PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 16 – Chap. 4 (F & W)

Analysis of motion near equilibrium –

Normal Mode Analysis

1. Normal modes of vibration for simple systems
2. Some concepts of linear algebra
3. Normal modes of vibration for more complicated systems

Physics Colloquium

Thursday

.

October 5th, 2023

Adaptive Optics and Interference Theory Enable Measurement of Retinal Function

Imaging of the retina has long been part of an ophthalmic exam, but the optics of the eye have aberrations that limit the quality of those images. Using adaptive optics, a technique originating in astronomy, researchers can measure and correct for the eye's optical aberrations thereby enabling diffraction limited imaging of the living retina. With this technology, individual photoreceptors and other retinal cells can be visualized noninvasively in the living human eye. My talk will provide an overview of adaptive optics imaging and will discuss how adaptive optics in combination with interference of light waves allows assessments of photoreceptor function.

4 pm - Olin 101

Refreshments will be served in the Olin lobby beginning at 3:30 pm

Jessica I. W. Morgan, Ph.D.

Associate Professor of Ophthalmology
Scheie Eye Institute
University of Pennsylvania

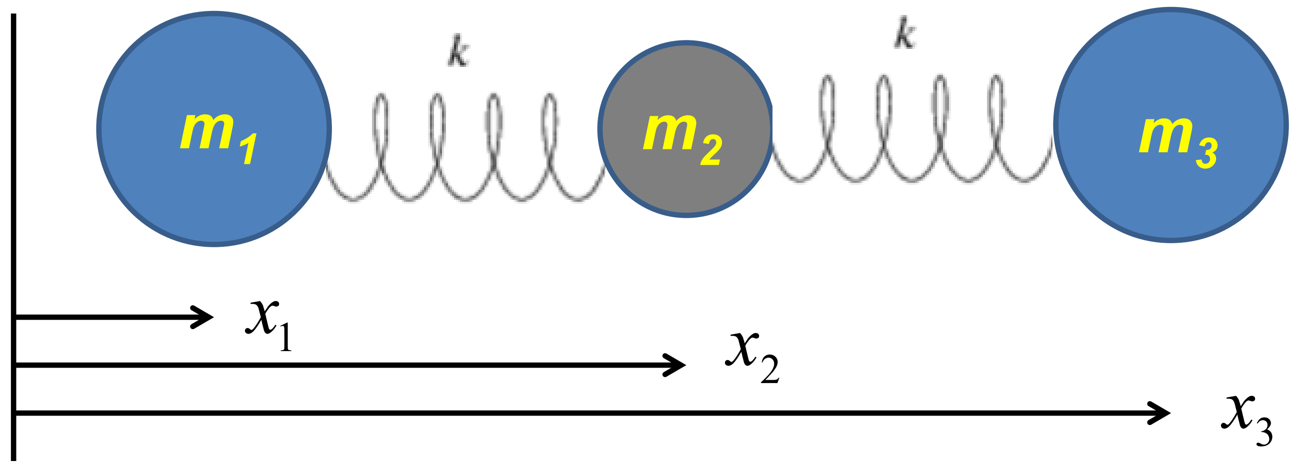
Wake Forest University

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space	
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	#8
10	Mon, 9/18/2023	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/20/2023	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/22/2023	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/25/2023	Chap. 1	Scattering analysis	#12
14	Wed, 9/27/2023	Chap. 1	Scattering analysis	#13
15	Fri, 9/29/2023	Chap. 1	Scattering analysis	#14
16	Mon, 10/2/2023	Chap. 4	Small oscillations near equilibrium	
17	Wed, 10/4/2023	Chap. 4	Normal mode analysis	Mid term start
18	Fri, 10/6/2023	Chap. 4	Normal mode analysis	
22	Mon, 10/9/2023	Chap. 7	Normal modes of continuous string	
20	Wed, 10/11/2023		Review and summary	Mid term due
	Fri, 10/13/2023	Fall Break		

From last time – example of system near equilibrium

Coupled oscillators --

Example – linear molecule



Analysis using linear algebra methods

General matrix form:

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for $m_1 = m_3 \equiv m_O$ and $m_2 \equiv m_C$ $\quad (CO_2)$

$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Finding eigenvalues/eigenvectors by hand --

$$\mathbf{M}\mathbf{y}^{\alpha} = \lambda^{\alpha}\mathbf{y}^{\alpha}$$

$$\left(\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right)\mathbf{y}^{\alpha} = 0$$

$$\left|\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right| \equiv \det\left(\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right) = 0 \quad \Rightarrow \text{polynomial for solutions } \lambda^{\alpha}$$

For each α and λ^{α} solve for the eigenvector coefficients $\mathbf{y}^{\alpha}$

Example

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right| = \begin{vmatrix} A - \lambda^{\alpha} & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B - \lambda^{\alpha} & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A - \lambda^{\alpha} \end{vmatrix} = \lambda^{\alpha}\left(\lambda^{\alpha} - A\right)\left(\lambda^{\alpha} - (A + 2B)\right) = 0$$

Example -- continued

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| = \begin{vmatrix} A-\lambda^\alpha & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B-\lambda^\alpha & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A-\lambda^\alpha \end{vmatrix} = \lambda^\alpha \left(\lambda^\alpha - A\right)\left(\lambda^\alpha - (A+2B)\right)$$

Solving for eigenvector corresponding to $\lambda^\alpha \equiv \lambda^1 = 0$

$$\begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \begin{pmatrix} y_{O1}^1 \\ y_C^1 \\ y_{O2}^1 \end{pmatrix} = 0 \qquad \Rightarrow \frac{y_{O1}^1}{y_C^1} = \frac{y_{O2}^1}{y_C^1} = \sqrt{\frac{B}{A}}$$

Note that the normalization of the eigenvector is arbitrary.

Digression on matrices -- continued

Eigenvalues of a matrix are “invariant” under a similarity transformation

Eigenvalue properties of matrix: $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Transformed matrix: $\mathbf{M}'\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

If $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$ then $\lambda'_\alpha = \lambda_\alpha$ and $\mathbf{S}^{-1}\mathbf{y}'_\alpha = \mathbf{y}_\alpha$

Proof $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

$$\mathbf{M}\left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right) = \lambda'_\alpha \left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right)$$

This means that if a matrix is “similar” to a Hermitian matrix, it has the same eigenvalues. The corresponding eigenvectors of **M** and **M’** are not the same but $\mathbf{y}_\alpha = \mathbf{S}^{-1}\mathbf{y}'_\alpha$

Example of a similarity transformation:

Original problem written in eigenvalue form:

$$\begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_3 & k/m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix}$$

Note that this matrix is not symmetric

Let $\mathbf{S} = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}$; $\mathbf{SMS}^{-1} = \begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix}$

Let $\mathbf{Y} \equiv \mathbf{SX}$

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Note that this matrix is symmetric

where $\kappa_{ij} = \kappa_{ji} \equiv \dfrac{k}{\sqrt{m_i m_j}}$

Note, here we have defined **S** as a transformation matrix (often called a similarity transformation matrix)

Sometimes, the similarity transformation is also unitary so that

$$\mathbf{U}^{-1} = \mathbf{U}^{H}$$

Example for 2x2 case --

$$\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \mathbf{U}^{-1} = \mathbf{U}^{H} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

How can you find a unitary transformation that also diagonalizes a matrix?

Example -- $\mathbf{M} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \qquad \mathbf{M}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Example -- $\mathbf{M}=\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ $\mathbf{M}'=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\mathbf{M}' = \mathbf{U}\mathbf{M}\mathbf{U}^H$ for $\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\mathbf{M}' = \begin{pmatrix} A\cos^2\theta + C\sin^2\theta + B\sin 2\theta & -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta \\ -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta & A\sin^2\theta + C\cos^2\theta - B\sin 2\theta \end{pmatrix}$$

$\Rightarrow$ choose $\theta = \frac{1}{2}\tan^{-1}\left(\frac{-2B}{C-A}\right)$

$\Rightarrow \lambda_1 = A\cos^2\theta + C\sin^2\theta + B\sin 2\theta$

$\Rightarrow \lambda_2 = A\sin^2\theta + C\cos^2\theta - B\sin 2\theta$

Note that this “trick” is special for 2x2 matrices, but numerical extensions based on the trick are possible.

