





**PHY 711 Classical Mechanics and  
Mathematical Methods  
10-10:50 AM MWF in Olin 103**

**Discussion for Lecture 16 – Chap. 4 (F & W)**

**Analysis of motion near equilibrium**

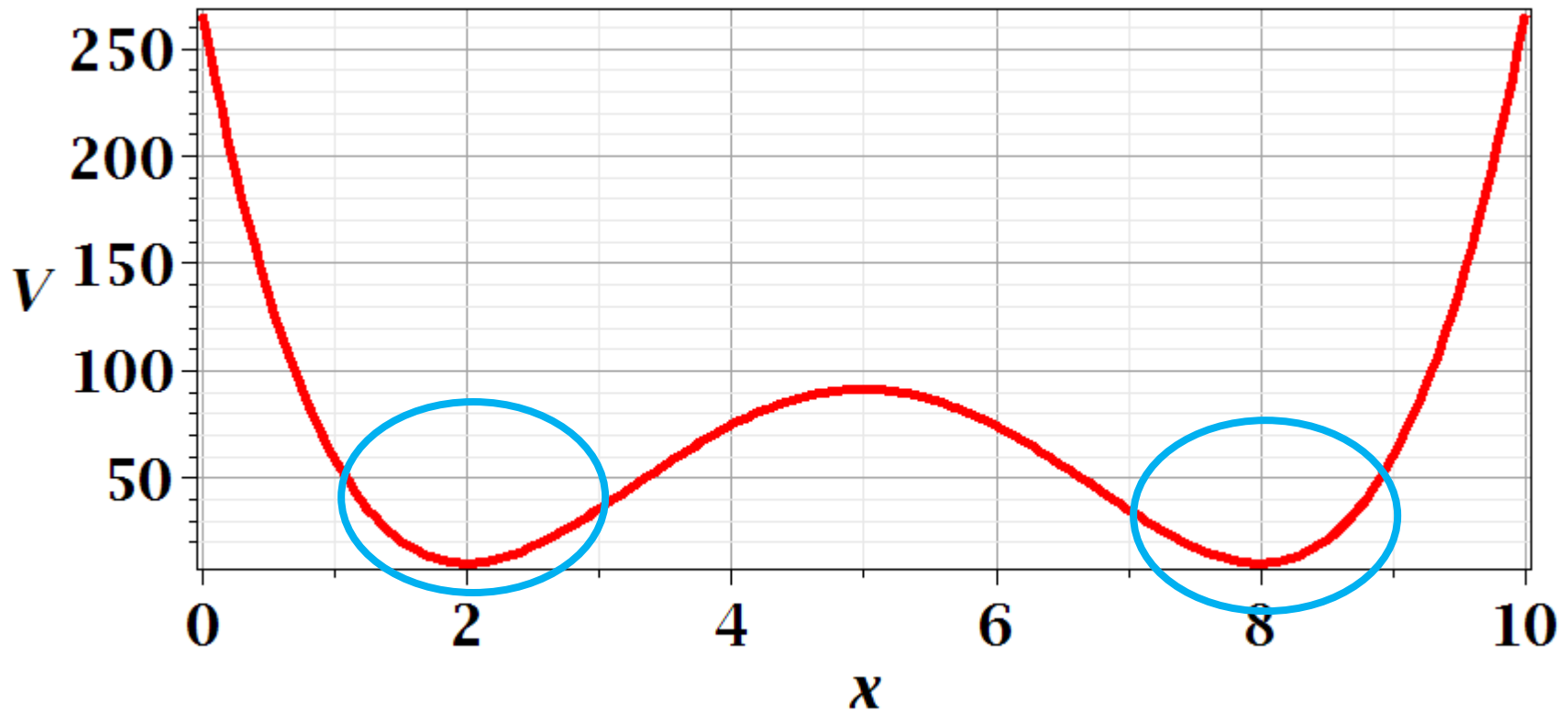
- 1. Small oscillations about equilibrium**
- 2. Normal modes of vibration**

