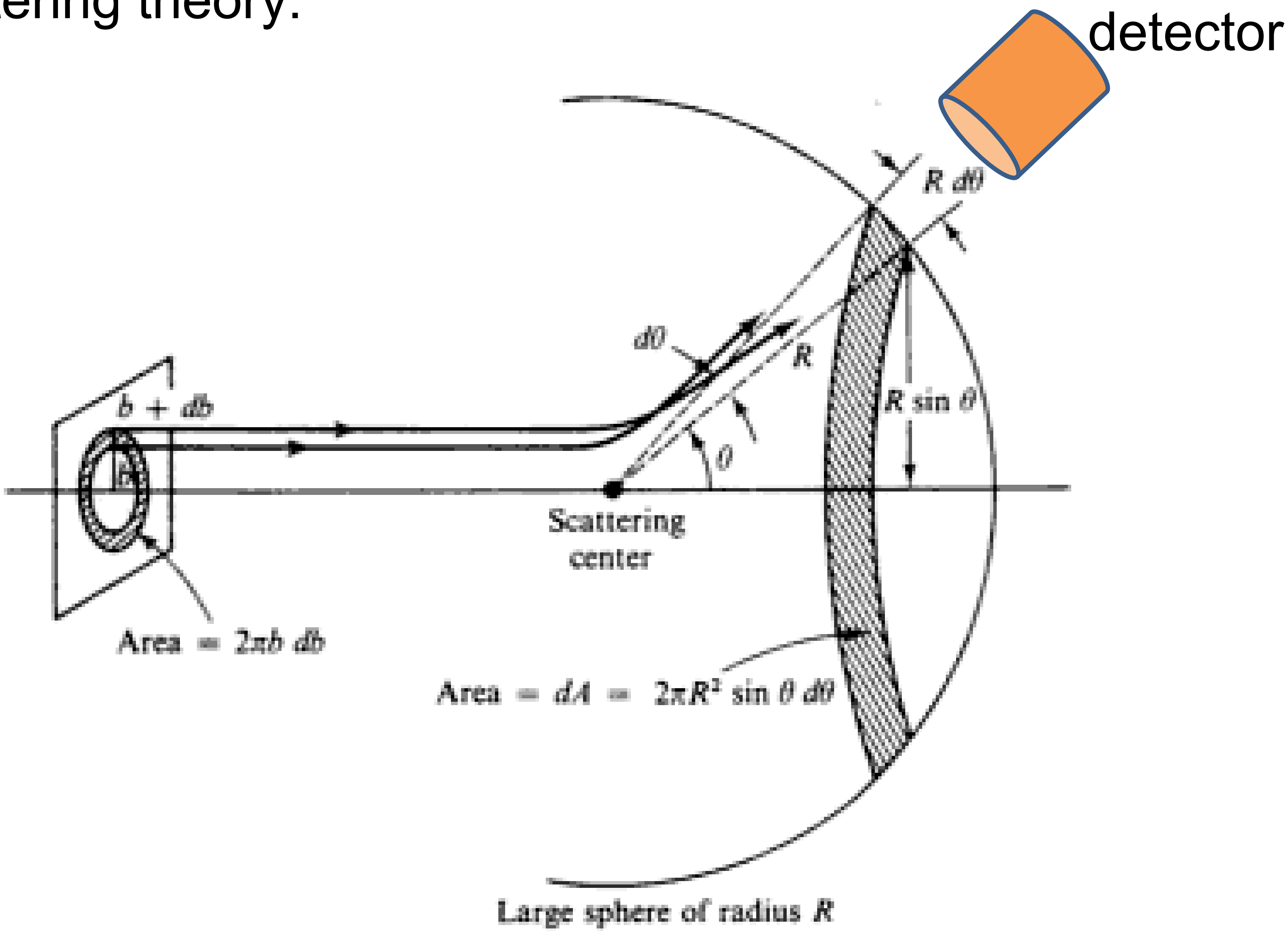


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Scattering theory:

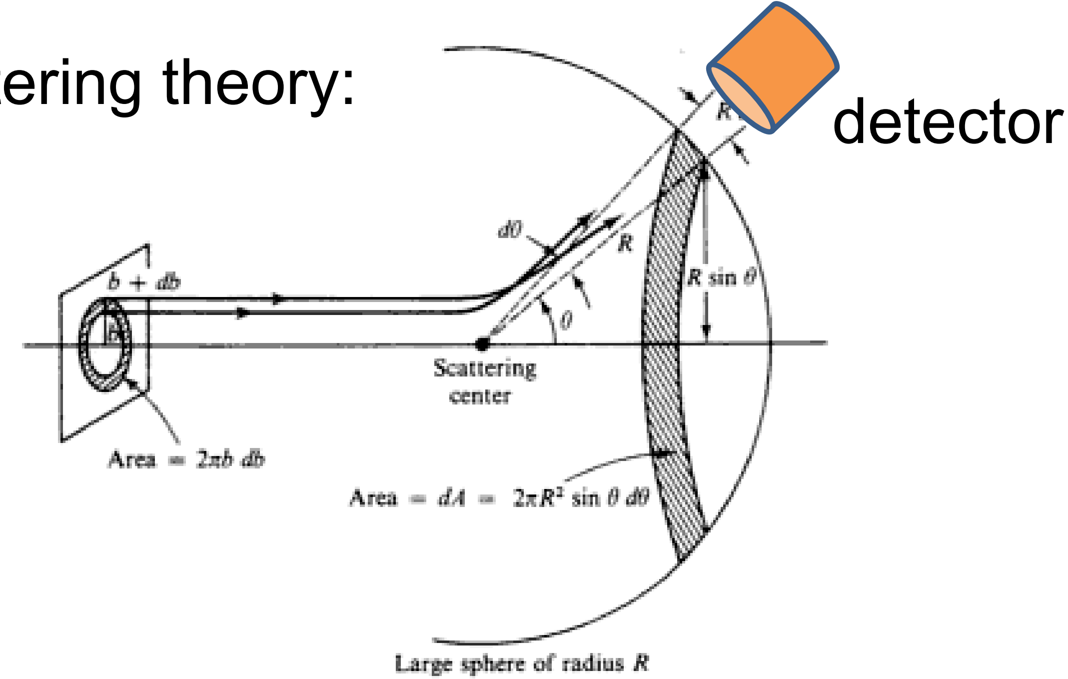


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Some reasons that scattering theory is useful:

1. It allows comparison between measurement and theory
2. The analysis depends on knowledge of the scattering particles when they are far apart
3. The scattering results depend on the interparticle interactions

Standard measure of differential cross section

Differential cross section

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle per unit time}}{\text{Number of incident particles per unit area per unit time per solid angle}}$$

$$= \text{Area of incident beam that is scattered into detector per solid angle at angle } \theta$$

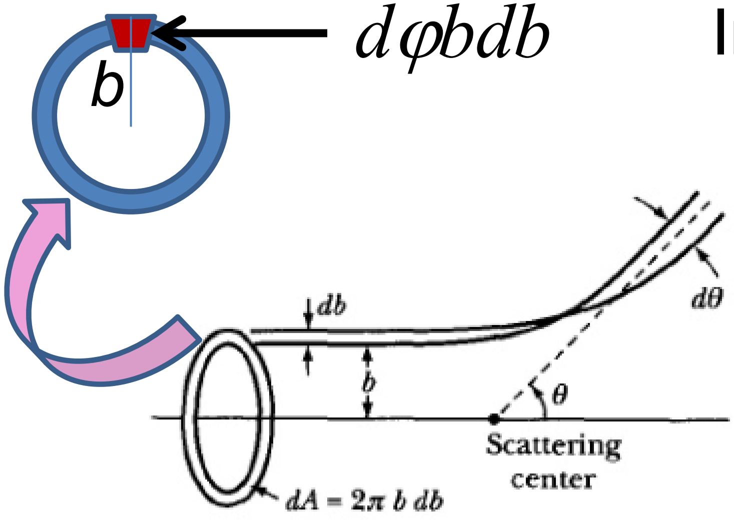
Standardization of scattering experiments --

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle per unit time}}{\text{Number of incident particles per unit area per unit time}}$$

$$= \text{Area of incident beam that is scattered into detector}$$

$$\text{at angle } \theta$$

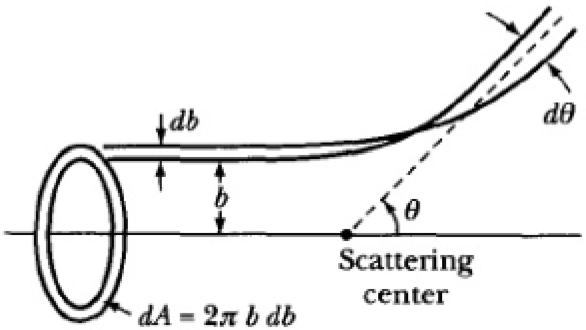


Impact parameter: *b*

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi\ b\ db}{d\phi\ \sin\theta\ d\theta} = \left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



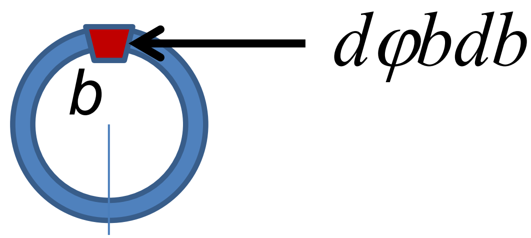


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi\ b\ db}{d\phi\ \sin\theta\ d\theta} = \left|\frac{b}{\sin\theta}\right|\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in φ

https://www.aps.org/publications/apsnews/200605/history.cfm

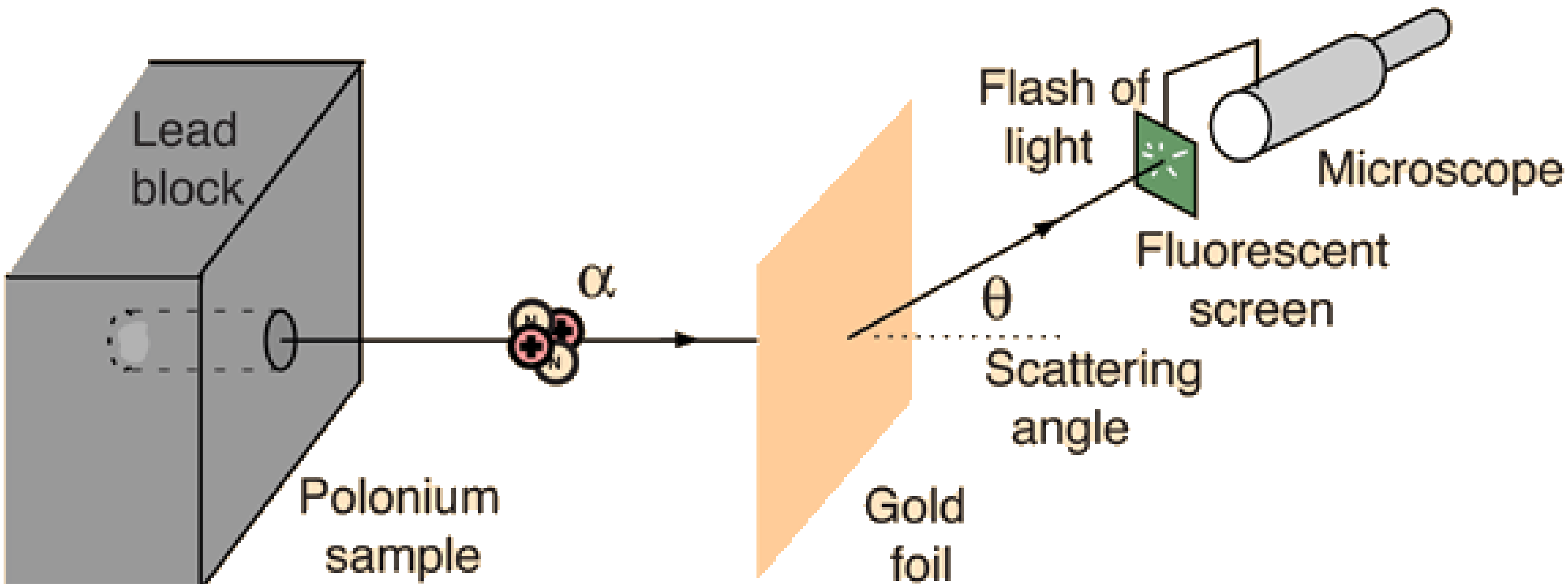
APSNews

May 2006 (Volume 15, Number 5)

