PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 19: Chap. 7 (F&W)

Mechanical motion of a continuous string

1. Comments on linear vs. non-linear differential equations – considering beyond harmonic oscillations
2. Back to linear analyses -- masses coupled by springs ←→mass continuum coupled by string
3. Mechanics one-dimensional continuous system
4. The wave equation

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space	
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	#8
10	Mon, 9/18/2023	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/20/2023	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/22/2023	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/25/2023	Chap. 1	Scattering analysis	#12
14	Wed, 9/27/2023	Chap. 1	Scattering analysis	#13
15	Fri, 9/29/2023	Chap. 1	Scattering analysis	#14
16	Mon, 10/2/2023	Chap. 4	Small oscillations near equilibrium	
17	Wed, 10/4/2023	Chap. 4	Normal mode analysis	Mid term start
18	Fri, 10/6/2023	Chap. 4	Normal mode analysis	
22	Mon, 10/9/2023	Chap. 7	Normal modes of continuous string	
20	Wed, 10/11/2023		Review and summary	Mid term due
	Fri, 10/13/2023	Fall Break		
21	Mon, 10/16/2023			

Your questions –

From Caela -- Can you please explain why we get the results on slide 21?

Wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

D'Alembert's solution:

$$\mu(x,t) = \frac{1}{2}\big(\varphi(x-ct) + \varphi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Digression – comment on linear vs non-linear equations

Linear oscillator equations (ODE example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$$

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

Euler-Lagrange equations: $\ddot{x} = -\omega^2 x$

Superposition property of linear equations: --

Suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions

$\Rightarrow Ax_1(t) + Bx_2(t)$ are also solutions (all A, B)

Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left.\frac{d^4V}{dx^4}\right|_{x_{eq}} + \cdots$$

$$\Rightarrow \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2}\varepsilon x^4 \right)$$

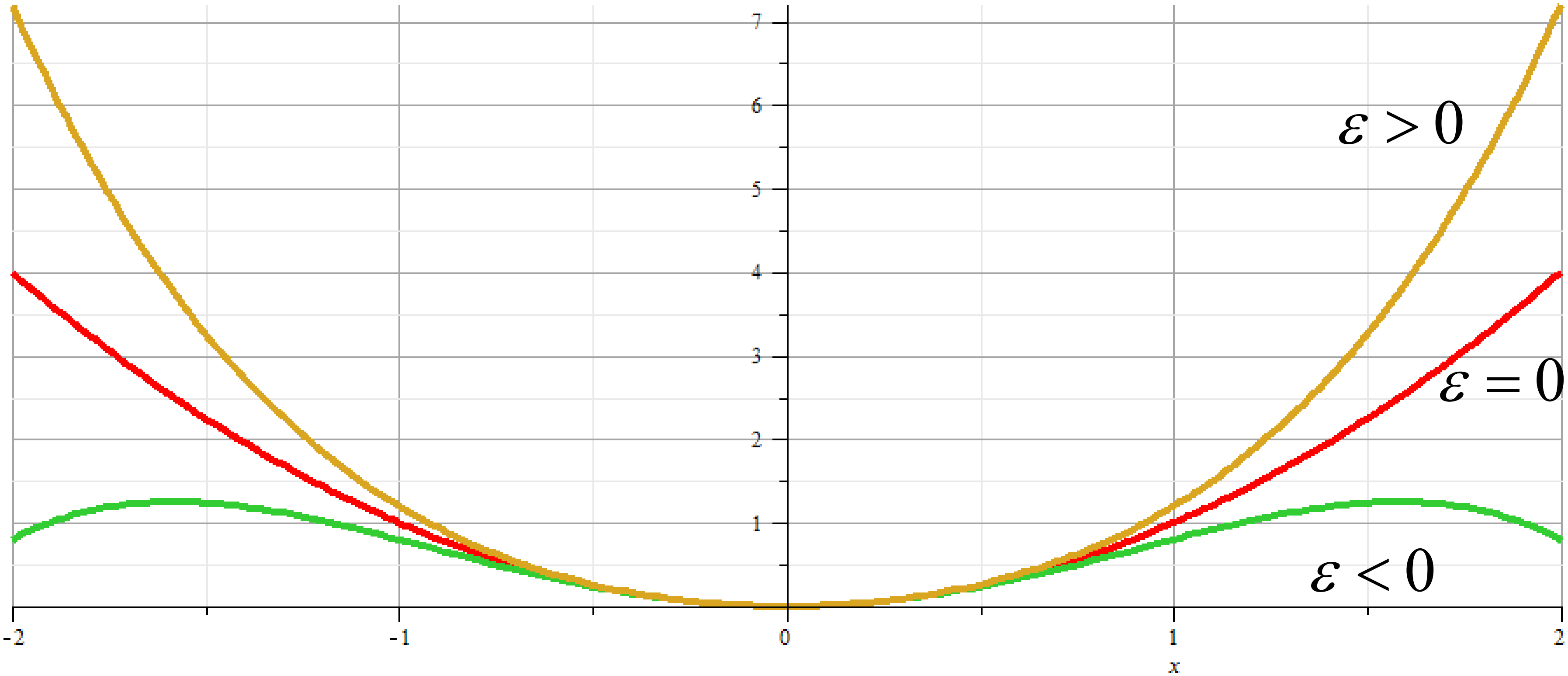
$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2}\varepsilon x^4 \right)$$

