PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 20: Review
Chap. 3,6,5,1,4,7 (F&W)

- Calculus of variation
- Lagrangian/Hamiltonian formalisms
- Phase space and the Liouville theorem
- Rigid body motion
- Scattering theory
- Small oscillations about equilibrium
- Wave motion in continuous media

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023	Chap. 3 & 6	Lagrangian equations of motion	#6
7	Mon, 9/11/2023	Chap. 3 & 6	Lagrangian to Hamiltonian formalism	#7
8	Wed, 9/13/2023	Chap. 3 & 6	Phase space	
9	Fri, 9/15/2023	Chap. 3 & 6	Canonical Transformations	#8
10	Mon, 9/18/2023	Chap. 5	Dynamics of rigid bodies	#9
11	Wed, 9/20/2023	Chap. 5	Dynamics of rigid bodies	#10
12	Fri, 9/22/2023	Chap. 5	Dynamics of rigid bodies	#11
13	Mon, 9/25/2023	Chap. 1	Scattering analysis	#12
14	Wed, 9/27/2023	Chap. 1	Scattering analysis	#13
15	Fri, 9/29/2023	Chap. 1	Scattering analysis	#14
16	Mon, 10/2/2023	Chap. 4	Small oscillations near equilibrium	
17	Wed, 10/4/2023	Chap. 4	Normal mode analysis	Mid term start
18	Fri, 10/6/2023	Chap. 4	Normal mode analysis	
22	Mon, 10/9/2023	Chap. 7	Normal modes of continuous string	
20	Wed, 10/11/2023		Review and summary	Mid term due
	Fri, 10/13/2023	Fall Break		
21	Mon, 10/16/2023			

Calculus of variation –

Consider a family of functions $y(x)$, with fixed end points

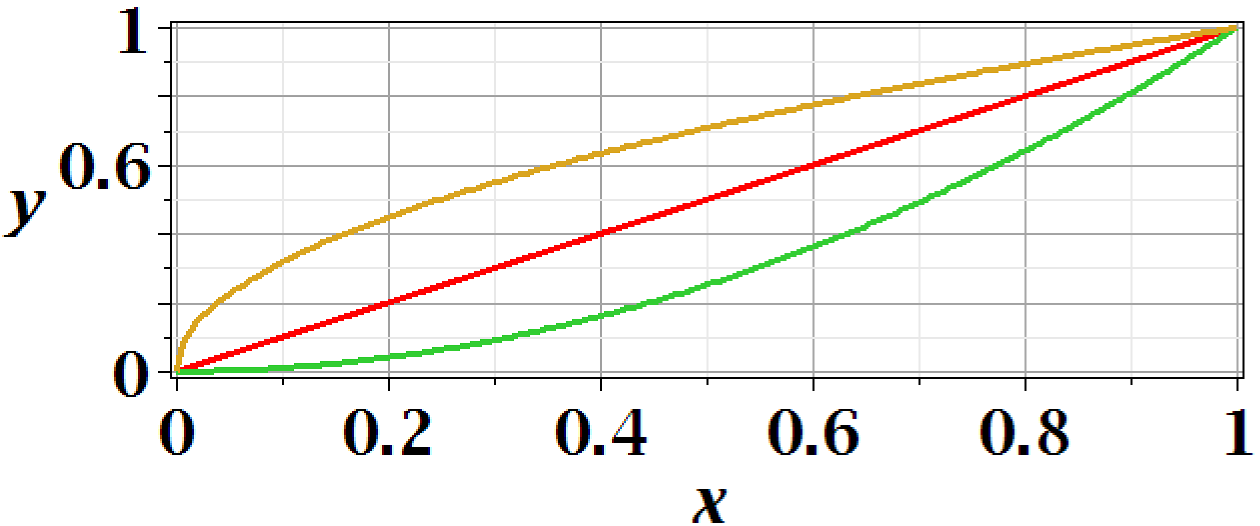
$y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral form $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y \, dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Note that this is a
"total" derivative

Summary --

Optimizing $I = \int_{x_i}^{x_f} f(y, \frac{dy}{dx}, x)dx$ -- for fixed $y(x_i) \equiv y_i$ and $y(x_f) \equiv y_f$

$$\delta I = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Euler-Lagrange equation