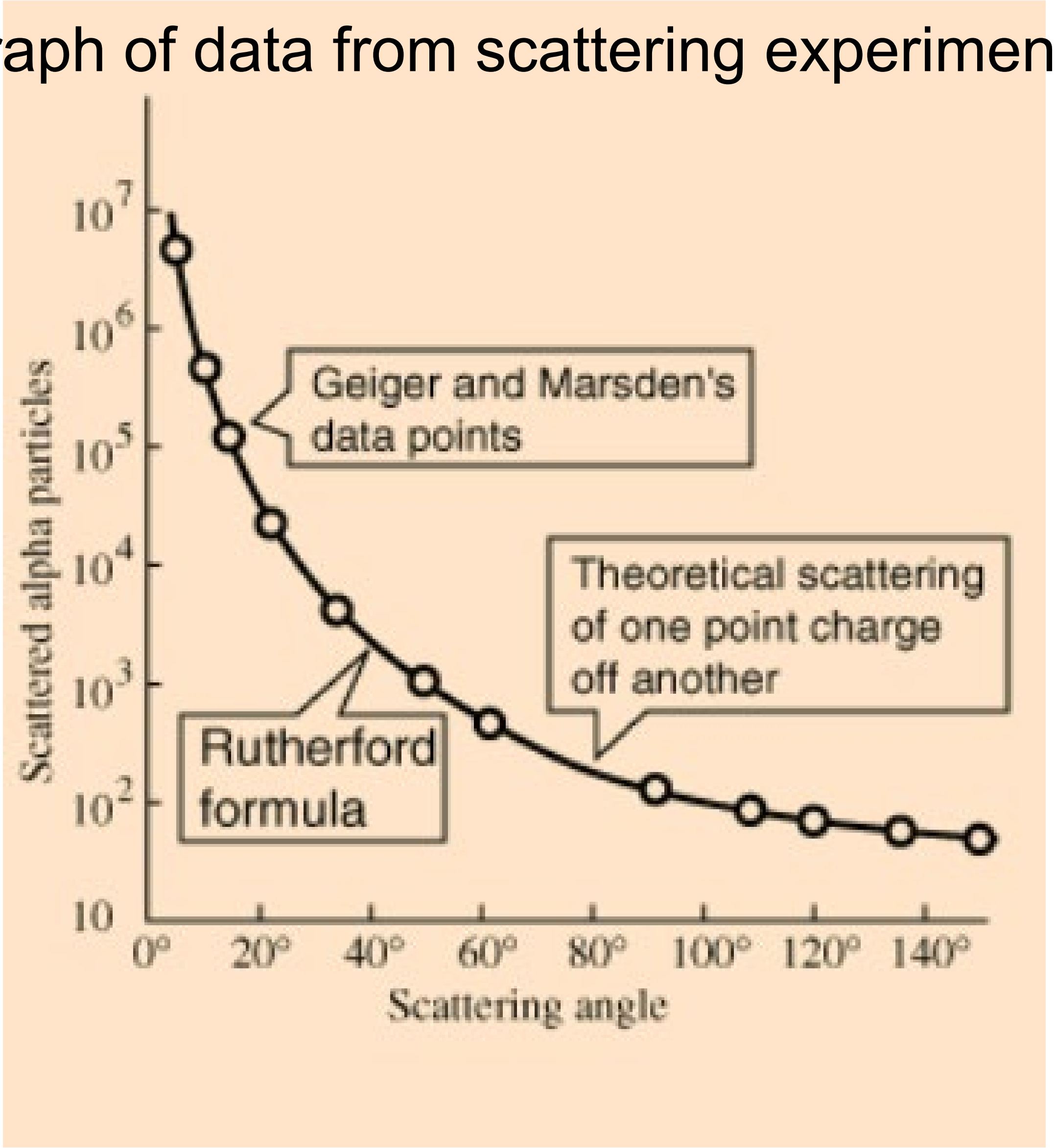
This Month in Physics History

May, 1911: Rutherford and the Discovery of the Atomic Nucleus

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Graph of data from scattering experiment



From website: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/rutsca2.html

How can we relate the differential scattering cross section values to information about the interaction potential $V(r)$ (assuming a central force interaction $\Rightarrow$ conserved angular momentum ℓ). In the following we will assume that the target particle is stationary (due to its large mass compared with the scattering particle).

Note that in the following slides, we reference the "center of mass reference frame" which will be discussed next time. For now, we can assume that the scattering particle has mass $\mu = m_1$ and the energy of interest is E_{rel}.