Note that transformations using unitary matrices are often convenient and they can be easily constructed from the eigenvalues of a matrix.

Suppose you have an $N \times N$ matrix $\mathbf{M}$ and find all N eigenvalues/vectors:

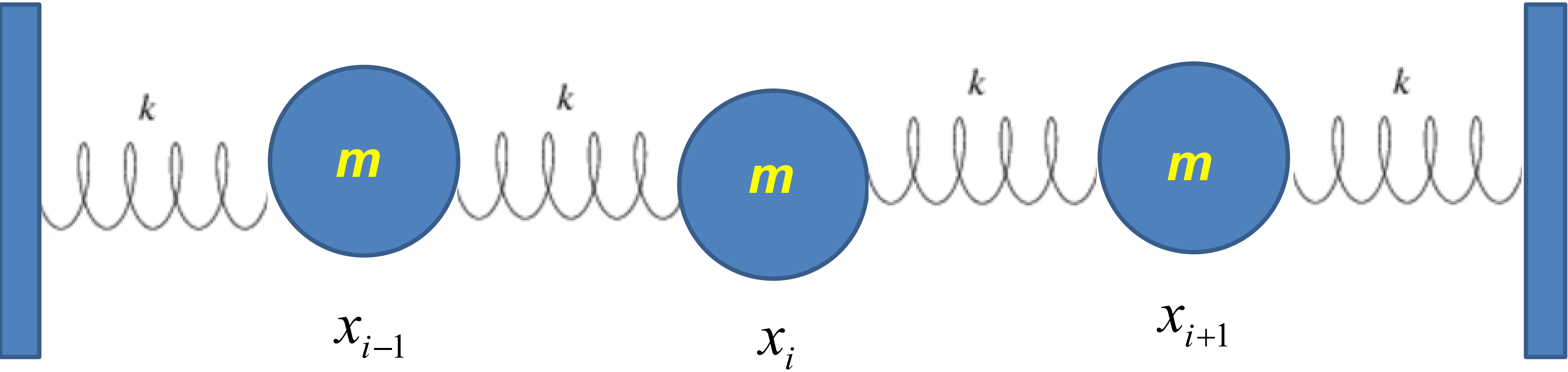
$\mathbf{M}\mathbf{y}^{\alpha} = \lambda^{\alpha}\mathbf{y}^{\alpha}$ orthonormalized so that $\left\langle \mathbf{y}^{\alpha} \middle| \mathbf{y}^{\beta} \right\rangle = \delta_{\alpha\beta}$

Now construct an $N \times N$ matrix $\mathbf{U}$ by listing the eigenvector columns:

$$\mathbf{U} \equiv \begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^N \\ y_2^1 & y_2^2 & \cdots & y_2^N \\ \vdots & \vdots & \cdots & \vdots \\ y_N^1 & y_N^2 & \cdots & y_N^N \end{pmatrix} \qquad \mathbf{U}^{-1} \equiv \begin{pmatrix} y_1^{1*} & y_2^{1*} & \cdots & y_N^{1*} \\ y_1^{2*} & y_2^{2*} & \cdots & y_N^{N*} \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{N*} & y_2^{N*} & \cdots & y_N^{N*} \end{pmatrix} \qquad \Rightarrow \text{ by construction } \mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$$

Also by construction $\mathbf{U}^{-1}\mathbf{M}\mathbf{U} = \begin{pmatrix} \lambda^1 & 0 & \cdots & 0 \\ 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda^N \end{pmatrix}$

Consider an extended system of masses and springs:



Note: each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2}m\sum_{i=1}^{N}\dot{x}_i^2 - \frac{1}{2}k\sum_{i=0}^{N}\left(x_{i+1} - x_i\right)^2$$

Note: In fact, we have N masses; x_0 and x_{N+1} will be treated using boundary conditions.