Euler - Lagrange equations :

$$\ddot{x} = -\omega^2 \left(x + \varepsilon x^3 \right)$$

Superposition-- no longer applies

$$V(x) \approx \frac{1}{2} m\omega^2 \left(x^2 + \frac{1}{2} \varepsilon x^4 \right)$$



x

Non - linear example - - continued

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2\left(x^2 + \frac{1}{2}\varepsilon x^4\right)$$

Euler - Lagrange equations :

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$$

Euler-Lagrange equations:

zero order (factor of ε^0): $\ddot{x}_0 + \omega^2 x_0 = 0$

first order (factor of ε^1): $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

Non-linear example -- continued

$$\ddot{x} + \omega^2\left(x + \varepsilon x^3\right) = 0$$

Initial conditions:

$$x(0) = X_0 \qquad \dot{x}(0) = 0$$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

Euler-Lagrange equations:

zero order: $\ddot{x}_0 + \omega^2 x_0 = 0$ $\qquad \Rightarrow x_0(t) = X_0 \cos(\omega t)$

first order: $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

$$\Rightarrow \ddot{x}_1(t) + \omega^2 x_1(t) = -X_0^3 \cos^3(\omega t) = -\frac{X_0^3}{4}\left(3\cos(\omega t) + \cos(3\omega t)\right)$$

$$\Rightarrow x_1(t) = -\frac{X_0^3}{8\omega^2}\left\{3\omega t \sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\}$$

$$x(t) = X_0 \cos(\omega t) - \varepsilon\frac{X_0^3}{8\omega^2}\left\{3\omega t \sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t)\right]\right\} + O\left(\varepsilon^2\right)$$

Non - linear example - - continued

$$\ddot{x} + \omega^2 \left(x + \varepsilon x^3 \right) = 0$$

Initial conditions:

$$x(0) = X_0 \qquad \dot{x}(0) = 0$$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$$

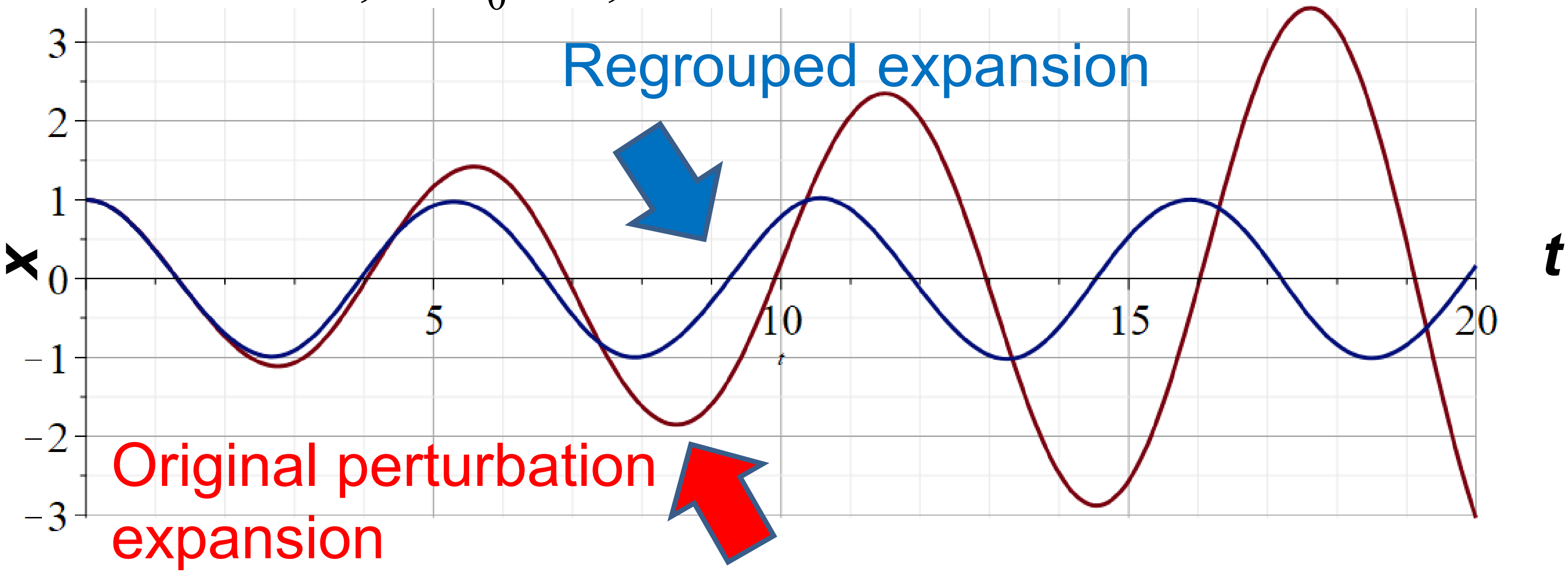
Previous result (blows up at large t):