More complete picture --

$$E_{total} = E_{\text{Center of mass}} + E_{rel}$$

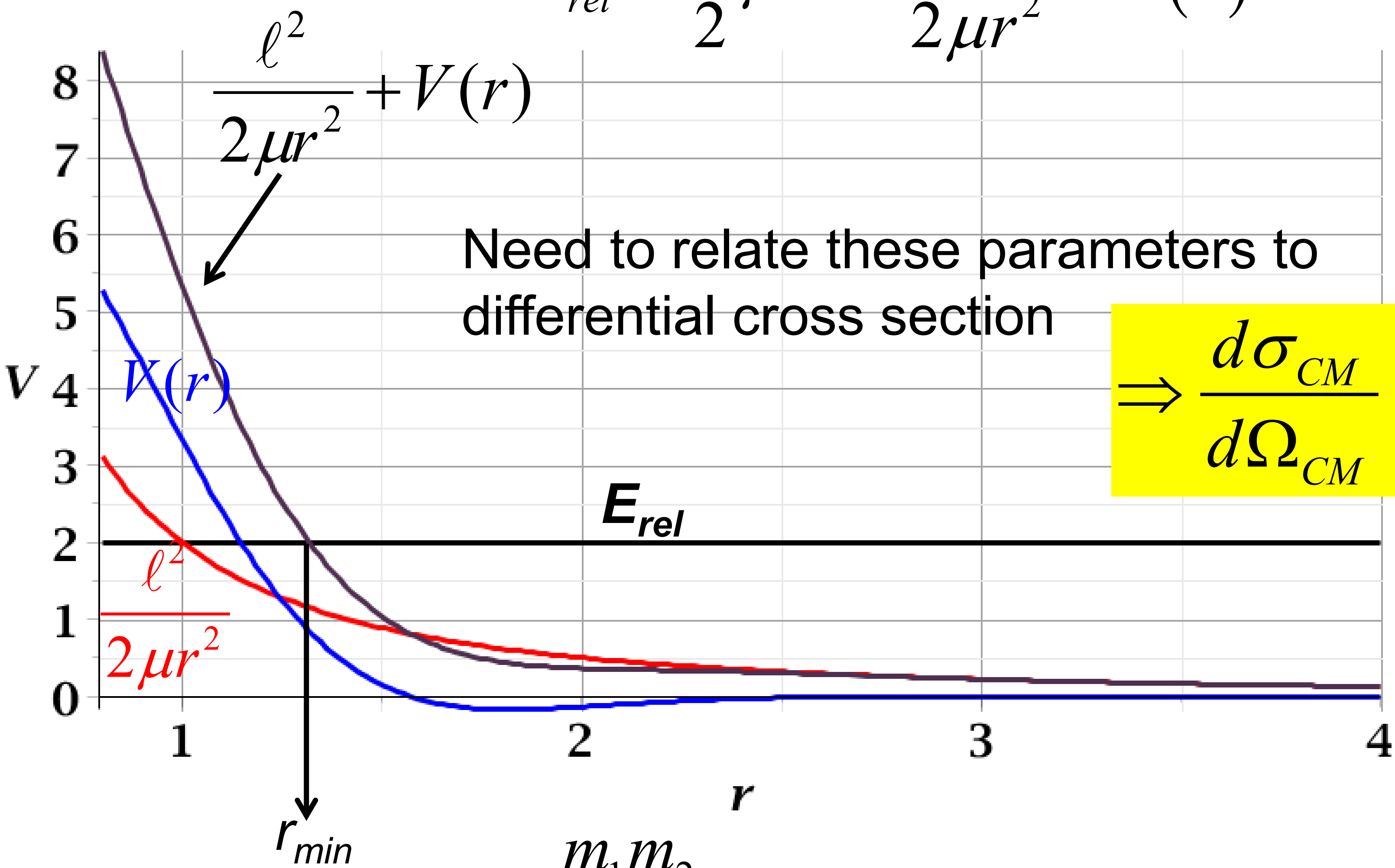
Energy of the center mass motion

Energy within the center of mass reference frame

In the following slides E_{rel} is written E

For a continuous potential interaction *V(r)*

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

More details

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Since **r**(t) represents motion in a plane, we will analyze the system in that plane and use polar coordinates.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$$

$$x(t) = r(t)\cos(\chi(t))$$

$$y(t) = r(t)\sin(\chi(t))$$

Note that $\left|\dot{\mathbf{r}}(t)\right|^2 = \dot{x}^2(t) + \dot{y}^2(t)$

$$= \dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)$$

Also note that the relative angular momentum of the system is a constant

$$\ell = \mu r^2 \dot{\chi}$$

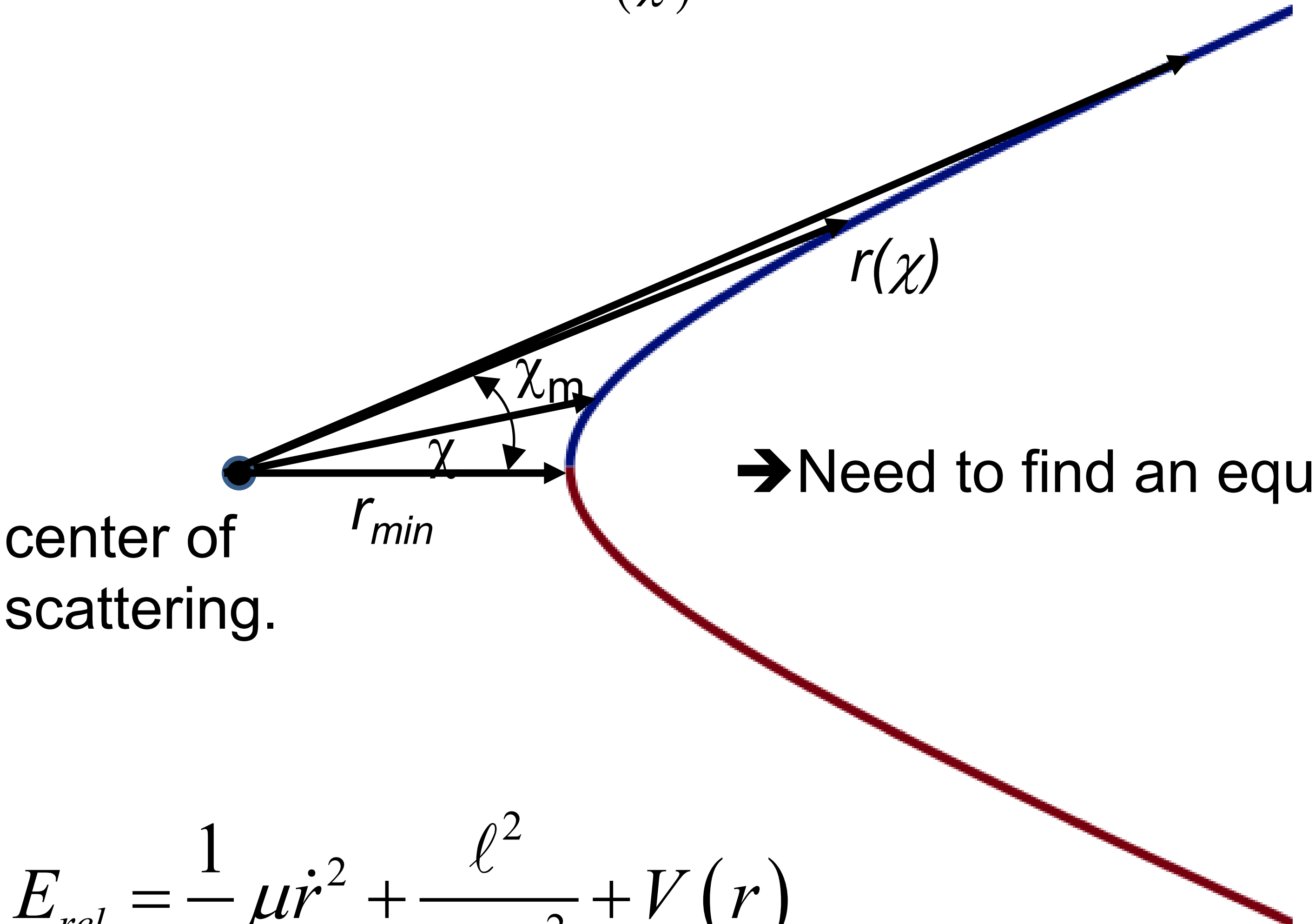
So that $\frac{1}{2}\mu\ |\dot{\mathbf{r}}(t)|^2 = \frac{1}{2}\mu\left(\dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)\right)$

