PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 23: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation ➔ various mathematical problems and techniques including:

1. Completeness property of Sturm-Liouville eigenfunctions
2. Construction of Green's functions
3. Fourier transforms
4. Laplace transforms
5. Complex variables
6. Contour integrals

22	Mon, 10/9/2023	Chap. 7	Normal modes of continuous string	
20	Wed, 10/11/2023		Review and summary	Mid term due
	Fri, 10/13/2023	Fall Break		
21	Mon, 10/16/2023	Chap. 7	The wave and other partial differential equations	#15
22	Wed, 10/18/2023	Chap. 7	Sturm-Liouville equations	#16
23	Fri, 10/20/2023	Chap. 7	Sturm-Liouville equations	#17
24	Mon, 10/23/2023			
25	Wed, 10/25/2023			
26	Fri, 10/27/2023			
27	Mon, 10/30/2023			
28	Wed, 11/01/2023			
29	Fri, 11/03/2023			

PHY 711 -- Assignment #17

Assigned: 10/20/2023 Due: 10/23/2023

Continue reading Chapter 7 in **Fetter & Walecka**.

1. Consider the function $f(x) = x^2 (1-x)$ in the interval $0 \leq x \leq 1$. Find the coefficients A_n of the Fourier series based on the terms sin($n \pi x$). Extra credit: Plot $f(x)$ and the Fourier series including 3 terms.

Review – Sturm-Liouville equations defined over a range of x.

For $x_a \leq x \leq x_b$

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right)\varphi_0(x) = 0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x) \right)\varphi(x) = F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a "complete" set over the range of x defined for the particular system.

Formal statement of the completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x') \quad \text{where} \quad N_n \equiv \int_{x_a}^{x_b} dx\sigma(x)\left(f_n(x)\right)^2$$

This means that within the interval $x_a \leq x \leq x_b$,

an arbitrary function $h(x)$ can be expanded: $h(x) = \sum_n C_n f_n(x)$.

Some details: Note that:

$$\int_a^b \sigma(x) f_n(x) f_m(x) dx = N_n \delta_{nm}$$

Now consider $K\left(\{C_n\}\right) \equiv \int_a^b \sigma(x)\left| h(x) - \sum_{n=1}^{N} C_n f_n(x)\right|^2 dx$

Some details: Note that:

$$\int_a^b \sigma(x) f_n(x) f_m(x) dx = N_n \delta_{nm}$$

Now consider $K\left(\{C_n\}\right) \equiv \int_a^b \sigma(x) \left| h(x) - \sum_{n=1}^{N} C_n f_n(x) \right|^2 dx$

Choosing the optimal $\{C_n\}$ which minimize $K\left(\{C_n\}\right)$:

$$\frac{\partial K\left(\{C_n\}\right)}{\partial C_m} = 0 \quad \Rightarrow C_m = \frac{1}{N_m} \int_a^b \sigma(x) h(x) f_m(x) dx$$

Formal completeness statement:

$$\sigma\left(x\right) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta\left(x - x'\right)$$

$$\int_a^b dx\ h(x) \sigma\left(x\right) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \int_a^b dx\ h(x) \delta\left(x - x'\right)$$

Formal completeness statement:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x')$$

$$\int_a^b dx\ h(x)\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \int_a^b dx\ h(x)\delta(x-x')$$

$$\sum_n \left(\frac{1}{N_n}\int_a^b dx h(x)\sigma(x) f_n(x)\right) f_n(x') = h(x')$$

Example Sturm-Liouville system --

Example for $\tau(x) = 1 = \sigma(x)$ and $v(x) = 0$ with

$0 \leq x \leq L$ and $f_n(0) = 0 = f_n(L)$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)\right)f_n(x) = \lambda_n \sigma(x) f_n(x) \quad \Rightarrow -\frac{d^2 f_n(x)}{dx^2} = \lambda_n f_n(x$$

In this case, the normalized eigenfunctions are

$$f_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, ...$$

Comment about normalized eigenfunctions:

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, ...$$

Check: $\int_0^L dx \left(f_n(x)\right)^2 = \frac{2}{L}\int_0^L dx \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 = \frac{2}{L}\int_0^L dx \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) = \frac{2}{L}\frac{L}{2} = 1$

Joseph Fourier

Jean-Baptiste Joseph Fourier

Born	21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France)
Died	16 May 1830 (aged 62)

Special case: $\tau(x) = 1 = \sigma(x) \qquad v(x) = 0$

$$-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{for } 0 \leq x \leq a, \quad \text{with } f_n(0) = f_n(a) = 0$$

$$f_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad \lambda_n = \left(\frac{n\pi}{a}\right)^2$$

Fourier series representation of function $h(x)$ in the interval $0 \leq x \leq a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$A_n = \sqrt{\frac{2}{a}} \int_0^a dx' \; h(x') \; \sin\left(\frac{n\pi x'}{a}\right)$$