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

$$x_0 \equiv 0 \text{ and } x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k\left(x_2 - 2x_1\right)$$

$$m\ddot{x}_2 = k\left(x_3 - 2x_2 + x_1\right)$$

....

$$m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

....

$$m\ddot{x}_N = k\left(x_{N-1} - 2x_N\right)$$

Matrix formulation --

Assume $x_i(t) = X_i e^{-i\omega t}$

$$\frac{m}{k}\omega^2 \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & -1 & 2 & -1 \\ \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

Can solve as an eigenvalue problem –

(Why did we not have to transform the equations as we did in the previous example?)

Because of its very regular form, this example also has an algebraic solution --

From Euler - Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try: $\quad x_j(t) = Ae^{-i\omega t + iqaj}$

Here "*a*" is the equilibrium length of a spring and *q* has the units of 1/length.

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

Is this treatment cheating?

a. Yes.
b. No cheating, but we are not done.

From Euler - Lagrange equations - - continued :

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try : $x_j(t) = Ae^{-i\omega t + iqaj}$ $\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

Note that : $x_j(t) = Be^{-i\omega t - iqaj}$ $\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

General solution :

$$x_j(t) = \Re\left(Ae^{-i\omega t + iqaj} + Be^{-i\omega t - iqaj}\right)$$

Impose boundary conditions :

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

Impose boundary conditions -- continued:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

$$\Rightarrow B = \text{-}A$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t}\left(e^{iqa(N+1)} - e^{-iqa(N+1)}\right)\right) = 0$$

$$\Rightarrow \sin\left(qa\left(N+1\right)\right) = 0$$

$$\Rightarrow \; qa\left(N+1\right) = \nu\pi \quad \text{where } \nu = 1,2\cdots N$$

$$qa = \frac{\nu\pi}{N+1}$$

Recap -- solution for integer parameter ν

$$x_j(t) = \Re\left(2iAe^{-i\omega_\nu t} \sin\left(\frac{\nu\pi j}{N+1} \right) \right)$$

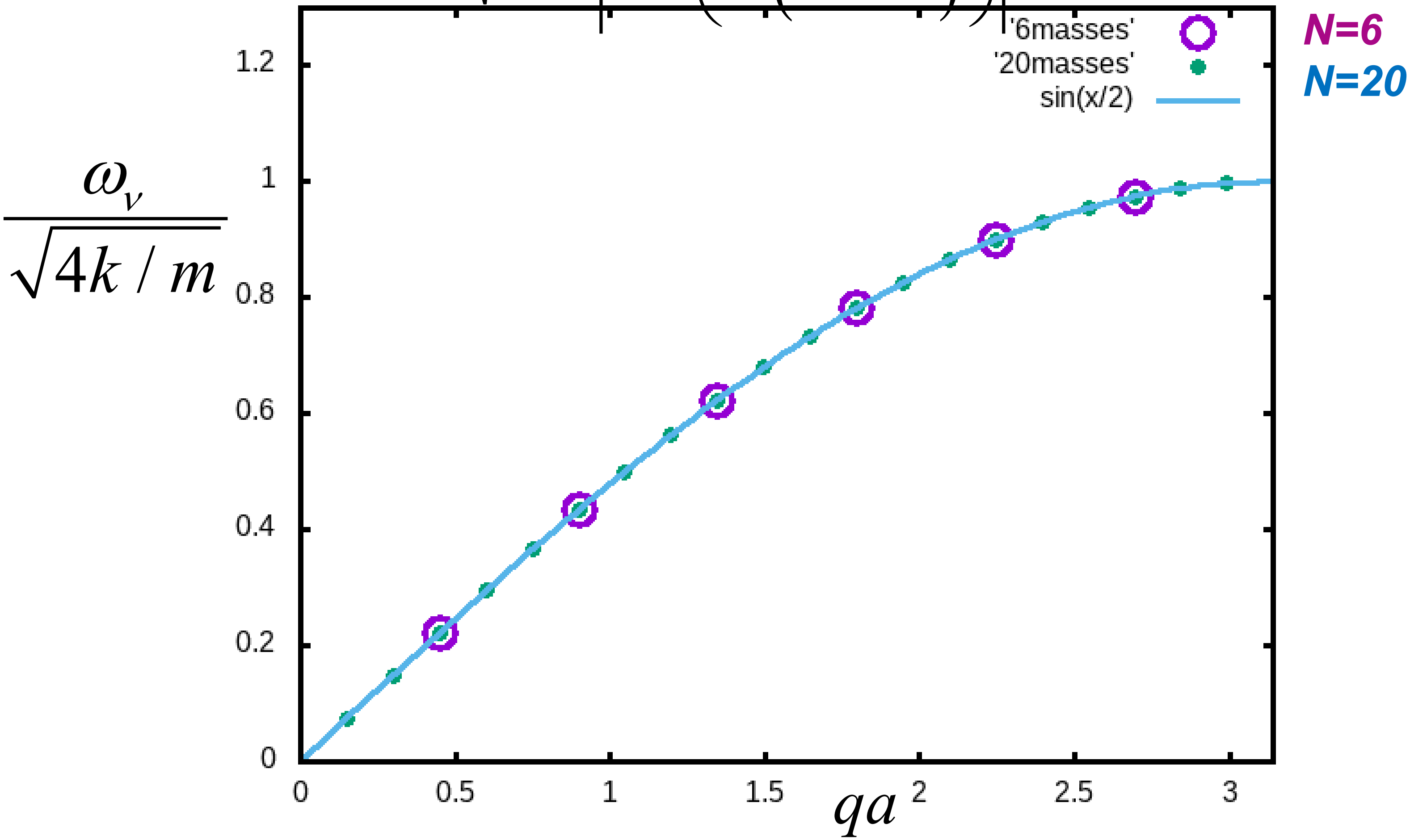
$$\omega_\nu^{\ 2} = \frac{4k}{m} \sin^2\left(\frac{\nu\pi}{2(N+1)} \right)$$

