PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 24: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation ➔ various mathematical problems and techniques including:

1. **Laplace transforms**
2. **Complex variables**
3. **Contour integrals**

	Fri, 10/13/2023	Fall Break		
21	Mon, 10/16/2023	Chap. 7	The wave and other partial differential equations	#15
22	Wed, 10/18/2023	Chap. 7	Sturm-Liouville equations	#16
23	Fri, 10/20/2023	Chap. 7	Sturm-Liouville equations	#17
24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023			
26	Fri, 10/27/2023			
27	Mon, 10/30/2023			
28	Wed, 11/01/2023			
29	Fri, 11/03/2023			

PHY 711 – Assignment #18

Assigned: 10/23/2023 Due: 10/30/2023

Continue reading Chapter 7 in **Fetter and Walecka**.

1. Consider the differential equation

$$-\frac{d^2}{dt^2}\phi(t) = F_0,$$

where F_0 is a given constant and where the initial values of $\phi(t)$ are given as $\phi(0) = 1$ and $\frac{d\phi}{dt}(t = 0) = 0$.

   (a) Find the Laplacian transformation of this system.

   (b) Find the inverse Laplacian transform to determine the function $\phi(t)$.

Last time, we introduced the Fourier Transform --

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\ f(t)\, e^{i\omega t}$$

Note that:

$$e^{\pm i\omega t} = \cos(\omega t) \pm i \sin(\omega t)$$

In this lecture, we will discuss a similar concept – the Laplace Transform.

$$i\omega \rightarrow -p$$

A brief introduction to Laplace transforms --

Laplace transforms are particularly useful in solving initial value problems.

The Lapace transform of the function $\phi(x)$ is defined:

$$L_{\phi(x)}(p) \equiv \int_0^\infty e^{-px}\phi(x)dx$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following identities can be shown:

$$L_{d\phi(x)/dx}(p) \equiv \int_0^\infty e^{-px}\frac{d\phi(x)}{dx}dx = -\phi(0) + pL_{\phi(x)}(p)$$

and

$$L_{d^2\phi(x)/dx^2}(p) \equiv \int_0^\infty e^{-px}\frac{d^2\phi(x)}{dx^2}dx = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2L_{\phi(x)}(p)$$

Some details (integrating by parts)--

Recall

$$L_{\phi(x)}(p) \equiv \int_0^\infty e^{-px}\phi(x)dx$$

Then:

$$L_{d\phi(x)/dx}(p) \equiv \int_0^\infty e^{-px}\frac{d\phi(x)}{dx}dx = \int_0^\infty \frac{d}{dx}\left(e^{-px}\phi(x)\right)dx + p\int_0^\infty e^{-px}\phi(x)dx$$

$$= -\phi(0) + pL_{\phi(x)}(p)$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty \mathrm{e}^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x\mathrm{e}^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{\mathrm{e}^{-\gamma x}} = \int_0^\infty \mathrm{e}^{-\gamma x}\mathrm{e}^{-px} dx = \frac{1}{p+\gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - \mathrm{e}^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\varphi}{dx^2} = F_0 e^{-\gamma x} \qquad \text{for} \quad \varphi(0) = 0 \qquad \text{and} \qquad \frac{d\varphi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \qquad \text{with} \quad \phi(0)=0, \ \frac{d\phi(0)}{dx}=0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2+\left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0\left(\frac{\pi/L}{(\pi/L)^2-1}\right)\left(\frac{1}{p^2+1}-\frac{1}{p^2+\left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\displaystyle\int_0^\infty \sin(at)e^{-pt}dt = \frac{a}{a^2+p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi/L)^2-1}\left(\sin\left(\frac{\pi x}{L}\right)-\frac{\pi}{L}\sin(x)\right)$$

Table of Laplace transforms

Laplace Transform Table

Largely modeled on a table in D'Azzo and Houpis, *Linear Control Systems Analysis and Design*, 1988