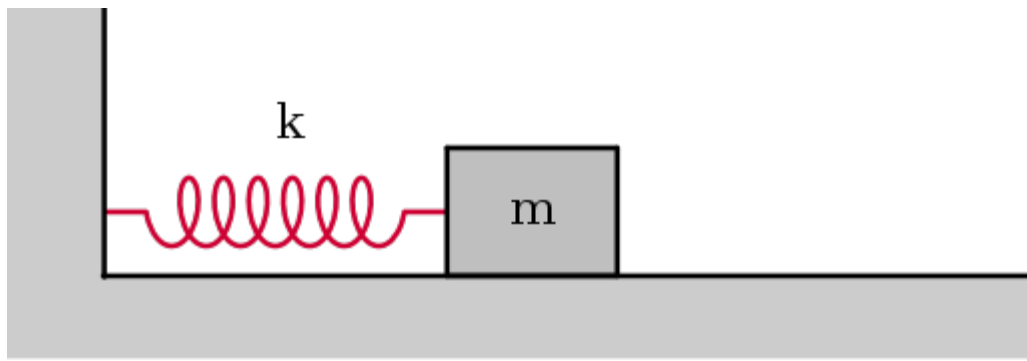


	Date	F&W	Topic	HW	
1	Mon, 8/28/2023		Introduction and overview	<a href="#">#1</a>	
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	<a href="#">#2</a>	
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	<a href="#">#3</a>	
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	<a href="#">#4</a>	
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	<a href="#">#5</a>	
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	<a href="#">#6</a>	
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	<a href="#">#7</a>	
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space		
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	<a href="#">#8</a>	
10	Mon, 9/18/2023	Chap. 5	Dynamics of rigid bodies	<a href="#">#9</a>	
11	Wed, 9/20/2023	Chap. 5	Dynamics of rigid bodies	<a href="#">#10</a>	
12	Fri, 9/22/2023	Chap. 5	Dynamics of rigid bodies	<a href="#">#11</a>	
13	Mon, 9/25/2023	Chap. 1	Scattering analysis	<a href="#">#12</a>	
14	Wed, 9/27/2023	Chap. 1	Scattering analysis	<a href="#">#13</a>	
15	Fri, 9/29/2023	Chap. 1	Scattering analysis	<a href="#">#14</a>	
	16	Mon, 10/2/2023	Chap. 4	Small oscillations near equilibrium	
	17	Wed, 10/4/2023	Chap. 4	Normal mode analysis	Mid term start
	18	Fri, 10/6/2023	Chap. 4	Normal mode analysis	
	22	Mon, 10/9/2023	Chap. 7	Normal modes of continuous string	
	20	Wed, 10/11/2023		Review and summary	Mid term due
		Fri, 10/13/2023	Fall Break		

Motivation for studying small oscillations – many interacting systems have stable and meta-stable configurations which are well approximated by:

$$V(x) \approx V(x_{eq}) + \frac{1}{2} (x - x_{eq})^2 \left. \frac{d^2V}{dx^2} \right|_{x_{eq}} = V(x_{eq}) + \frac{1}{2} k (x - x_{eq})^2$$





Equations of motion for a single oscillator:

Let  $k \equiv m\omega^2$

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

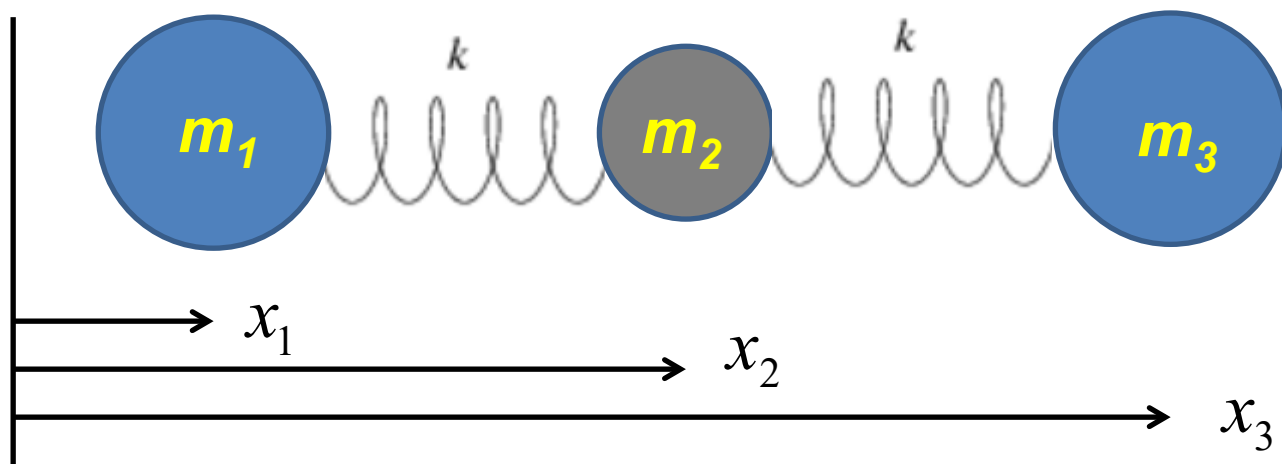
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \Rightarrow \quad m\ddot{x} = -m\omega^2 x$$

$$x(t) = A \sin(\omega t + \varphi)$$



# Coupled oscillators --

## Example – linear molecule



$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$- \frac{1}{2} k (x_2 - x_1 - \ell_{12})^2 - \frac{1}{2} k (x_3 - x_2 - \ell_{23})^2$$