Question -- what would be the Euler-Lagrange-type relation for

optimizing $I = \int_{x_i}^{x_f} f(y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, x)dx$ -- for fixed $y(x_i) \equiv y_i$ and $y(x_f) \equiv y_f$

We are now going to shift notation in order to apply the calculus of variation formalism to Hamilton's principle and Lagrangian mechanics.

$$x \to t$$

$$y(x) \to q(t)$$

$$\frac{dy}{dx} \to \dot{q}(t)$$

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$S = \int_{t_1}^{t_2} L\left(\{q,\dot{q}\};t\right)dt \equiv \int_{t_1}^{t_2} L\left(\left\{y,\frac{dy}{dt}\right\};t\right)dt$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g.*

Newton's formulation: $\quad m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $\quad y(t) = y_i + v_i t - \frac{1}{2}gt^2$

Lagrangian for this case:

$$L = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) dt \quad \text{is minimized for physical } y(t):$$

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Extension of these ideas to multiple coordinates due to multiple dimensions and/or multiple particles.

1 particle with 1 degree of freedom ➔ **many particles with multiple degrees of freedom**

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \quad \Rightarrow \quad S = \int_{t_1}^{t_2} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)\, dt$$

for example: $L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$

1 particle+3 Cartesian dimensions

or $L(\{x_i\}, \{y_i\}, \{z_i\}, \{\dot{x}_i\}, \{\dot{y}_i\}, \{\dot{z}_i\}, t)$

N particles+3 Cartesian dimensions

Another example: $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}\left(I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}\left(I_1\dot{\beta}\right) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}\left(I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\right) = 0$$

Modification of Lagrangian due to electric and magnetic fields

For a single particle of charge q using cartesian coordinates and cgs units.

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi\left(\mathbf{r},t\right) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$$

where $\mathbf{E}\left(\mathbf{r},t\right) = -\nabla\Phi\left(\mathbf{r},t\right) - \frac{1}{c}\frac{\partial\mathbf{A}\left(\mathbf{r},t\right)}{\partial t} \qquad \mathbf{B}\left(\mathbf{r},t\right) = \nabla\times\mathbf{A}\left(\mathbf{r},t\right)$

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi\left(\mathbf{r},t\right) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r},t\right)$$

Introducing the Hamiltonian --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$

2. Compute generalized momenta : $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$

3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$

4. Form Hamiltonian function : $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Example problem --

PHY 711 – Assignment #7

Assigned: 09/11/2023 Due: 09/18/2023

The material for this exercise is covered in the lecture notes and in Chapters 3 and 6 of Fetter and Walecka.

1. A particle of mass m and charge q is subjected to a vector potential $\mathbf{A}(\mathbf{r}, t) = -(E_0 ct + B_0 x)\hat{\mathbf{z}}$. (Note that we are using the cgs Gaussian units of your text book.) Here E_0 denotes a constant electric field amplitude and B_0 denotes a constant magnetic field amplitude. The initial particle position is $\mathbf{r}(0) = 0$ and the initial particle velocity is $\dot{\mathbf{r}}(0) = 0$.

   (a) Determine the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ which describes the particle's motion.

   (b) Write the Euler-Lagrange equations for this system.

   (c) Find the particle trajectories $x(t)$, $y(t)$, $z(t)$ by solving the equations and imposing the given initial conditions.

   (d) Determine the Hamiltonian for this system and evaluate dH/dt.

Steps for tackling a problem –

1. What are the basic concepts the apply to this problem.
2. Write down the fundamental equations
3. Solve
4. Check.

In this case, we expect that we should use the Lagrangian formalism and thus we need to know how to represent electric and magnetic fields in the Lagrangian.