$F(s)$	$f(t) \quad 0 \le t$	
1. 1	$\delta(t)$	unit impulse at $t=0$
2. $\frac{1}{s}$	1 or $u(t)$	unit step starting at $t=0$
3. $\frac{1}{s^2}$	$t \cdot u(t)$ or t	ramp function
4. $\frac{1}{s^n}$	$\frac{1}{(n-1)!}t^{n-1}$	n = positive integer
5. $\frac{1}{s}e^{-as}$	$u(t-a)$	unit step starting at $t=a$
6. $\frac{1}{s}(1-e^{-as})$	$u(t)-u(t-a)$	rectangular pulse
7. $\frac{1}{s+a}$	e^{-at}	exponential decay
8. $\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	n = positive integer
9. $\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$	
10. $\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}(1-\frac{b}{b-a}e^{-at}+\frac{a}{b-a}e^{-bt})$	
11. $\frac{s+\alpha}{s(s+a)(s+b)}$	$\frac{1}{ab}[\alpha-\frac{b(\alpha-a)}{b-a}e^{-at}+\frac{a(\alpha-b)}{b-a}e^{-bt}]$	
12. $\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	
13. $\frac{s}{(s+a)(s+b)}$	$\frac{1}{a-b}(ae^{-at}-be^{-bt})$	

https://www.dartmouth.edu/~sullivan/22files/New%20Laplace%20Transform%20Table.pdf

Inverse Laplace transform :

$$\mathcal{L}_{\phi}(p) = \int_0^{\infty} e^{-pt}\phi(t)dt$$

In order to evaluate these integrals, we need to use complex analysis.

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_{\phi}(p)dp$$

Check: $$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_{\varphi}(p)dp = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}dp\int_0^{\infty} e^{-pu}\varphi(u)du$$

$$\frac{1}{2\pi i}\int_0^{\infty}\varphi(u)du\int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)}dp = \frac{1}{2\pi i}\int_0^{\infty}\varphi(u)du\int_{-\infty}^{\infty} e^{\lambda(t-u)}e^{is(t-u)}ids$$

$$= \frac{1}{2\pi i}\int_0^{\infty}\varphi(u)du\left(e^{\lambda(t-u)}2\pi i\ \delta(t-u)\right)$$

$$= \begin{cases} \varphi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In general – to calculate inverse Laplace transforms, we need to introduce concepts of complex numbers and contour integration

Introduction to complex variables

1. Basic properties
2. Notion of an analytic complex function
3. Cauchy integral theory
4. Analytic functions and functions with poles
5. Evaluating integrals of functions in the complex plane

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \quad \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Examples of analytic functions

$$e^{z} = e^{x+iy} = e^{x}\cos(y) + ie^{x}\sin(y)$$

$$\frac{\partial u}{\partial x} = e^{x}\cos(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = e^{x}\sin(y) = -\frac{\partial u}{\partial y} \quad \checkmark$$

$$z^{2} = \left(x+iy\right)^{2} = \left(x^{2}-y^{2}\right) + 2ixy \equiv u(x,y) + iv(x,y)$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} \quad \checkmark$$

Examples of non-analytic functions

Note that $z = \rho e^{i\phi} = \rho e^{i\phi + i2\pi n}$ for any integer n

$$\Rightarrow \ln z = \ln \rho + i\left(\phi + 2\pi n\right)$$

$\ln z$ is not analytic because it is multivalued

$$\Rightarrow z^{\alpha} = \rho^{\alpha} e^{i\alpha\phi} e^{i2\pi n\alpha}$$

z^{α} is not analytic for non-integer α because it is multivalued

Behavior of $f(z) = \dfrac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Behavior of $f(z) = \frac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

This observation helps us to focus on a special kind of singularity called a "pole"