Equilibrium lengths



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1 - \ell_{12})^2 - \frac{1}{2}k(x_3 - x_2 - \ell_{23})^2$$

Let:  $x_1 \rightarrow x_1 - x_1^0$      $x_2 \rightarrow x_2 - x_1^0 - \ell_{12}$      $x_3 \rightarrow x_3 - x_1^0 - \ell_{12} - \ell_{23}$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2$$

Coupled equations of motion :

$$m_1\ddot{x}_1 = k(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3\ddot{x}_3 = -k(x_3 - x_2)$$

Coupled equations of motion :

$$m_1 \ddot{x}_1 = k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3 \ddot{x}_3 = -k(x_3 - x_2)$$

Let  $x_i(t) = X_i^\alpha e^{-i\omega_\alpha t}$  where  $X_i^\alpha$  and  $\omega_\alpha$  are to be determined

$$-\omega_\alpha^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k(X_3^\alpha - X_2^\alpha)$$

Coupled linear equations:

$$-\omega_\alpha^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k(X_3^\alpha - X_2^\alpha)$$

Matrix form:

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$



Matrix form :

$$\begin{pmatrix} k - \omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k - \omega_\alpha^2 m_2 & -k \\ 0 & -k & k - \omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form :

Let  $Y_i \equiv \sqrt{m_i} X_i$  Equations for  $Y_i$  take the form :

$$\begin{pmatrix} \kappa_{11} - \omega_\alpha^2 & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} - \omega_\alpha^2 & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} - \omega_\alpha^2 \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = 0$$

where  $\kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$

Digression:

Eigenvalue properties of matrices

$$\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$$

Hermitian matrix :  $H_{ij} = H_{ji}^*$

Theorem for Hermitian matrices :

$$\lambda_\alpha \text{ have real values and } \mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

Unitary matrix :  $\mathbf{U}\mathbf{U}^H = \mathbf{I}$

$$|\lambda_\alpha| = 1 \text{ and } \mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

## Digression on matrices -- continued

Eigenvalues of a matrix are “invariant” under a similarity transformation

Eigenvalue properties of matrix:  $\mathbf{M}\mathbf{y}'_{\alpha} = \lambda'_{\alpha}\mathbf{y}'_{\alpha}$

Transformed matrix:  $\mathbf{M}'\mathbf{y}'_{\alpha} = \lambda'_{\alpha}\mathbf{y}'_{\alpha}$

If  $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$  then  $\lambda'_{\alpha} = \lambda_{\alpha}$  and  $\mathbf{S}^{-1}\mathbf{y}'_{\alpha} = \mathbf{y}_{\alpha}$

Proof  $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_{\alpha} = \lambda'_{\alpha}\mathbf{y}'_{\alpha}$

$$\mathbf{M}(\mathbf{S}^{-1}\mathbf{y}'_{\alpha}) = \lambda'_{\alpha}(\mathbf{S}^{-1}\mathbf{y}'_{\alpha})$$

## Example of transformation:

Original problem written in eigenvalue form:

$$\begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_3 & k/m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix}$$

$$\text{Let } \mathbf{S} = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}; \quad \mathbf{S}\mathbf{M}\mathbf{S}^{-1} = \begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix}$$

Let  $\mathbf{Y} \equiv \mathbf{S}\mathbf{X}$

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

where  $\kappa_{ij} = \kappa_{ji} \equiv \frac{k}{\sqrt{m_i m_j}}$

In our case :

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for  $m_1 = m_3 \equiv m_O$  and  $m_2 \equiv m_C$  ( $\text{CO}_2$ )

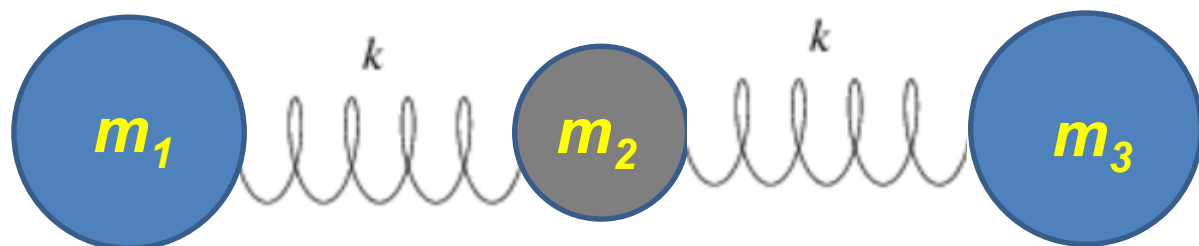
$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