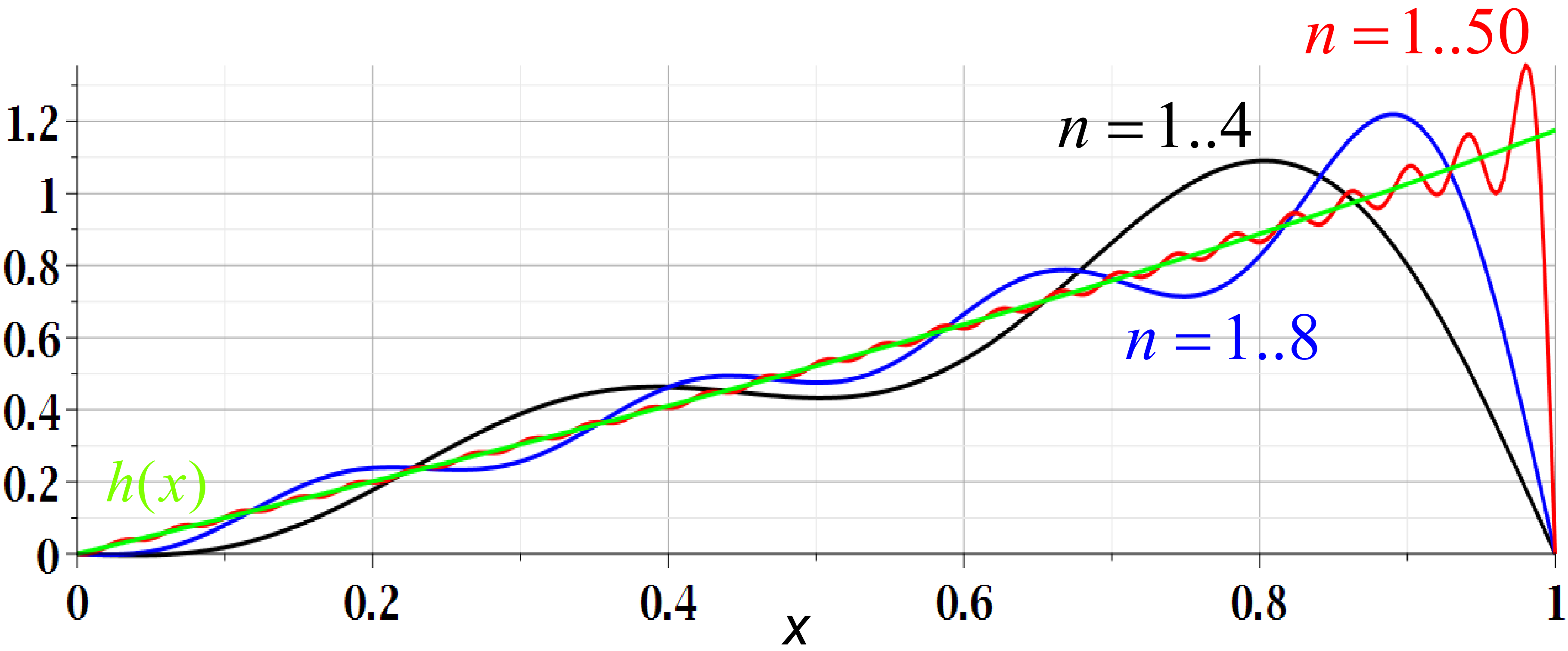
*Note that if $h(x)$ does not vanish at $x = 0$ and $x = a$, the more general

expression applies: $$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) + \sum_{n=0}^{\infty} B_n \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right)$$

(with some restrictions).

Example

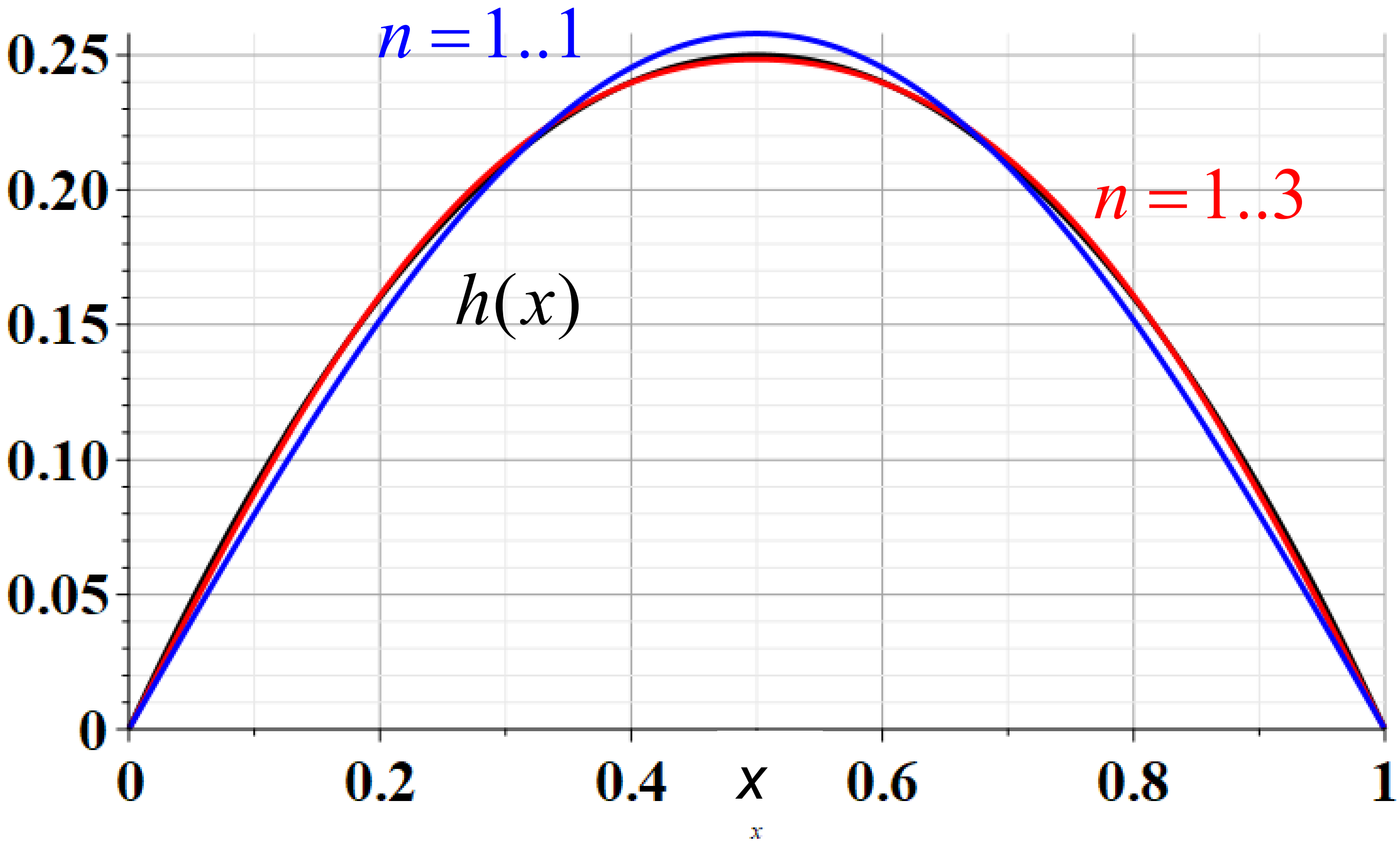
$$h(x) = \sinh(x) = \sqrt{2}\pi \sinh(1)\left(\frac{\sin(\pi x)}{\pi^2+1} - \frac{2\sin(2\pi x)}{4\pi^2+1} + \ldots - (-1)^n n \frac{\sin(n\pi x)}{n^2\pi^2+1} + \ldots\right)$$