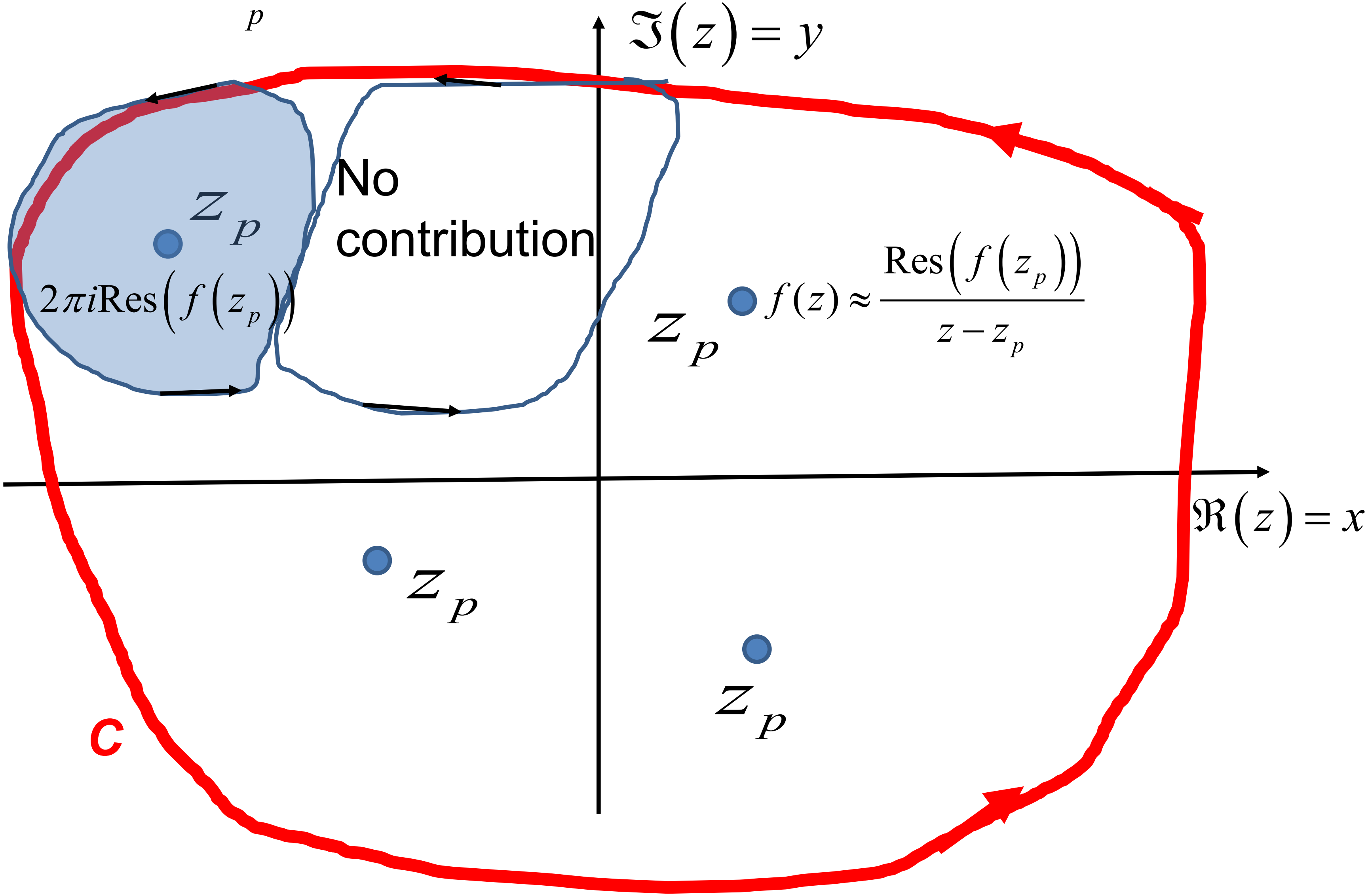
For $f(z)$ in the vicinity of $z = z_p$: $f(z) \approx \frac{g(z_p)}{z - z_p}$

Therefore: $\oint f(z) dz = 0$ or $\oint f(z) dz = g(z_p) \oint \frac{dz}{z - z_p} = 2\pi i g(z_p)$

Integration does not include z_p

Integration does include z_p

$$\oint_C f(z)dz = 2\pi i \sum_p \text{Res}\left(f(z_p)\right)$$



General formula for determining residue:

Suppose that in the neighborhood of z_p, $f(z) \approx \frac{h(z)}{\left(z-z_p\right)^m} \underset{z\to z_p}{\equiv} \frac{\text{Res}\left(f(z_p)\right)}{z-z_p}$