Note that non-trivial, unique values are

$$\nu = 1, 2, \cdots N$$

Examples

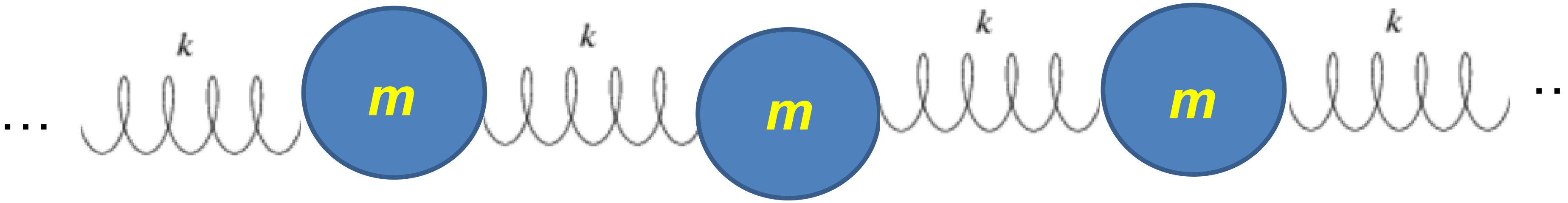
$$\omega_\nu = \sqrt{\frac{4k}{m}} \left| \sin\left(\frac{\nu\pi}{2(N+1)} \right) \right|$$



Note that solution form remains correct for $N \to \infty$

$$\omega(qa) = \sqrt{4k/m} \left| \sin\left(\frac{qa}{2} \right) \right|$$

For extended (infinite) chain without boundaries:



x_{i-1} x_i x_{i+1}

From Euler-Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{for all } x_j$$

Try: $x_j(t) = Ae^{-i\omega t + iqaj}$

$$-\omega^2 Ae^{-i\omega t + iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t + iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