$$x(t) = X_0 \cos(\omega t) - \varepsilon \frac{X_0^3}{8\omega^2} \left\{ 3\omega t \, \sin(\omega t) + \frac{1}{4}\left[\cos(\omega t) - \cos(3\omega t) \right] \right\} + O\left(\varepsilon^2\right)$$

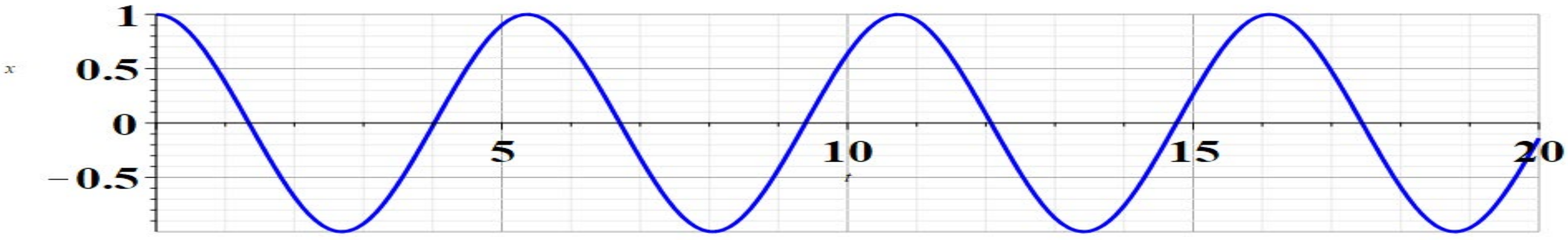
By rearranging terms (allowing effective frequency to vary):

$$x(t) = X_0 \cos\left(\omega \left(1 + \varepsilon \frac{3X_0^2}{8\omega} \right) t \right) - \varepsilon \frac{X_0^3}{32\omega^2} \left\{ \cos(\omega t) - \cos(3\omega t) \right\} + O\left(\varepsilon^2\right)$$

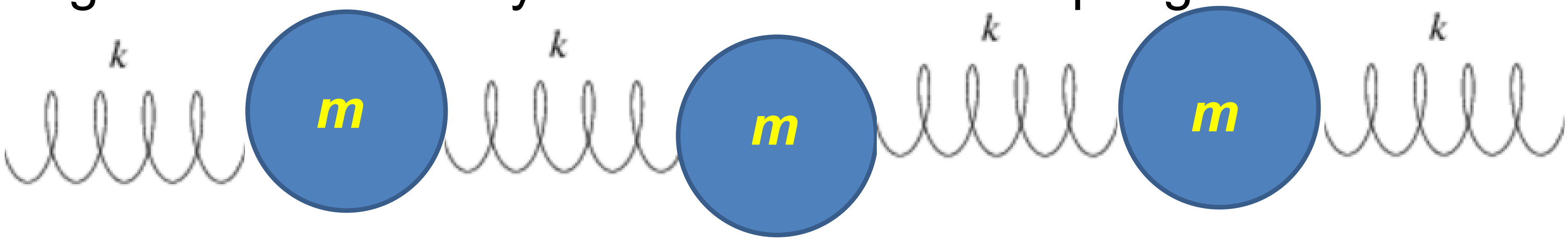
For $\omega = 1, \quad X_0 = 1, \quad \epsilon = 0.5$



Numerical solution according to Maple



Back to linear equations –
Longitudinal case: a system of masses and springs:



$$x_{i-1} \qquad x_i \qquad x_{i+1}$$

$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - x_i \right)^2$$

$$\Rightarrow m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1} \right)$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x,t) \qquad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2$$