# Eigenvalues and eigenvectors :

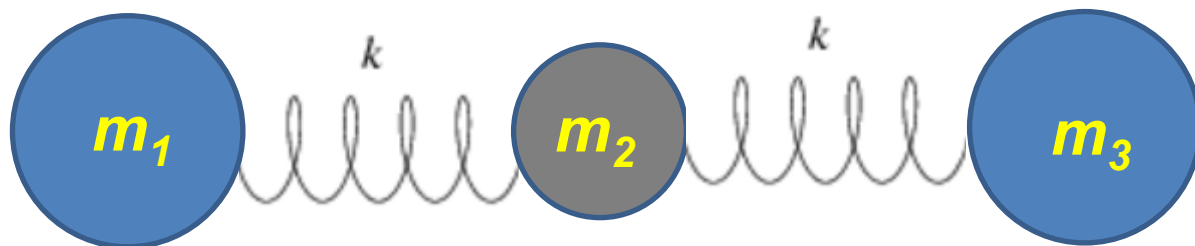
$$\omega_1^2 = 0 \quad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_O}{m_C}} \\ 1 \\ \sqrt{\frac{m_O}{m_C}} \end{pmatrix}, \quad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m_O} \quad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

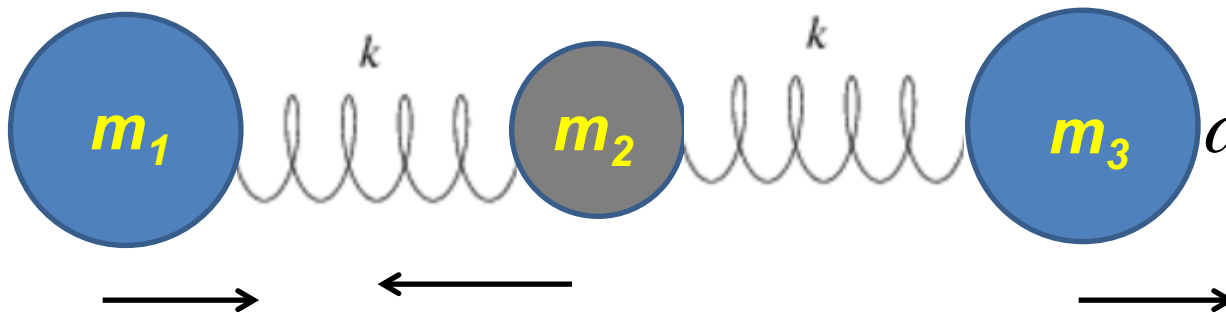
$$\omega_3^2 = \frac{k}{m_O} + \frac{2k}{m_C} \quad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_O}{m_C}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$



$$\omega_1 = 0$$



$$\omega_2 = \sqrt{\frac{k}{m_o}}$$



$$\omega_3 = \sqrt{\frac{k}{m_o} + \frac{2k}{m_c}}$$



General solution :

$$x_i(t) = \Re \left( \sum_{\alpha} C^{\alpha} X_i^{\alpha} e^{-i\omega_{\alpha} t} \right)$$

For example, normal mode amplitudes

$C^{\alpha}$  can be determined from initial conditions



Comment on solving for eigenvalues and eigenvectors – while it is reasonable to find these analytically for 2x2 or 3x3 matrices, it is prudent to use Maple or Mathematica for larger systems.

[Maple example](#)

[Mathematica example](#)

## Additional digression on matrix properties

### Singular value decomposition

It is possible to factor any real matrix  $\mathbf{A}$  into unitary matrices  $\mathbf{V}$  and  $\mathbf{U}$  together with positive diagonal matrix  $\mathbf{\Sigma}$  :

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathbf{H}}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}$$