Example

$$h(x) = x(1-x)$$

$$= \sum_{n=1}^{\infty} A_n \sqrt{2}\sin(n\pi x) \qquad A_n = \begin{cases} \dfrac{4\sqrt{2}}{n^3\pi^3} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$



Fourier series representation of function $h(x)$ in the interval $0 \le x \le a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \text{with} \qquad A_n = \sqrt{\frac{2}{a}} \int_0^a dx' \; h(x') \; \sin\left(\frac{n\pi x'}{a}\right)$$

Can show that the series converges provided that $h(x)$ is

piecewise continuous.

Note that this analysis can also apply to time dependent functions. In the remainder of the lecture, we will consider time dependent functions.

$$x \to t \qquad a \to T \qquad 0 \le t \le T \qquad \frac{n\pi}{a} \to \frac{n\pi}{T} \equiv \omega_n$$

$$h(t) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{T}} \sin(\omega_n t) \qquad A_n = \sqrt{\frac{2}{T}} \int_0^T dt' \; h(t') \; \sin(\omega_n t')$$

Note that for this finite time range, Fourier series is discrete in frequency and continuous in time.

Comment on complex functions –

We can also treat complex functions, typically separately considering the real and imaginary parts.

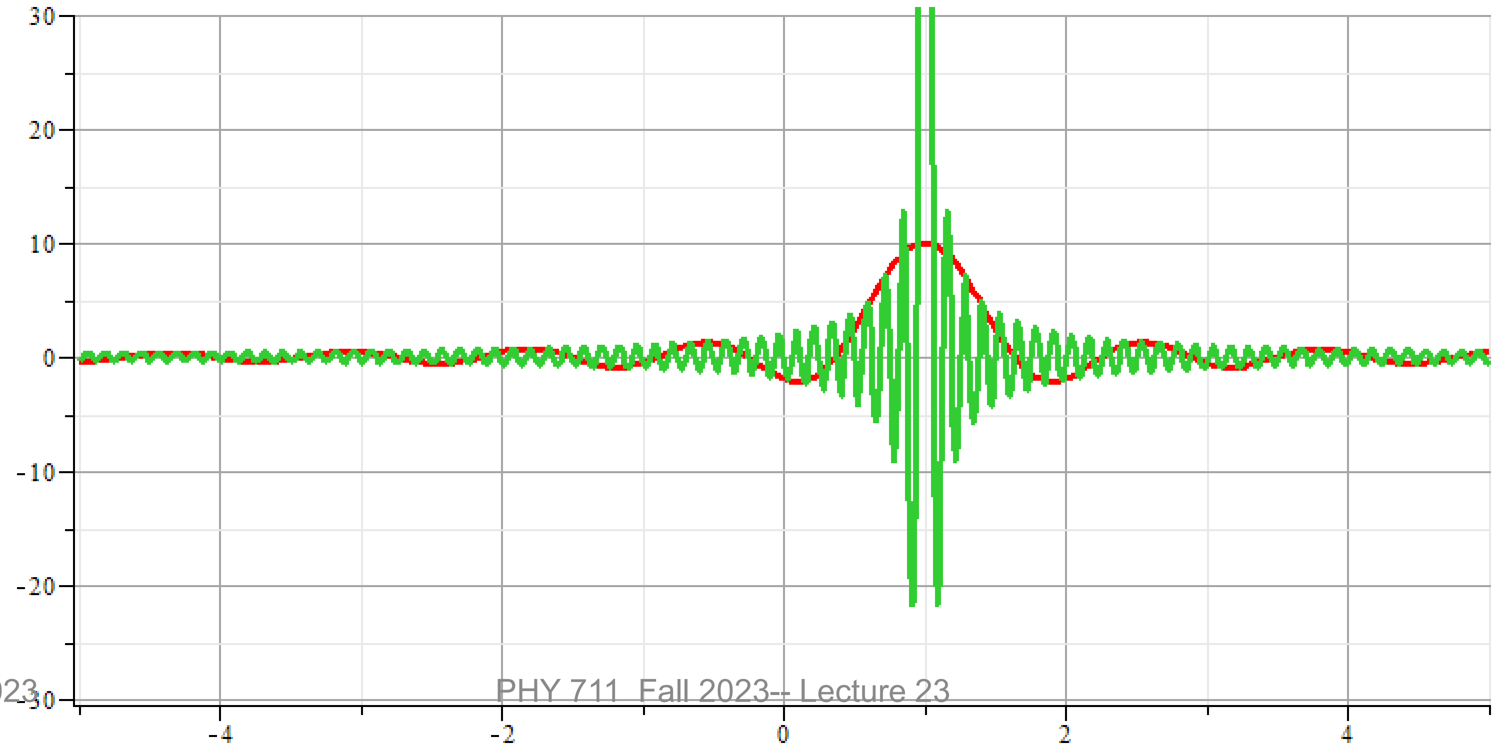
Generalization to infinite range -- Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\ e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt\ e^{-i(\omega-\omega_0)t} = \frac{2\sin\left[(\omega-\omega_0)T\right]}{\omega-\omega_0} \underset{T\rightarrow\infty}{\approx} 2\pi\delta(\omega-\omega_0)$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega \, F(\omega) \, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, f(t) \, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \, f(t') \, e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' \, f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \;\, e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' \, f(t') \, \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Question about forward and backward transforms –