Discrete equation : $m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$

Continuum equation : $m\dfrac{\partial^2 \mu}{\partial t^2} = k\left(\Delta x\right)^2 \dfrac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m/\Delta x}\right)\frac{\partial^2 \mu}{\partial x^2}$$

system parameter with units of (velocity)^2

For transverse oscillations on a string

with tension τ and mass/length σ:

$$\left(\frac{k\Delta x}{m/\Delta x}\right) \Rightarrow \frac{\tau}{\sigma}$$

More details

Longitudinal case

Consider Taylor's series (focussing on x-dependence)

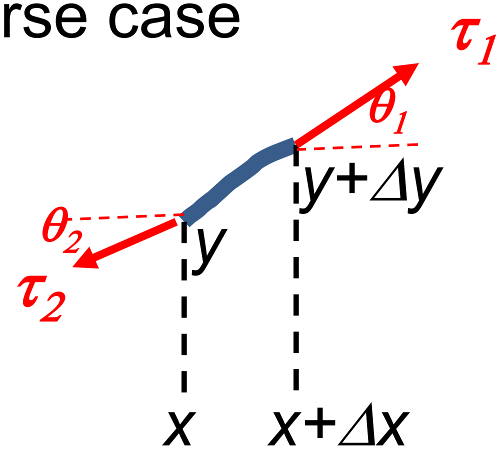
$$\mu(x+\Delta x)=\mu(x)+\Delta x\left.\frac{d\mu}{dx}\right|_x+\frac{1}{2}(\Delta x)^2\left.\frac{d^2\mu}{dx^2}\right|_x+\frac{1}{6}(\Delta x)^3\left.\frac{d^3\mu}{dx^3}\right|_x+\frac{1}{24}(\Delta x)^4\left.\frac{d^4\mu}{dx^4}\right|_x+.....$$

$$\mu(x-\Delta x)=\mu(x)-\Delta x\left.\frac{d\mu}{dx}\right|_x+\frac{1}{2}(\Delta x)^2\left.\frac{d^2\mu}{dx^2}\right|_x-\frac{1}{6}(\Delta x)^3\left.\frac{d^3\mu}{dx^3}\right|_x+\frac{1}{24}(\Delta x)^4\left.\frac{d^4\mu}{dx^4}\right|_x+.....$$

Therefore $\quad (\Delta x)^2\left.\frac{d^2\mu}{dx^2}\right|_x=\mu(x+\Delta x)+\mu(x-\Delta x)-2\mu(x)-\frac{1}{12}(\Delta x)^4\left.\frac{d^4\mu}{dx^4}\right|_x+.....$

$$\Rightarrow\left.\frac{d^2\mu}{dx^2}\right|_x\approx\frac{\mu(x+\Delta x)+\mu(x-\Delta x)-2\mu(x)}{(\Delta x)^2}$$

More details
Transverse case



Net vertical force on increment of string:

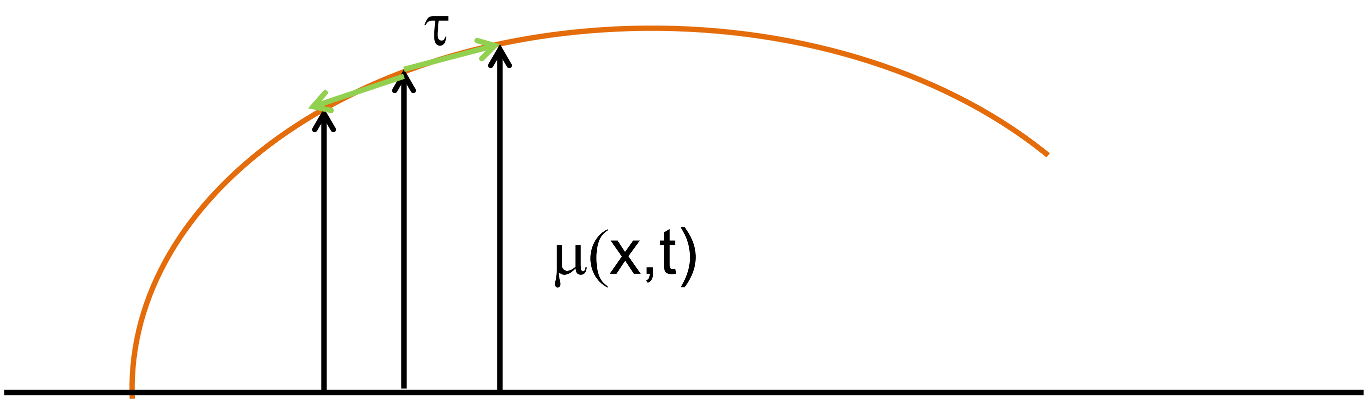
$$\tau_1 \sin\theta_1 - \tau_2 \sin\theta_2 \approx \tau_1 \tan\theta_1 - \tau_2 \tan\theta_2$$

$$\approx \tau \left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x} \right)$$

$$= \tau \Delta x \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \tau \left(\Delta x \frac{d^2 y}{dx^2} \right)$$