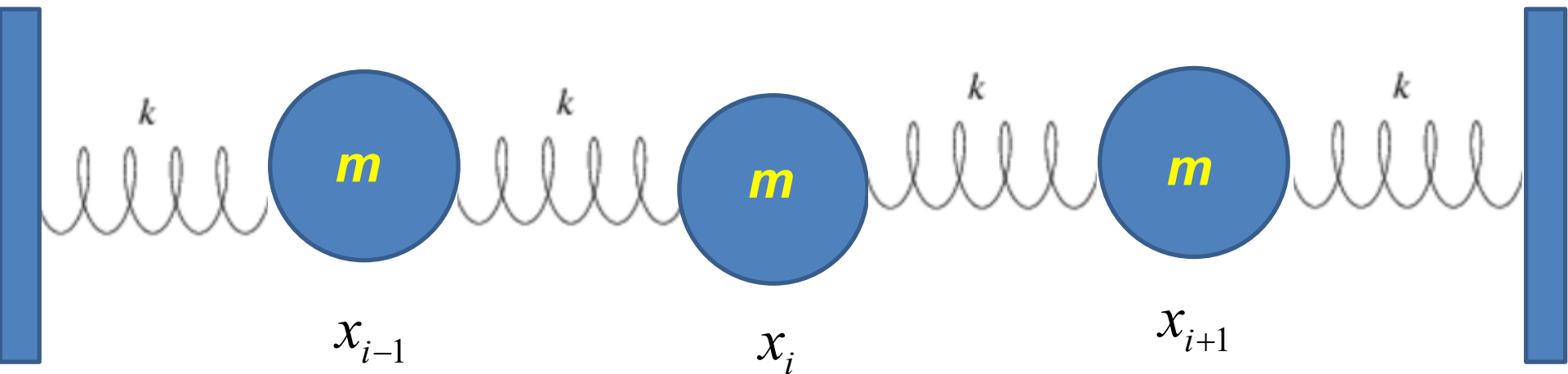
## Singular value decomposition -- continued

Consider using SVD to solve a singular linear algebra problem  $\mathbf{AX} = \mathbf{B}$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

$$\mathbf{X} = \sum_{i \text{ for } \sigma_i > \varepsilon} \mathbf{v}_i \frac{\langle \mathbf{u}_i^H | \mathbf{B} \rangle}{\sigma_i}$$


Consider an extended system of masses and springs:



Note: each mass coordinate is measured relative to its equilibrium position  $x_i^0$

$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

Note: In fact, we have  $N$  masses;  $x_0$  and  $x_{N+1}$  will be treated using boundary conditions.


$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

$$x_0 \equiv 0 \text{ and } x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k(x_2 - 2x_1)$$

$$m\ddot{x}_2 = k(x_3 - 2x_2 + x_1)$$

.....

$$m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$$

.....

$$m\ddot{x}_N = k(x_{N-1} - 2x_N)$$

Matrix formulation --

Assume  $x_i(t) = X_i e^{-i\omega t}$

$$\frac{m}{k} \omega^2 \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & -1 & 2 & -1 \\ \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

Can solve as an eigenvalue problem --



> *with(LinearAlgebra);*

=

$$\text{> } A := \begin{bmatrix} 5 & -1 & 0 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 \\ 0 & -1 & 5 & -1 & 0 \\ 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & -1 & 5 \end{bmatrix};$$

⌈ > *Eigenvalues(A);*

$$\begin{bmatrix} 5 \\ 6 \\ 4 \\ 5 - \sqrt{3} \\ 5 + \sqrt{3} \end{bmatrix}$$



This example also has an algebraic solution --

From Euler - Lagrange equations :

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

Try :  $x_j(t) = Ae^{-i\omega t + iqa_j}$

$$-\omega^2 Ae^{-i\omega t + iqa_j} = \frac{k}{m} (e^{iqa} - 2 + e^{-iqa}) Ae^{-i\omega t + iqa_j}$$

$$-\omega^2 = \frac{k}{m} (2 \cos(qa) - 2)$$

$$\Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$



From Euler-Lagrange equations -- continued:

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

$$\text{Try: } x_j(t) = Ae^{-i\omega t + iqaj} \quad \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

$$\text{Note that: } x_j(t) = Be^{-i\omega t - iqaj} \quad \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

General solution:

$$x_j(t) = \Re\left(Ae^{-i\omega t + iqaj} + Be^{-i\omega t - iqaj}\right)$$

Impose boundary conditions:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

Impose boundary conditions -- continued:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

$$\Rightarrow B = -A$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t} \left(e^{iqa(N+1)} - e^{-iqa(N+1)}\right)\right) = 0$$

$$\Rightarrow \sin\left(qa(N+1)\right) = 0$$

$$\Rightarrow qa(N+1) = \nu\pi \quad \text{where } \nu = 0, 1, 2, \dots$$

$$qa = \frac{\nu\pi}{N+1}$$



## Summary of results:

$$\Rightarrow \omega_v^2 = \frac{4k}{m} \sin^2 \left( \frac{v\pi}{2(N+1)} \right)$$

$$v = 0, 1, \dots, N$$

$$x_n = \Re \left( 2iA \sin \left( \frac{v\pi n}{N+1} \right) \right)$$

$$n = 1, 2, \dots, N$$