Comment – Nomenclature for forward and backward can be confusing, but the point is that if we know the time dependence, we can determine the frequency dependence and if we know the frequency dependence, we can determine the time dependence.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt \left(f(t)\right)^* f(t) = 2\pi \int_{-\infty}^{\infty} d\omega \left(F(\omega)\right)^* F(\omega)$$

Check:

$$\int_{-\infty}^{\infty} dt \left(f(t)\right)^* f(t) = \int_{-\infty}^{\infty} dt \left(\left(\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right)$$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t}$$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega')\, 2\pi\delta(\omega' - \omega)$$

$$= 2\pi \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$

Note that for an infinite time range, the Fourier transform is continuous in both time and frequency.

Use of Fourier transforms to solve wave equation

Wave equation: $$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Suppose $u(x,t) = e^{-i\omega t}\tilde{F}(x,\omega)$ where $\tilde{F}(x,\omega)$ satisfies the equation:

$$\frac{\partial^2 \tilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2}\tilde{F}(x,\omega) \equiv -k^2\tilde{F}(x,\omega)$$

More generally:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(x,\omega) e^{-i\omega t}$$

Further assume that fixed boundary conditions apply: $0 \le x \le L$

with $\tilde{F}(0,\omega) = 0$ and $\tilde{F}(L,\omega) = 0$

For $n = 1,2,3\cdots$

$$\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t}\sin(k_n x) = e^{-i\omega_n t}\frac{\left(e^{ik_n x} - e^{-ik_n x}\right)}{2i} = \frac{\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition: Suppose $u(x,t) = \sum_n A_n e^{-i\omega_n t} \tilde{F}_n(x, \omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \tilde{F}(x,\omega_n) \equiv -k_n^2 \tilde{F}(x,\omega_n)$$

For $\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right)$ $\quad k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n A_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{A_n}{2i} e^{-i\omega_n t}\left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{A_n}{2i}\left(e^{ik_n (x-ct)} - e^{-ik_n (x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

Note that at this point, we do not know the coefficients A_n; however, it clear that the solutions are consistent with D'Alembert's analysis of the wave equation.

Now consider the Fourier transform for a time periodic function:

Suppose $f(t+nT) = f(t)$ for any integer n

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ f(t) e^{i\omega t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{0}^{T} dt\ f(t) e^{i\omega(t+nT)}\right)$$

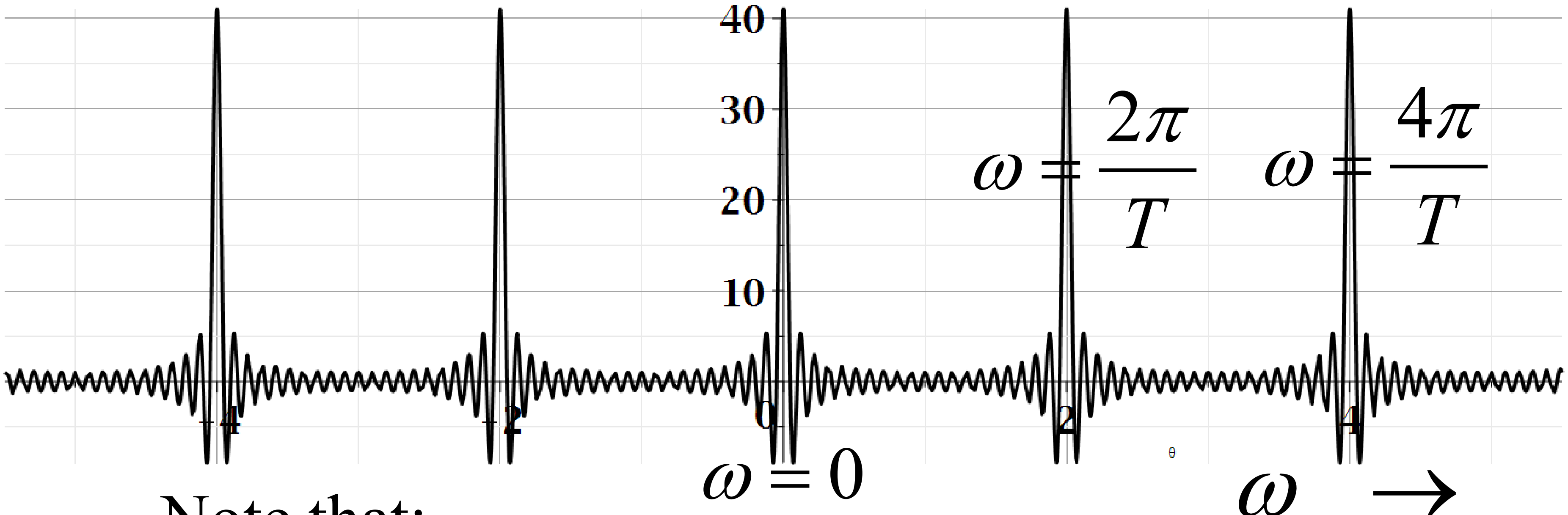
Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