Transverse displacement:



$$\frac{\partial^2 \mu}{\partial t^2} = \frac{\tau}{\sigma}\frac{\partial^2 \mu}{\partial x^2}$$

Wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

Lagrangian for continuous system:

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle:

$$\delta \int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx\, L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x} \frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Euler - Lagrange equations for continuous system:

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Example:

$$L = \frac{\sigma}{2}\left(\frac{\partial \mu}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial \mu}{\partial x}\right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for} \quad c^2 = \frac{\tau}{\sigma}$$

Note that this is an example of a partial differential equation

General solutions $\mu(x,t)$ to the wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Initial value solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

These functions would be given

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

Solution -- continued: $\mu(x,t) = f(x-ct)+g(x+ct)$

then: $\mu(x,0) = \varphi(x) = f(x)+g(x)$

$$\frac{\partial\mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx}-\frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x)-g(x) = -\frac{1}{c}\int^{x}\psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\varphi(x)-\frac{1}{c}\int^{x}\psi(x')dx'\right)$$

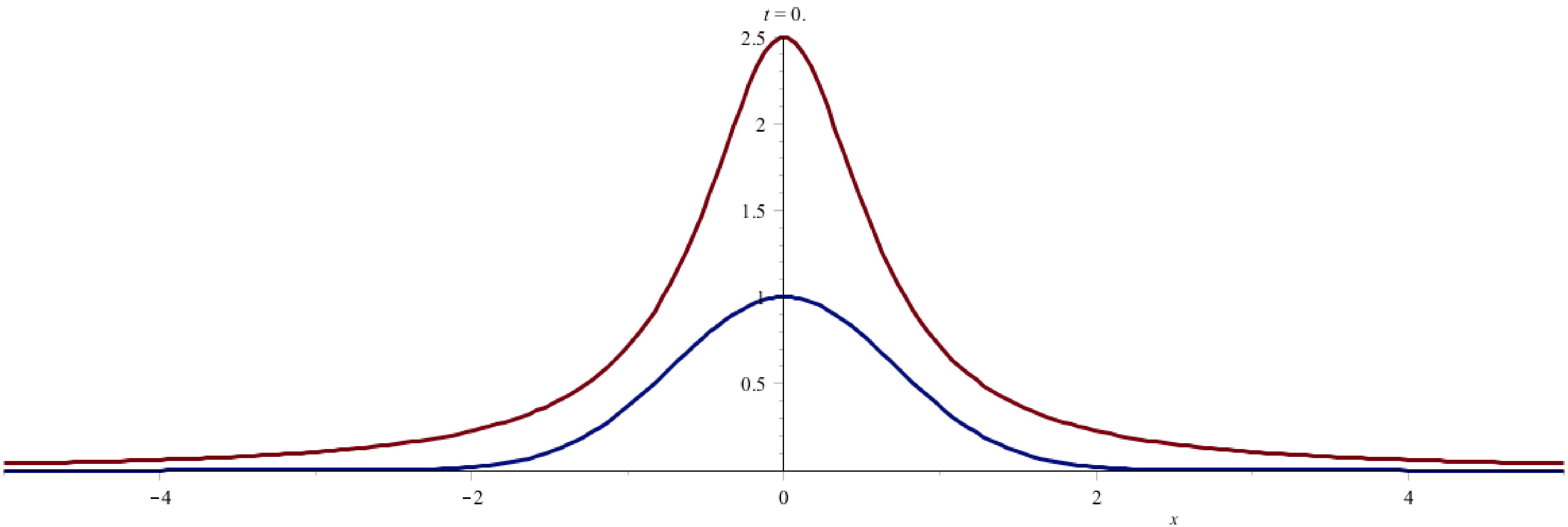
$$g(x) = \frac{1}{2}\left(\varphi(x)+\frac{1}{c}\int^{x}\psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\varphi(x-ct)+\varphi(x+ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x')dx'$$

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x},\mathrm{t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$



Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $\displaystyle \frac{\partial \mu(x,t)}{\partial t} = -\frac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$