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = L_{mech}(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) - q\left(\Phi(\mathbf{r},t) - \frac{1}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\right)$$

In this example: $\Phi(\mathbf{r},t) = 0$

$$\mathbf{A}(\mathbf{r},t) = -\hat{\mathbf{z}}\left(E_0 ct + B_0 x\right)$$

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = L_{mech}(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) - q\left(\Phi(\mathbf{r},t) - \frac{1}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\right)$$

In this example: $\Phi(\mathbf{r},t) = 0$

$$\mathbf{A}(\mathbf{r},t) = -\hat{\mathbf{z}}\left(E_0 ct + B_0 x\right)$$

Note that this corresponds to an electric field:

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = E_0\hat{\mathbf{z}}$$

and a magnetic field:

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) = B_0\hat{\mathbf{y}}$$

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = \frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2 - \frac{q\dot{z}}{c}\left(E_0 ct + B_0 x\right)$$

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t)=\frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2-\frac{q\dot{z}}{c}\left(E_0ct+B_0x\right)$$

Digression on forming the Hamiltonian for this case:

$$p_x=\frac{\partial L}{\partial\dot{x}}=m\dot{x}$$

$$p_y=\frac{\partial L}{\partial\dot{y}}=m\dot{y}$$

$$p_z=\frac{\partial L}{\partial\dot{z}}=m\dot{z}-\frac{q}{c}\left(E_0ct+B_0x\right)$$

$$H=p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-L=\frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2$$

Is this correct?

$$H(\mathbf{r}(t),\mathbf{p}(t),t) = \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}\left(p_z + qE_0 t + \frac{q}{c}B_0 x\right)^2$$

Note that $\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{qE_0}{m}\left(p_z + qE_0 t + \frac{q}{c}B_0 x\right)$ why?

Comment on solving equations of motion

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = \frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2 - \frac{q\dot{z}}{c}\left(E_0 ct + B_0 x\right)$$

Initial values: $\mathbf{r}(0) = 0 \qquad \dot{\mathbf{r}}(0) = 0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}m\dot{x} + \frac{qB_0\dot{z}}{c} \qquad \Rightarrow \dot{x} + \frac{qB_0 z}{mc} = K = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \qquad \Rightarrow y(t) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0 = \frac{d}{dt}\left(m\dot{z} - \frac{q}{c}\left(E_0 ct + B_0 x\right)\right) \qquad \Rightarrow m\dot{z} - \frac{q}{c}\left(E_0 ct + B_0 x\right) = K' = 0$$

Use these two equations to decouple x(t) and z(t)

Introduction to rigid body analysis

Moment of inertia tensor:

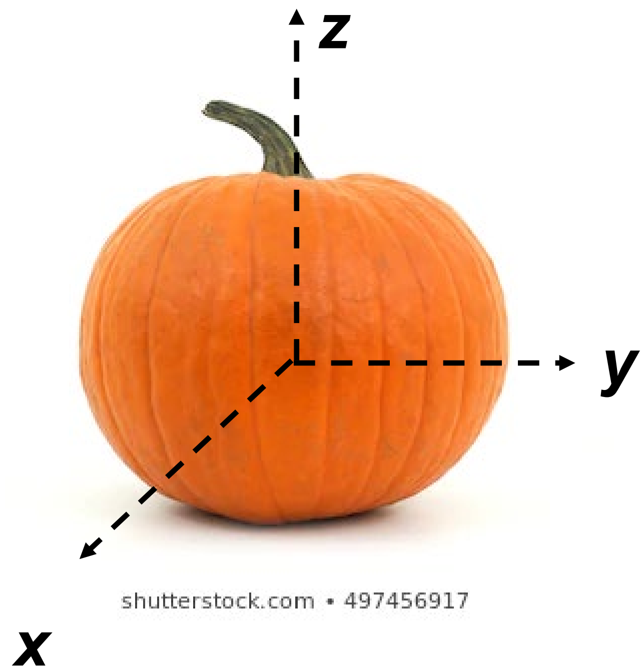
$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p \right) \qquad \text{(dyad notation)}$$