Details:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \lim_{N \to \infty} \sum_{n=-N}^{N} e^{in\omega T} = \lim_{N \to \infty} \frac{\sin\left(\left(N + \frac{1}{2}\right)\omega \mathrm{T}\right)}{\sin\left(\frac{1}{2}\omega \mathrm{T}\right)}$$

$$\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega \mathrm{T}\right)}{\sin\left(\frac{1}{2}\omega \mathrm{T}\right)}$$



Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

Geometric summation: $$\sum_{n=-N}^{N} e^{in\omega T} = \frac{\sin\left(\left(N+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{N\to\infty}\left(\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right) = 2\pi\sum_{\nu}\delta\left(\omega T-\nu\Omega T\right) = \frac{2\pi}{T}\sum_{\nu}\delta\left(\omega-\nu\Omega\right)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ f(t) e^{i\omega t} = \frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty}\Omega\delta\left(\omega-\nu\Omega\right)\left(\int_{0}^{T} dt\ f(t) e^{i\omega t}\right)$$

Thus, for a time periodic function

$$f(t) = \int_{-\infty}^{\infty} d\omega\ F(\omega)\ e^{-i\omega t} = \sum_{\nu=-\infty}^{\infty}\bar{F}\left(\nu\Omega\right)e^{-i\nu\Omega t},$$

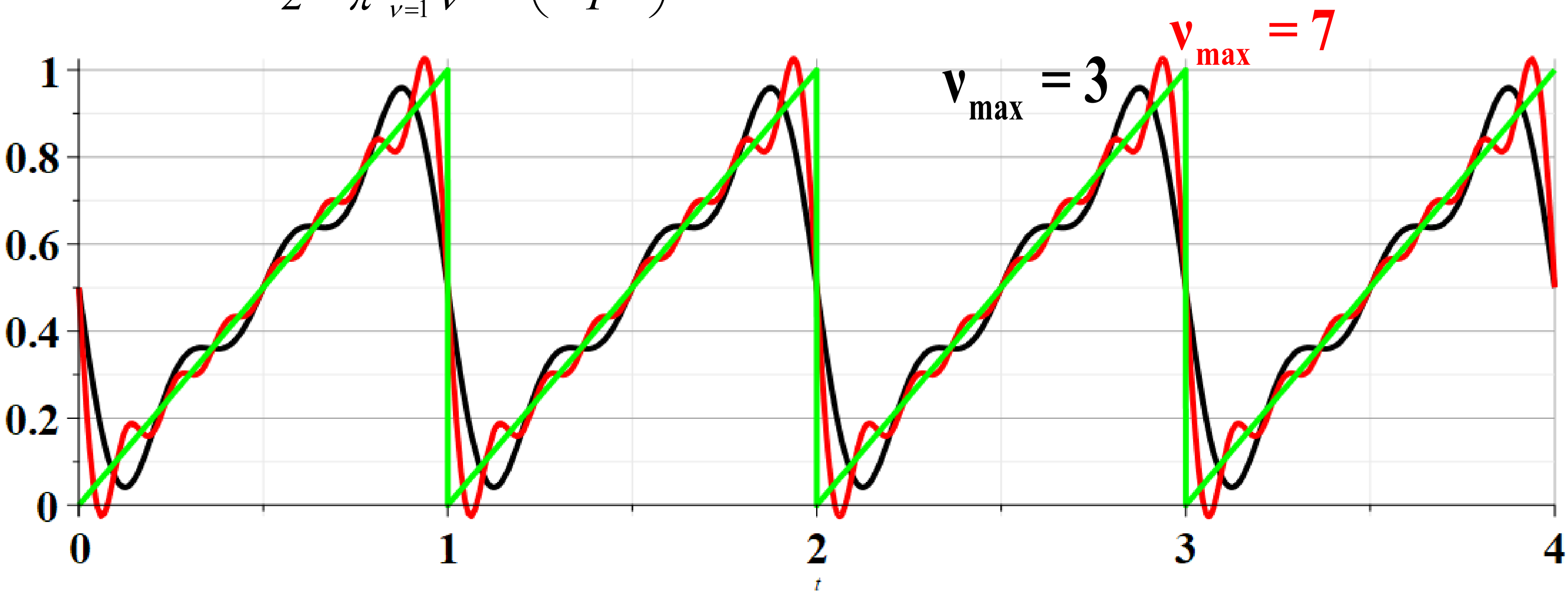
where $\bar{F}\left(\nu\Omega\right) = \frac{1}{T}\int_{0}^{T} dt\ f(t) e^{i\nu\Omega t}$

Example:

Suppose: $\quad f(t)=\dfrac{t-nT}{T}$ for $nT\le t\le (n+1)T;\quad n=\ldots-3,-2,-1,0,1,2,3\ldots$

$$\bar{F}(\nu\Omega)=\frac{1}{T}\int_0^T\frac{t}{T}\,e^{i\frac{\nu 2\pi t}{T}}dt=\bar{F}^*(-\nu\Omega)=\frac{-i}{2\pi\nu}\ \text{for}\ \nu=1,2,3\ldots\quad \bar{F}(0)=\frac{1}{2}$$

$$f(t)=\frac{1}{2}-\frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\sin\left(\frac{2\pi\nu t}{T}\right)$$