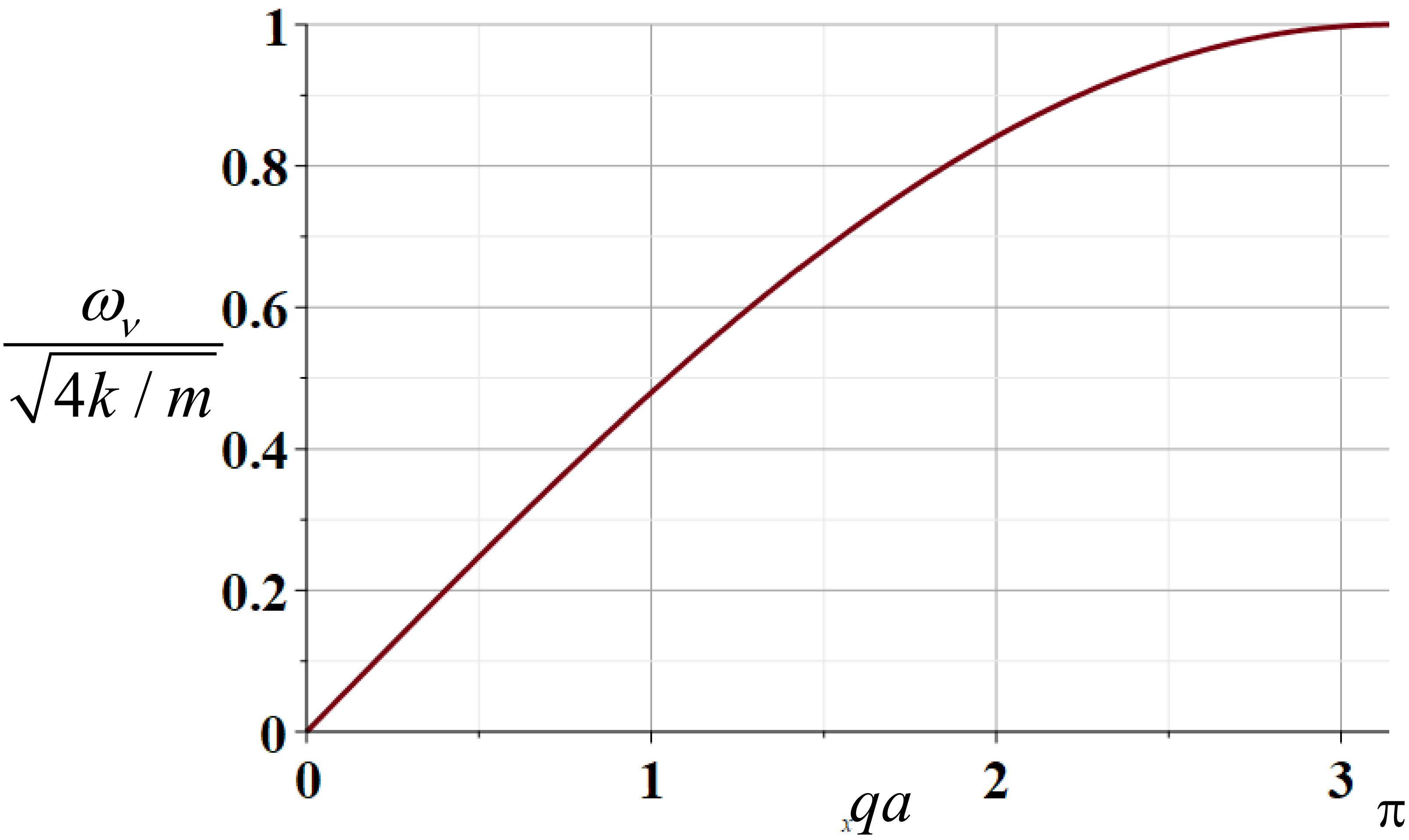
$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

distinct values for $0 \le qa \le \pi$

Note that we are assuming that all masses and springs are identical here.