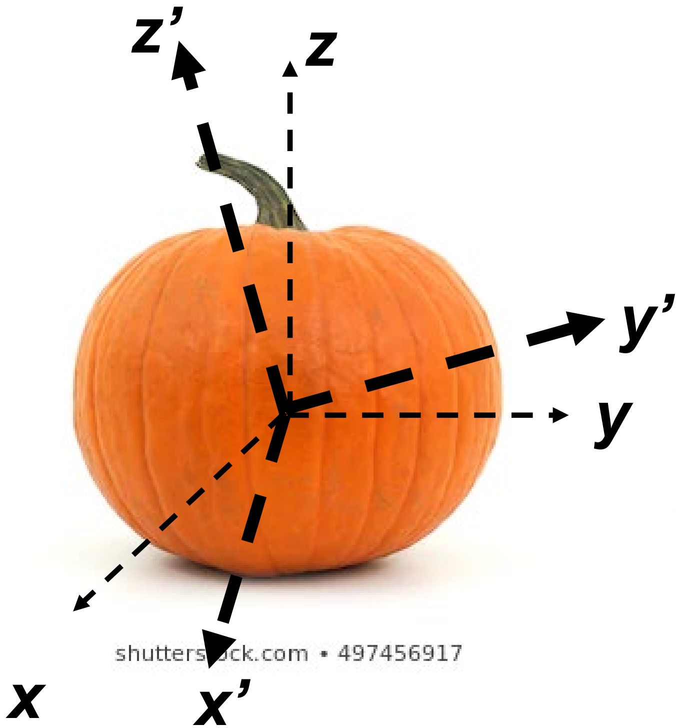
Note: For a given object and a given coordinate system, one can find the moment of inertia matrix

Matrix notation :

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$





Moment of inertia in original coordinates

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

Moment of inertia in principal axes ***(x',y',z')***

$$\overleftrightarrow{\mathbf{I}} \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Descriptions of rotation about a given origin; analyzed in the body fixed frame

Torque equation applied to the angular momentum of body:

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1$$

$$+\tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3 = 0$$

Torqueless Euler equations for rotation in body fixed frame:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$

$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

➔ Solution for symmetric object with $I_2 = I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_1\right) = 0$$

$$I_1\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$

$$I_3\dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = \text{(constant)}$$

Define: $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega$$

$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

Solution of Euler equations for symmetric object continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

where $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

Solution:
$$\tilde{\omega}_1(t) = A\cos(\Omega t + \phi)$$
$$\tilde{\omega}_2(t) = A\sin(\Omega t + \phi)$$
$$\tilde{\omega}_3(t) = \tilde{\omega}_3 \quad \text{(constant)}$$

$$T = \frac{1}{2}\sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$
$$= I_1 A\left(\cos\left(\Omega t + \varphi\right)\hat{\mathbf{e}}_1 + \sin\left(\Omega t + \varphi\right)\hat{\mathbf{e}}_2\right) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Scattering theory:

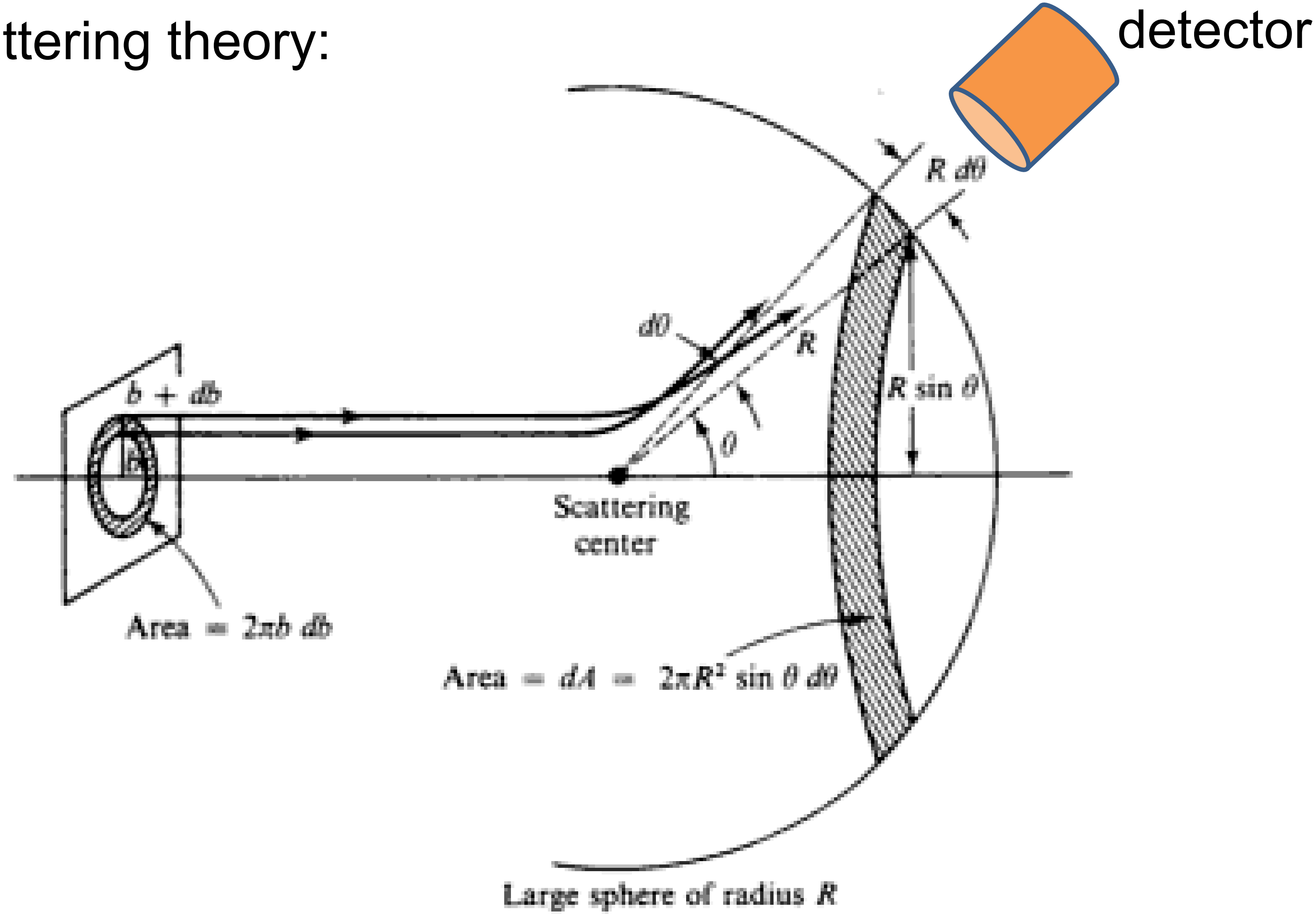
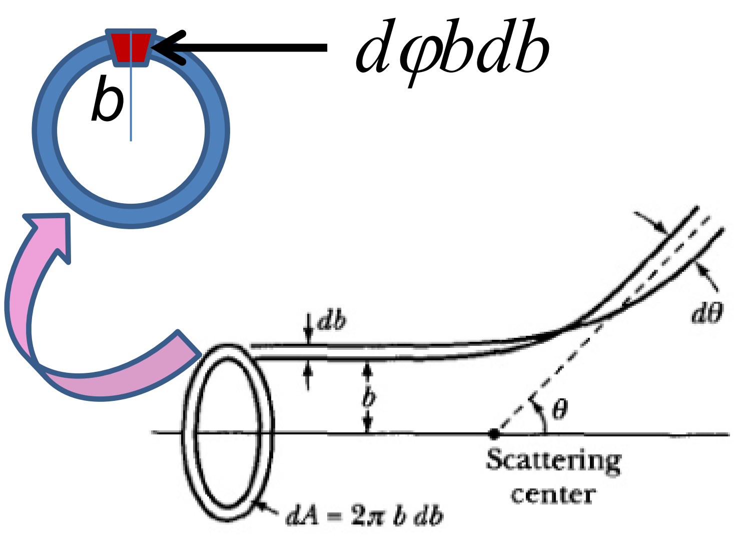


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area per solid angle}}$$

$=$ Area of incident beam that is scattered into detector at angle θ



$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

General formula relating b and θ:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

$\Rightarrow$ There are relatively few forms of $V(1/u)$ for which the integral has an analytic result.

A problem mentioned in F&W for an example:

$$V(r) = \frac{\gamma}{r^2} \text{ where } \frac{d\sigma}{d\Omega} = \frac{\gamma\pi^2}{E\sin\theta} \frac{(\pi - \theta)}{\theta^2 (2\pi - \theta)^2}$$

More generally, it is possible to use numerical integration methods (with care) to evaluate $b(\theta)$ to relate exp and theory.