Summary –

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega \; F(\omega) \; e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \;\; f(t) \, e^{i\omega t}$$

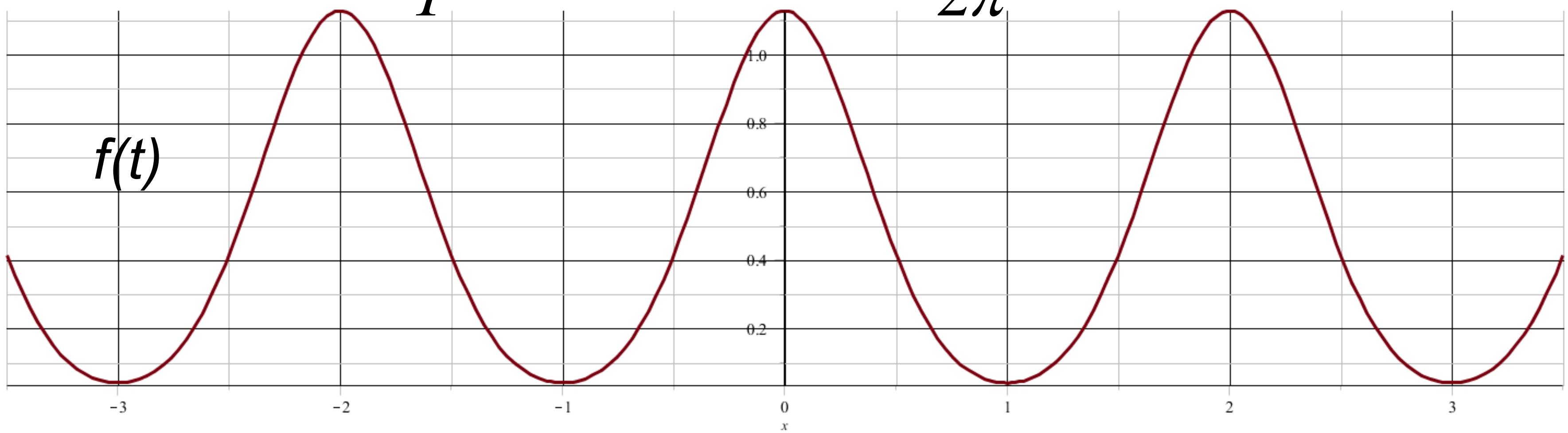
Find discrete frequencies ω for functions *f(t)* over finite time domain of for functions *f(t)* which are periodic*: f(t)=f(t+nT)*

➔Numerically, there is an advantage of tabulating double discrete Fourier transforms (discrete in ω and in *t*).

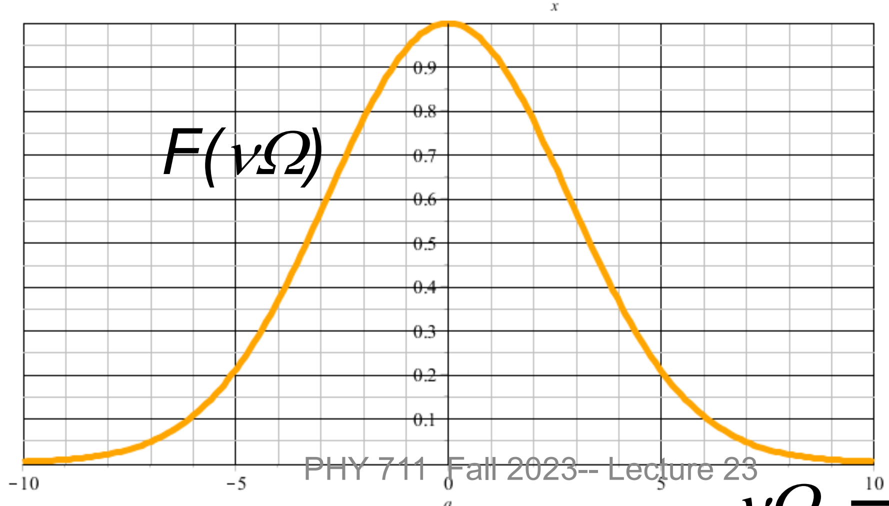
Example:

Suppose: $$f(t) = \frac{1}{a\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

where $\Omega \equiv \frac{2\pi}{T}$ and $F(\nu\Omega) = \frac{1}{2\pi} e^{-a^2\nu^2\Omega^2/4}$