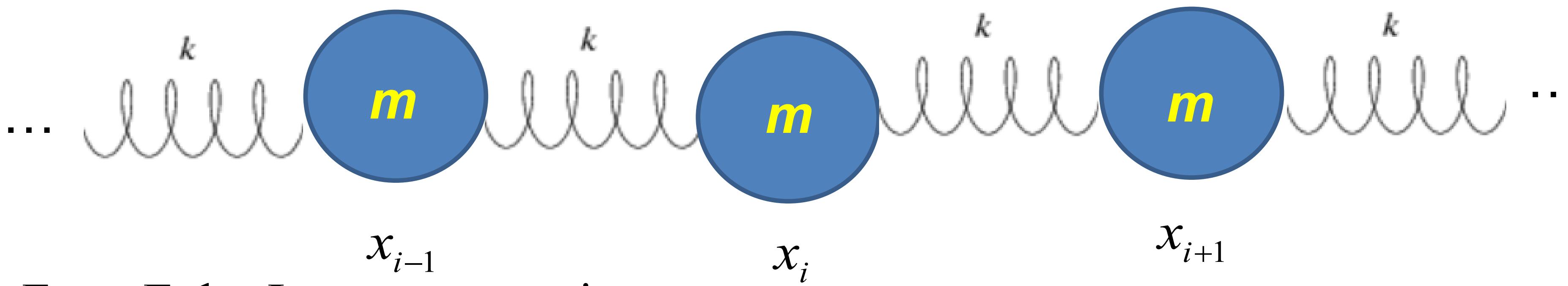
Here "*a*" is the equilibrium length of a spring and *q* has the units of 1/length.

Plot of distinct values of $\omega_\nu(q)$



Note that for $N \to \infty$, q becomes a continuous variable within the range $0 < qa < \pi$.

For extended (infinite) chain without boundaries:



From Euler-Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{for all } x_j \qquad \text{Try:} \quad x_j(t) = Ae^{-i\omega t + iqaj}$$

$$-\omega^2 Ae^{-i\omega t + iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t + iqaj} \qquad -\omega^2 = \frac{k}{m}\left(2\cos\left(qa\right) - 2\right)$$

$$\Rightarrow \omega = \sqrt{\frac{4k}{m}}\left|\sin\left(\frac{qa}{2}\right)\right| \qquad \text{distinct values for } 0 \le qa \le \pi$$

Note that there are an infinite number of normal mode frequencies!

Does this make sense?

(A) Yes

(B) No

Consider an infinite system of masses and springs now with two kinds of masses:

$$x_i \qquad y_i \qquad x_{i+1} \qquad y_{i+1} \qquad x_{i+2}$$

Note: each mass coordinate is measured relative to its equilibrium position $x_i^0 \equiv 0, y_i^0 \equiv 0, \cdots$

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - y_i \right)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(y_i - x_i \right)^2$$

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - y_i\right)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(y_i - x_i\right)^2$$

Euler - Lagrange equations :

$$m\ddot{x}_j = k\left(y_{j-1} - 2x_j + y_j\right)$$

$$M\ddot{y}_j = k\left(x_j - 2y_j + x_{j+1}\right)$$

Trial solution :

$$x_j(t) = Ae^{-i\omega t + i2qaj}$$

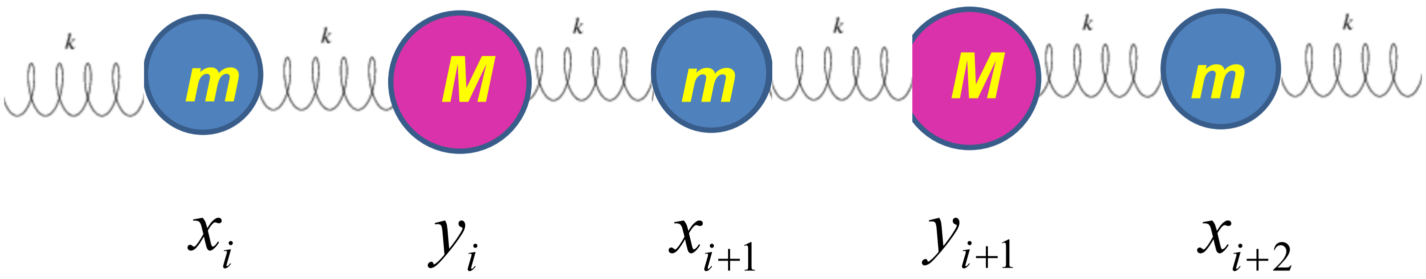
$$y_j(t) = Be^{-i\omega t + i2qaj}$$

Note that 2qa is an unknown parameter.