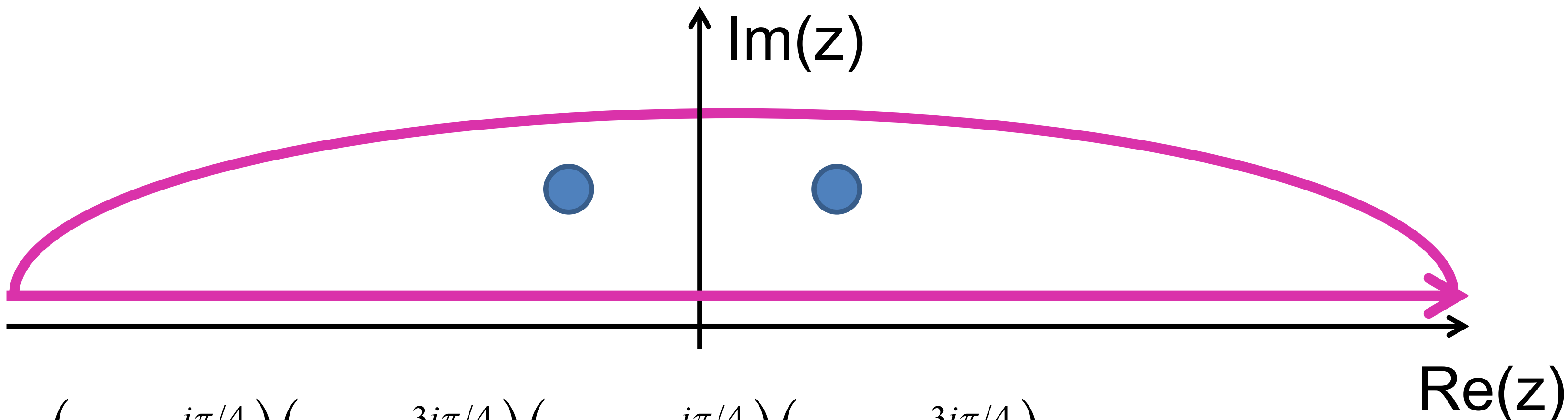
Since $h(z) \equiv \left(z-z_p\right)^m f(z)$ is analytic near z_p, we can make a Taylor exansion

about z_p : $$h(z) \approx h(z_p) + \left(z-z_p\right)\frac{dh(z_p)}{dz} + \ldots + \frac{\left(z-z_p\right)^{m-1}}{(m-1)!}\frac{d^{m-1}h(z_p)}{dz^{m-1}} + \ldots.$$

$$\Rightarrow \text{Res}\left(f(z_p)\right) = \lim_{z\to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}\left(\left(z-z_p\right)^m f(z)\right)}{dz^{m-1}}\right\}$$

In the following examples *m=1*

Example: $$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx + 0 = \oint \frac{z^2}{1+z^4}dz$$



$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\text{Res}\left(z_p = e^{i\pi/4}\right) + \text{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

Note:
m=1

$$\text{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \qquad \text{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{2}\left(\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right) - \left(-\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right)\right) = \frac{\pi}{\sqrt{2}}$$

Some details:

Note that: $e^{i\pi} = -1 = e^{-i\pi}$

$$e^{-3i\pi/4} = e^{i\pi/4 - i\pi} = -e^{i\pi/4}$$

$$f(z) = \frac{z^2}{1+z^4}$$

$$\text{Res}\left(f(z = e^{i\pi/4}\right) = \frac{\left(e^{i\pi/4}\right)^2}{\left(e^{i\pi/4} - e^{3i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-3i\pi/4}\right)}$$

$$= \frac{e^{i\pi/2}}{\left(e^{i\pi/4} + e^{-i\pi/4}\right)\left(e^{i\pi/4} - e^{-i\pi/4}\right)\left(e^{i\pi/4} + e^{i\pi/4}\right)}$$

$$= \frac{e^{i\pi/4}}{2\left(i - (-i)\right)} = \frac{e^{i\pi/4}}{4i}$$

Question – Could we have chosen the contour in the lower half plane?