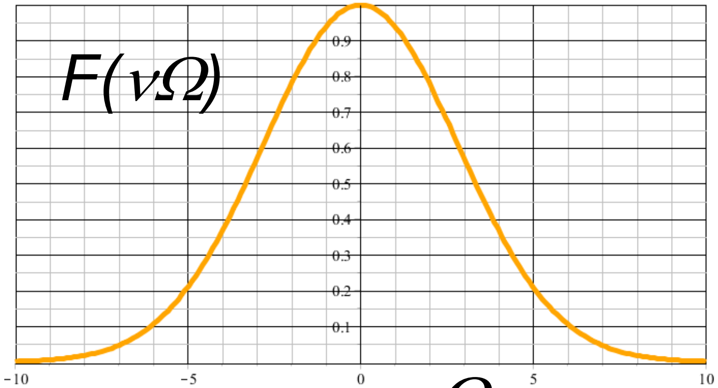


$t \longrightarrow$



$\nu\Omega \longrightarrow$

Continued: $$f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$$



$\nu\Omega$ ⟶

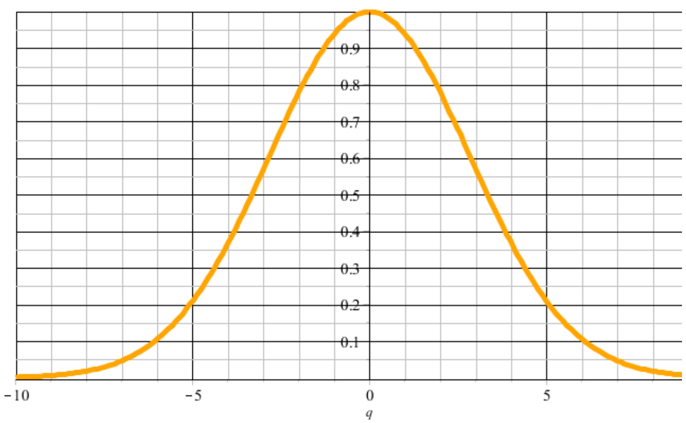
$$\Omega=\frac{2\pi}{T}$$

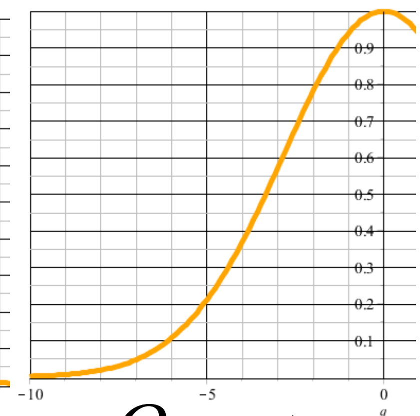
Note:

$$f(t)\approx\sum_{\nu=-M}^{M} F(\nu\Omega)e^{-i\nu\Omega t}$$

because $F(\nu'\Omega)\approx 0$

for $|\nu'|>M$





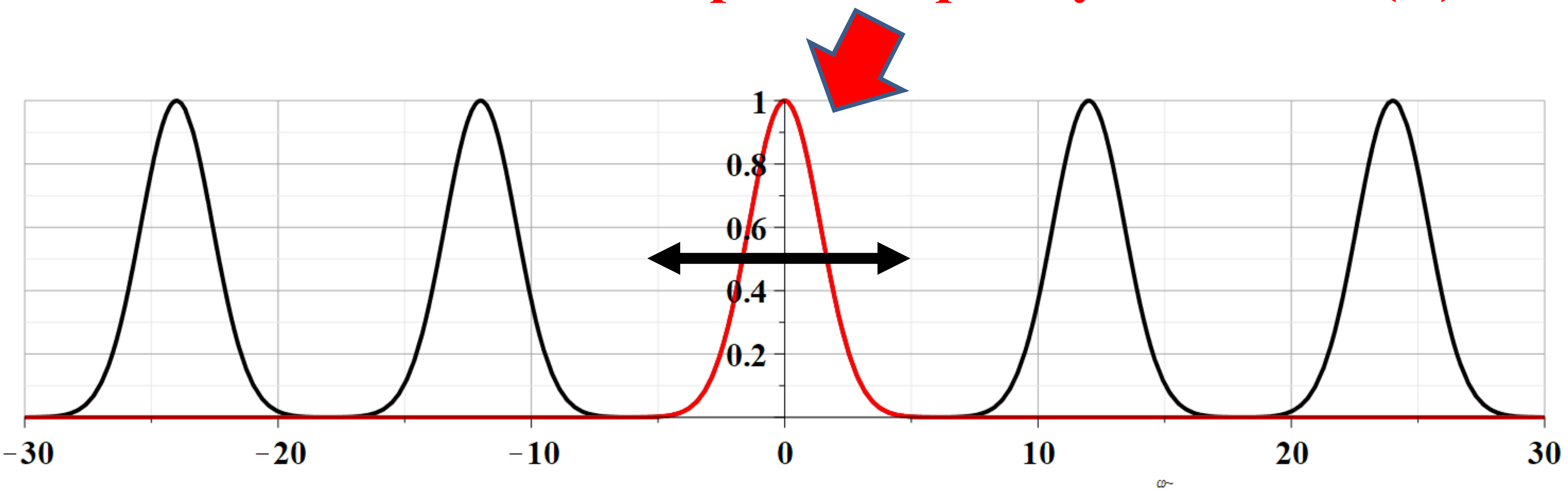
$\nu\Omega$ ⟶

$\nu=-M$

$\nu=M$

Constructed frequency periodic function --

Envelope of frequency function F(ω)



Falsely periodic frequency function $\tilde{F}(\omega)$

Thus, for any periodic function: $f(t)=\sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$

Now suppose that the transformed function is bounded;