$$= \frac{1}{2}\mu\dot{r}^2(t) + \frac{\ell^2}{2\mu r^2}$$

$$\Rightarrow E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Trajectory of relative vector in center of mass frame

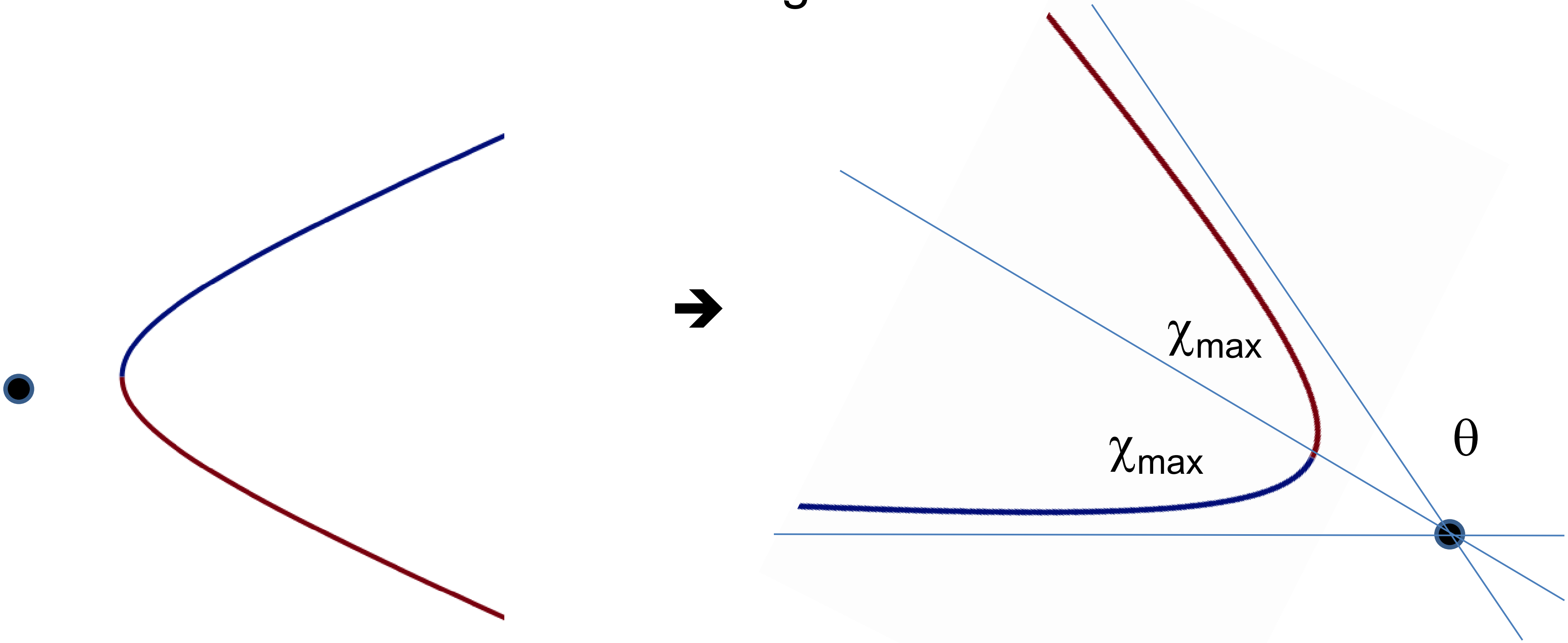
$r(\chi)$



➔Need to find an equation for $r(\chi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

How is this related to scattering?



Note that here θ measures the scattering angle

Evaluation of constants far from scattering center --

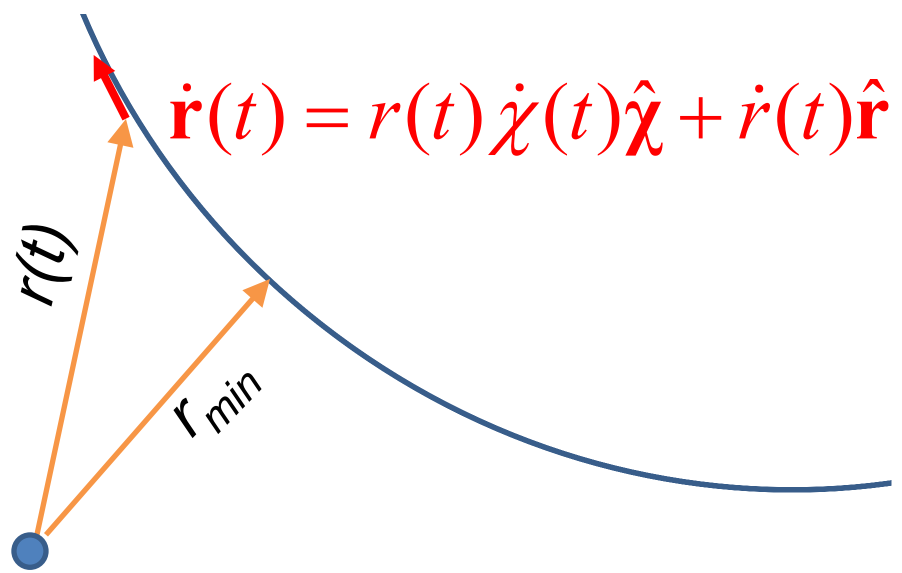
$$\ell = \mathbf{r} \times (\mu \dot{\mathbf{r}}) = r \mu r \frac{d\chi}{dt} = \mu r^2 \frac{d\chi}{dt}$$

$$\text{also: } \ell = b\mu \dot{r}(t = -\infty)$$

Note that E_{rel} and ℓ can be evaluated from $\dot{r}(t)$ at $t = -\infty$ or $t = \infty$.