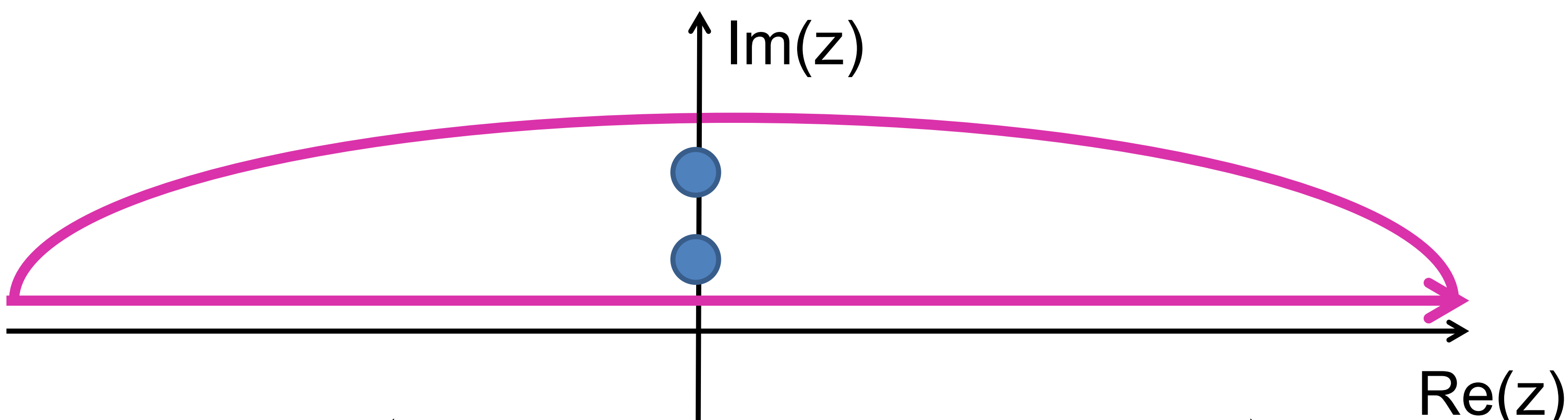
a. Yes b. No

Another example: $I = \int_0^\infty \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx$

$$\int_0^\infty \frac{\cos(ax)}{4x^4 + 5x^2 + 1} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{4x^4 + 5x^2 + 1} dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4 + 5z^2 + 1} dz$$

$$4z^4 + 5z^2 + 1 = 4\left(z - i\right)\left(z - \tfrac{i}{2}\right)\left(z + i\right)\left(z + \tfrac{i}{2}\right)$$

Note:
m=1



$$I = 2\pi i\left(\mathrm{Res}\left(z_p = i\right) + \mathrm{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx = \frac{1}{2}\oint \frac{e^{iaz}}{4z^4+5z^2+1}dz$$

$$= 2\pi i\left(\text{Res}\left(z_p = i\right) + \text{Res}\left(z_p = \tfrac{i}{2}\right)\right)$$

$$= \frac{\pi}{6}\left(-e^{-a} + 2e^{-a/2}\right)$$

Question – Could we have chosen the contour in the lower half plane?

a. Yes

b. No

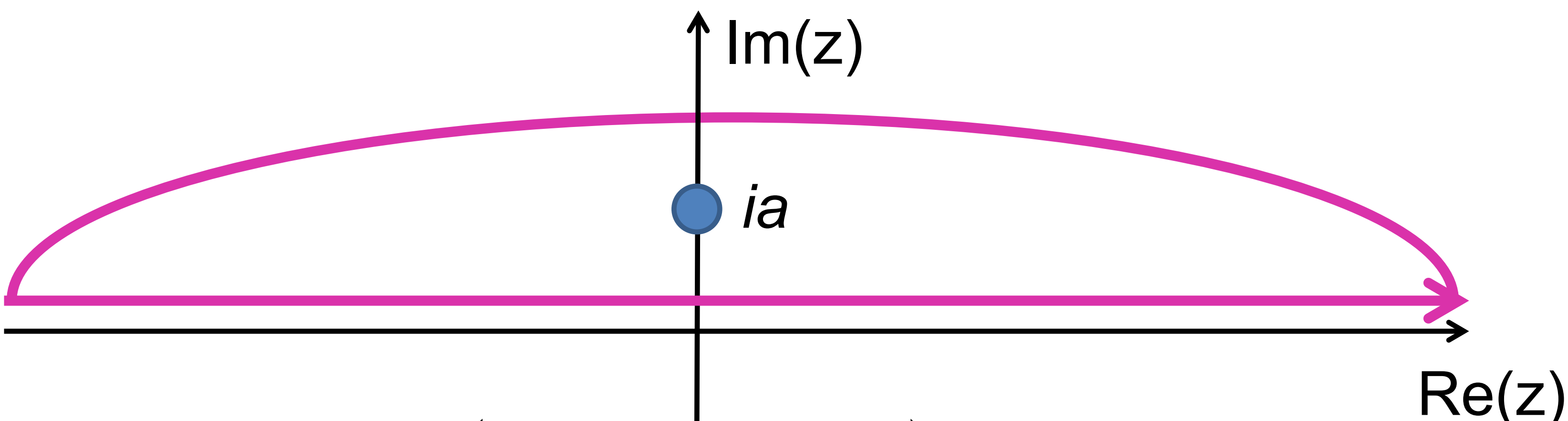
Note that for $a > 0$ and $z_I > 0$

in the lower half plane: $e^{iaz} = e^{iaz_R}e^{az_I}$

Another example: $I = \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx \quad \text{for } k > 0 \text{ and } a > 0$