$\left|F(\nu\Omega)\right| \le \varepsilon$ for $|\nu| \ge N$

Define a periodic transform function

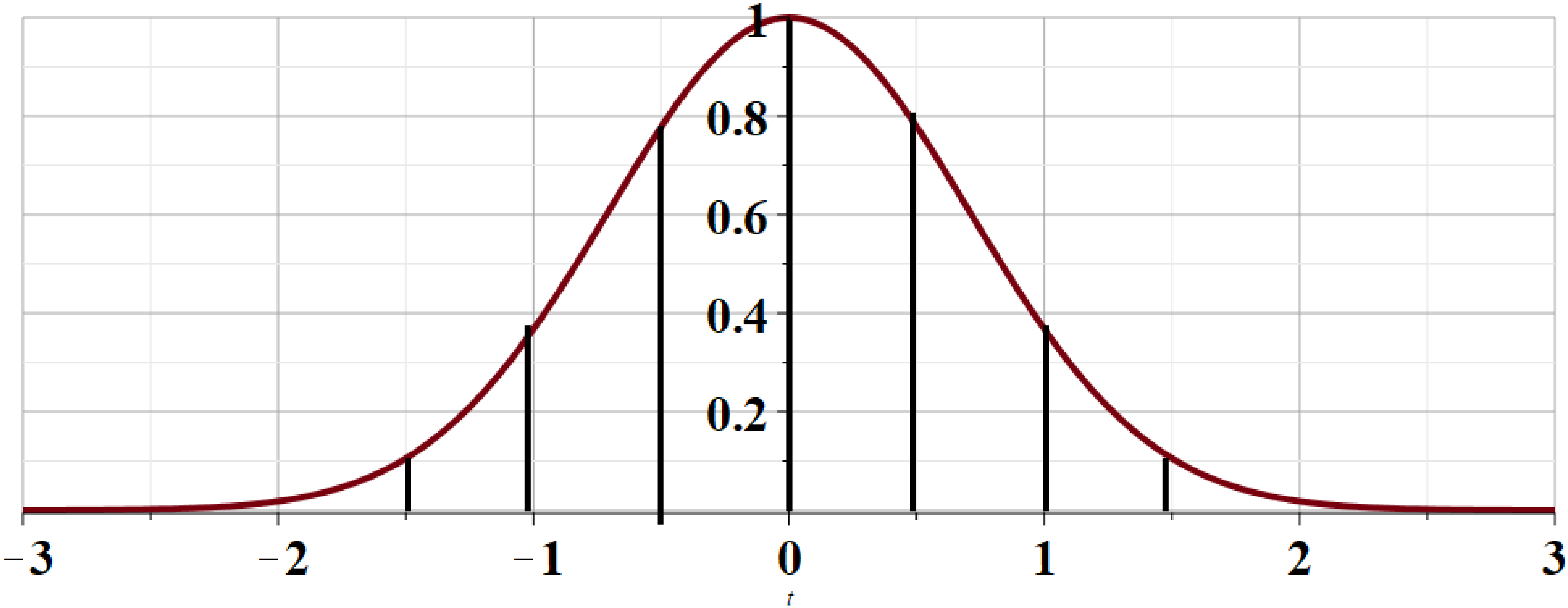
$\tilde{F}(\nu\Omega+\sigma W) \equiv \tilde{F}(\nu\Omega)$ for $\sigma = ...-3,-2,-1,0,1,2,3...$ where $W \equiv ((2N+1)\Omega)$

Recall that: $\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega-\nu\Omega)$, where $\Omega \equiv \frac{2\pi}{T}$

$$f(t)=\sum_{\nu=-\infty}^{\infty} \tilde{F}(\nu\Omega)e^{-i\nu\Omega t} = \frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N} \tilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

For $t=\frac{m}{2N+1}T$: $\Rightarrow f\left(\frac{mT}{2N+1}\right)=\sum_{\nu=-M}^{M} F(\nu\Omega)e^{-i2\pi\nu m/(2N+1)}$

Falsely discretized time function $\bar{f}(t)$



t →

$$t = \frac{m}{2N+1}T:$$

Doubly periodic functions

$$t \rightarrow \frac{\mu T}{2N+1}$$

$$\widetilde{f}_\mu = \frac{1}{2N+1}\sum_{\nu=-N}^{N}\widetilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\widetilde{F}_\nu = \sum_{\mu=-N}^{N}\widetilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$$2N+1 \rightarrow M$$

$$\widetilde{f}_\mu = \frac{1}{M}\sum_{\nu=0}^{M-1} \widetilde{F}_\nu e^{-i2\pi\nu\mu/M}$$

$$\widetilde{F}_\nu = \sum_{\mu=0}^{M} \widetilde{f}_\mu e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\widetilde{F}_0 = \widetilde{f}_0 W^0 + \widetilde{f}_1 W^0 + \widetilde{f}_2 W^0 + \widetilde{f}_3 W^0 + \cdots$$

$$\widetilde{F}_1 = \widetilde{f}_0 W^0 + \widetilde{f}_1 W^1 + \widetilde{f}_2 W^2 + \widetilde{f}_3 W^3 + \cdots$$

$$\widetilde{F}_2 = \widetilde{f}_0 W^0 + \widetilde{f}_1 W^2 + \widetilde{f}_2 W^4 + \widetilde{f}_3 W^6 + \cdots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, “An algorithm for machine calculation of complex Fourier series” Math. Computation 19, 297-301 (1965) http://www.fftw.org/

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Introduction

FFTW is a C subroutine library for computing the discrete Fourier transform (DFT) in one or more dimensions, of arbitrary input size, and of both real and complex data (as well as of even/odd data, i.e. the discrete cosine/sine transforms or DCT/DST). We believe that FFTW, which is free software, should become the FFT library of choice for most applications.

The latest official release of FFTW is version **3.3.10**, available from our download page. Version 3.3 introduced support for the AVX x86 extensions, a distributed-memory implementation on top of MPI, and a Fortran 2003 API. Version 3.3.1 introduced support for the ARM Neon extensions. See the release notes for more information.

The FFTW package was developed at MIT by Matteo Frigo and Steven G. Johnson.

Our benchmarks, performed on on a variety of platforms, show that FFTW's performance is typically superior to that of other publicly available FFT software, and is even competitive with vendor-tuned codes. In contrast to vendor-tuned codes, however, FFTW's performance is *portable*: the same program will perform well on most architectures without modification. Hence the name, "FFTW," which stands for the somewhat whimsical title of "**Fastest Fourier Transform in the West**."

Subscribe to the fftw-announce mailing list to receive release announcements (or use the web feed).

Fourier series and Fourier transforms are useful for solving and analyzing a wide variety of functions, also beyond the Sturm-Liouville context.

Next time, we will consider a related concept – the Laplace transform.