$$E_{rel} = \frac{1}{2}\mu\left(\dot{r}(t = -\infty)\right)^2$$

$$\Rightarrow \ell = b\sqrt{2\mu E_{rel}}$$



Questions:

1. How can we find $r(\chi)$?
2. If we find $r(\chi)$, how can we relate χ to θ?
3. How can we find $b(\theta)$?

Recall --

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables from t to angle:

$$r(t) \Leftrightarrow r(\chi)$$

$$\frac{dr}{dt} = \frac{dr}{d\chi}\frac{d\chi}{dt} = \frac{dr}{d\chi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\chi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\chi) \Leftrightarrow \chi(r)$:

From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\chi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

When the dust clears:

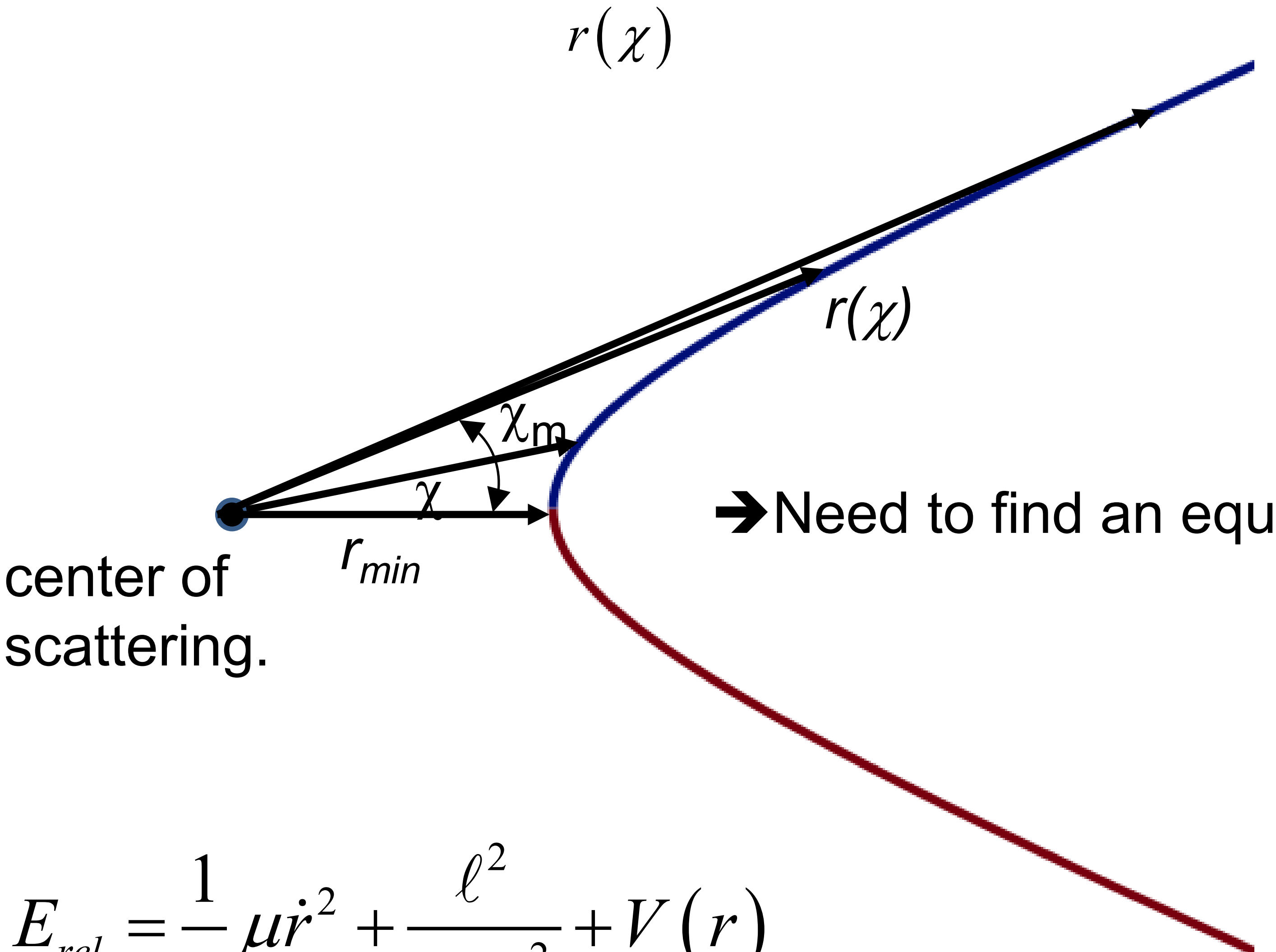
$$d\chi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

$$\ell = b\sqrt{2\mu E}$$

$$d\chi = dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\Rightarrow \chi_{\max}(b, E) = \chi\left(r \to \infty\right) - \chi\left(r = r_{\min}\right)$$

Trajectory of relative vector in center of mass frame



➔Need to find an equation for $r(\chi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