Does this form seem reasonable?

$$\begin{pmatrix} m\omega^2 - 2k & k\left(e^{-i2qa} + 1\right) \\ k\left(e^{i2qa} + 1\right) & M\omega^2 - 2k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Comment on notation --



Trial solution:

$$x_j(t) = Ae^{-i\omega t + i2qaj}$$

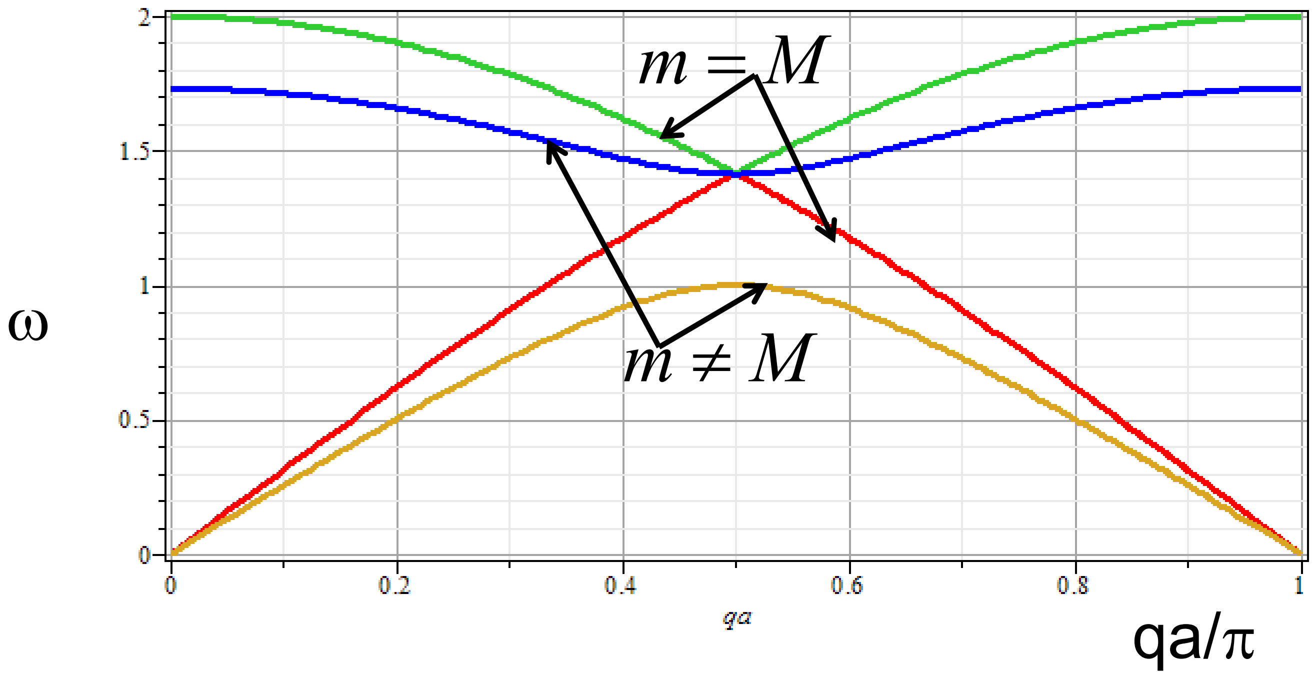
$$y_j(t) = Be^{-i\omega t + i2qaj}$$

Using 2qa as our unknown parameter is a convenient choice so that we can easily relate our solution to the m=M case.

$$\begin{pmatrix} m\omega^2 - 2k & k\left(e^{-i2qa}+1\right) \\ k\left(e^{i2qa}+1\right) & M\omega^2 - 2k \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Solutions :

$$\omega_{\pm}^{\ 2} = \frac{k}{m} + \frac{k}{M} \pm k\sqrt{\frac{1}{m^2} + \frac{1}{M^2} + \frac{2\cos(2qa)}{mM}}$$



Next time –

1. Extension of these ideas to 2 and 3 dimensions
2. Extension of these ideas to continuous elastic media.