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$



$$I = 2\pi i \left(\text{Res}\left(z_p = ia \right) \right) = \pi e^{-ka}$$

Some details --

$$\int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{i} \int_{-\infty}^{\infty} \frac{x e^{ikx}}{x^2 + a^2} dx = \frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz$$

$$z^2 + a^2 = (z - ia)(z + ia)$$

$$\frac{1}{i} \oint \frac{z e^{ikz}}{z^2 + a^2} dz = 2\pi i \frac{1}{i} \lim_{z \to ia} \left((z - ia) \frac{z e^{ikz}}{z^2 + a^2} \right)$$

$$= 2\pi i \frac{1}{i} \frac{ia e^{-ka}}{2ia} = \pi e^{-ka}$$

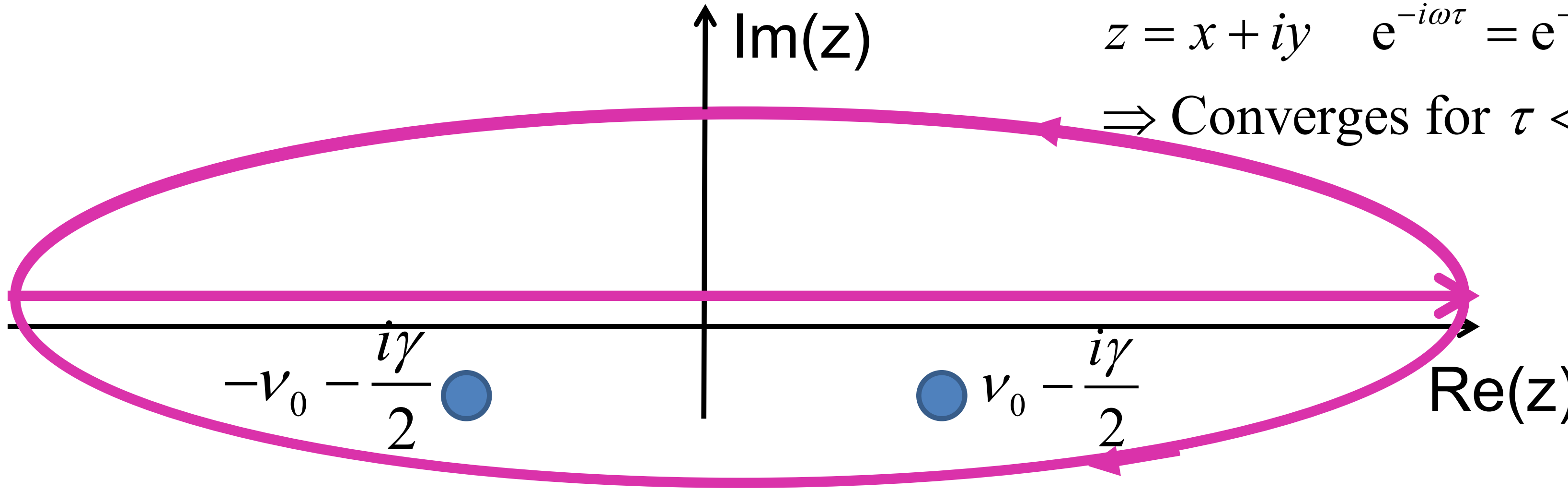
From the Drude model of dielectric response --

$$G(\tau) = \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$ where ω_p, ω_0, and γ are positive constants

Upper hemisphere:

$z = x + iy \quad e^{-i\omega\tau} = e^{-ix\tau + y\tau}$

$\Rightarrow$ Converges for $\tau < 0$



Lower hemisphere:

$z = x - iy \quad e^{-i\omega\tau} = e^{-ix\tau - y\tau}$

$\Rightarrow$ Converges for $\tau > 0$

$$\nu_0 \equiv \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

From the Drude model of dielectric response -- continued --

$$G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$ where ω_p, ω_0, and γ are positive constants

$$G(\tau) = \omega_p^2 \begin{cases} 0 & \text{for } \tau < 0 \\ e^{-\gamma\tau/2} \dfrac{\sin \nu_0 \tau}{\nu_0} & \text{for } \tau > 0 \end{cases}$$