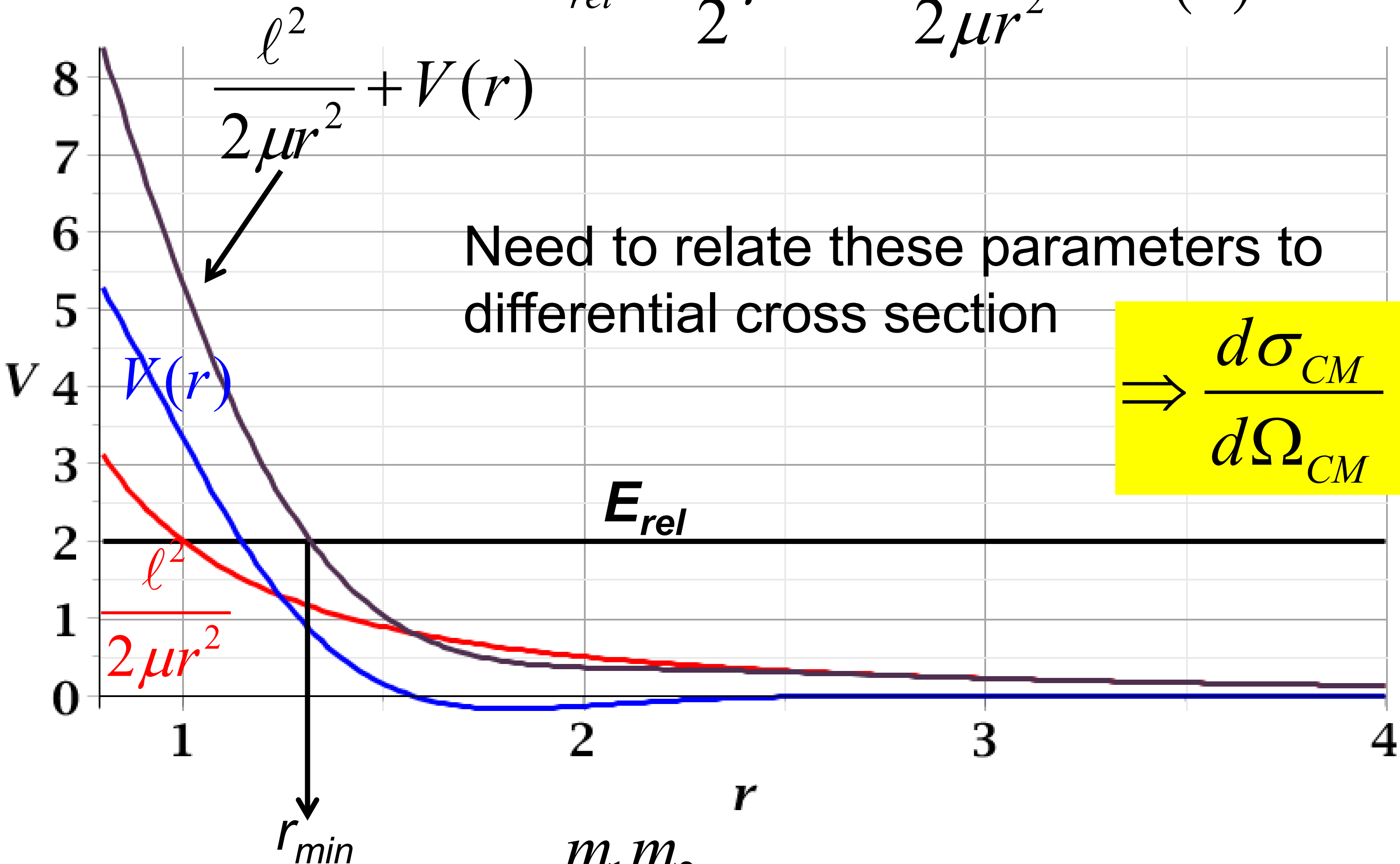
$$\int_{0}^{\chi_{\max}} d\chi = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

For a continuous potential interaction *V(r)*

$$E_{rel} = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

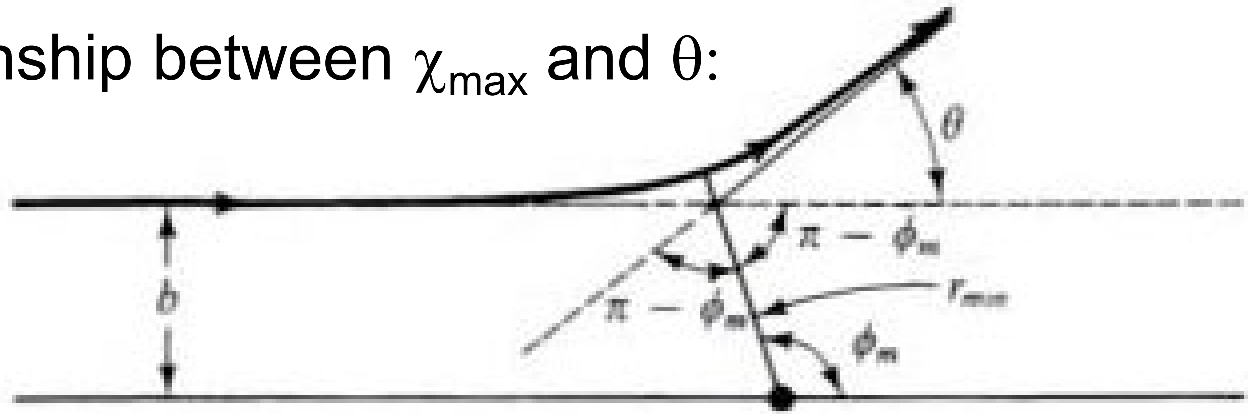
$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

Close up of repulsive interaction for a particular trajectory;
Also visualizing impact parameter *b*

Relationship between χ_{max} and θ:



$$2(\pi - \chi_{\max}) + \theta = \pi$$

$$\Rightarrow \chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

Here, θ represents the scattering angle in the center of mass frame and ϕ is used instead of χ.

General equations for central potential V(r)

$$\chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

➔These equations relate scattering angle to impact parameter *b*

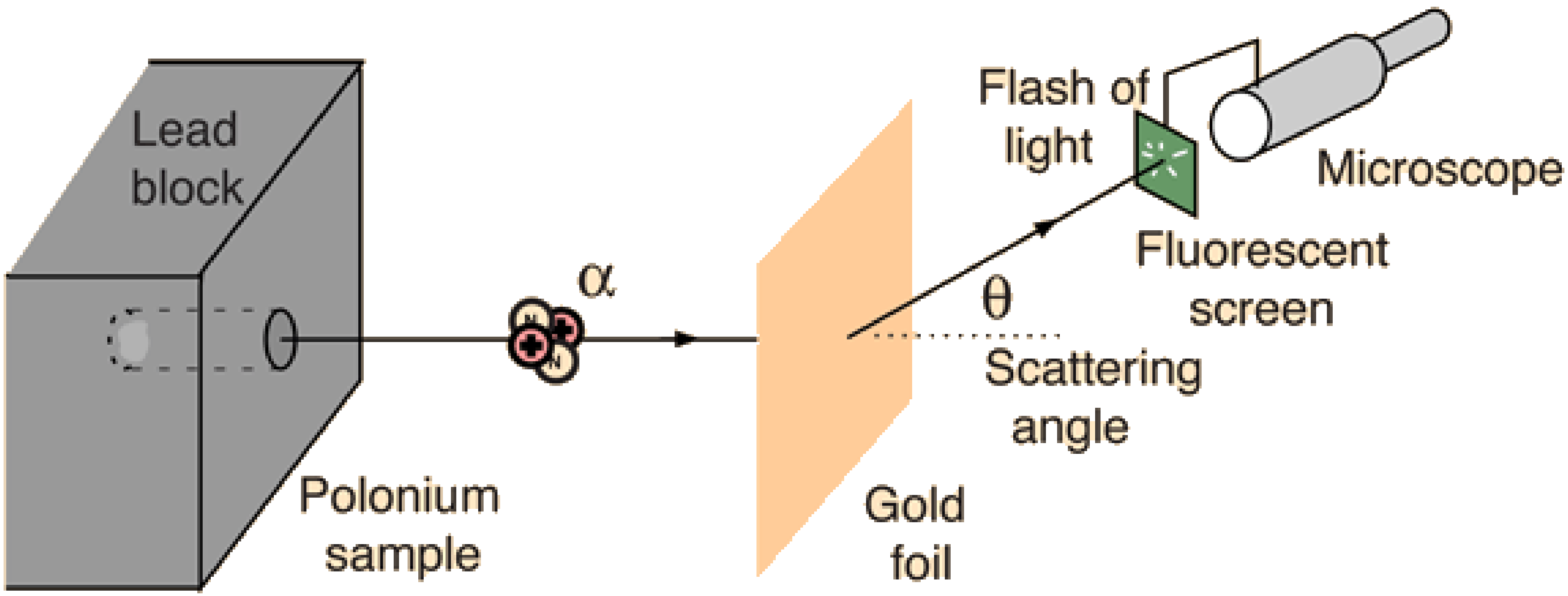
$$\chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

To go further, we need to know *V(r)*

Example: Diagram of Rutherford scattering experiment http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2\mathrm{b}} + \sqrt{\left(\frac{\kappa}{2\mathrm{b}}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

Some details –

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r}$$

SI units

$$V(r) = \frac{zZe^2}{4\pi\epsilon_0 r}$$

where e represents the elementary charge in Coulombs

r represents the particle separation in meters

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad \Rightarrow \kappa = \frac{zZe^2}{4\pi\epsilon_0 E}$$

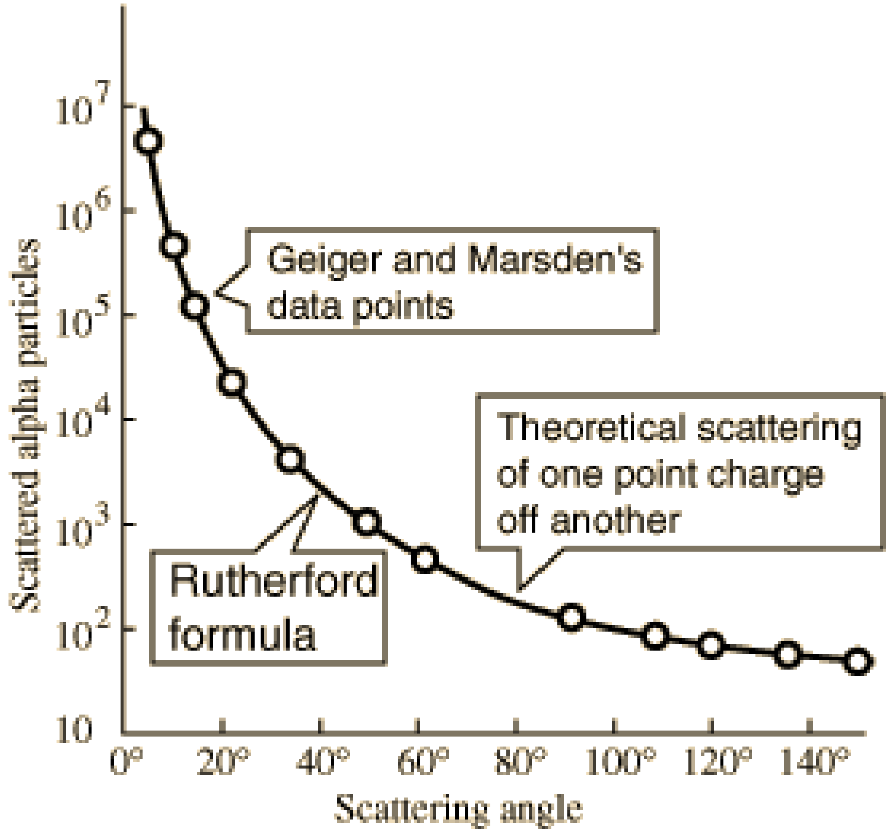
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\text{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\text{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

(in SI units)

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

General formula relating b and θ:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

$\Rightarrow$ There are relatively few forms of $V(1/u)$ for which the integral has an analytic result.

A problem mentioned in Cline for an example:

$$V(r) = \frac{\gamma}{r^2} \text{ where } \frac{d\sigma}{d\Omega} = \frac{\gamma\pi^2}{E\sin\theta} \frac{(\pi-\theta)}{\theta^2(2\pi-\theta)^2}$$

More generally, it is possible to use numerical integration methods (with care) to evaluate $b(\theta)$.

Transformation between lab and center of mass results:

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $$\tan\theta=\frac{\sin\psi}{\cos\psi+m_1/m_2}$$

For elastic scattering