



**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF in Olin 103
Notes for Lecture 26 – Chap. 8 (F & W)**

Motions of elastic membranes

- 1. Review of standing waves on a string**
- 2. Standing waves on a two dimensional membrane.**
- 3. Boundary value problems**

	Fri, 10/13/2023	Fall Break		
21	Mon, 10/16/2023	Chap. 7	The wave and other partial differential equations	#15
22	Wed, 10/18/2023	Chap. 7	Sturm-Liouville equations	#16
23	Fri, 10/20/2023	Chap. 7	Sturm-Liouville equations	#17
24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023			
28	Wed, 11/01/2023			
29	Fri, 11/03/2023			
31	Mon, 11/06/2023			
32	Wed, 11/08/2023			
33	Fri, 11/10/2023			

PHY 711 -- Assignment #20

Assigned: 10/27/2023 Due: 10/30/2023

Read Chapter 8 in **Fetter & Walecka**.

1. In the last few slides of Lecture 20, we considered an annular membrane stretched between the circular radii $a \leq r \leq b$. Find a wave form $f(r)$ for the case that $a=0.5$ and $b=1$, assuming angular uniformity ($m=0$) and that $f(a)=f(b)=0$.



Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Initial value problem : $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int^x \psi(x') dx'$$

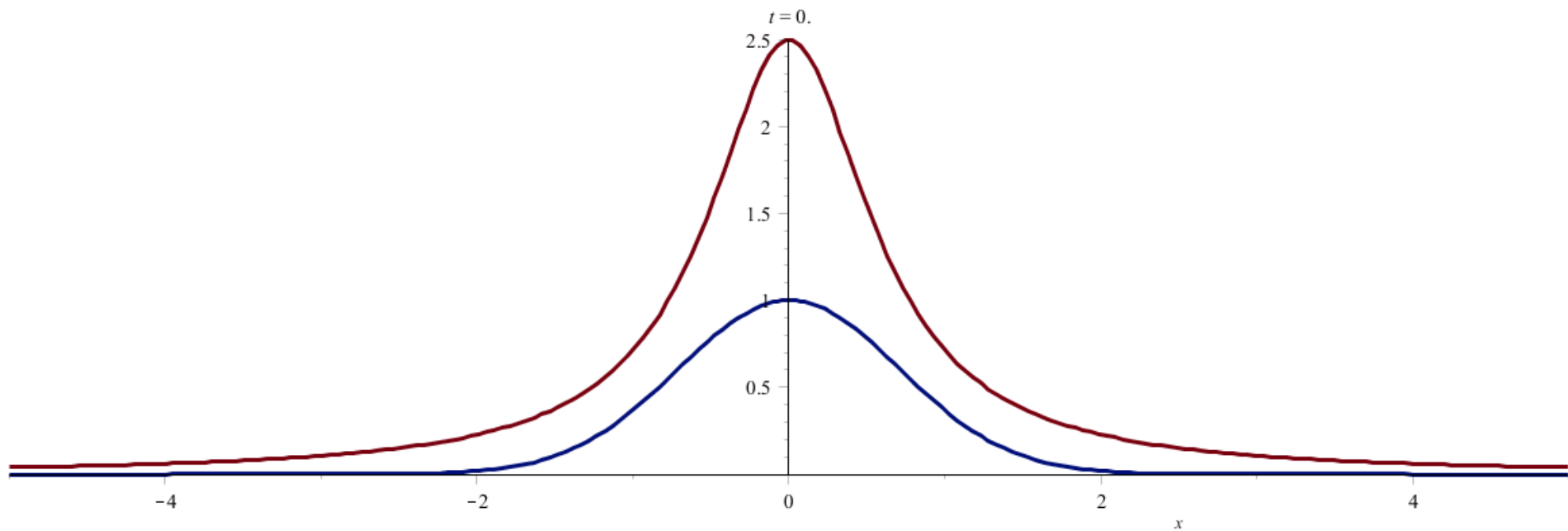
For each x , find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2} \left(\phi(x) - \frac{1}{c} \int^x \psi(x') dx' \right)$$

$$g(x) = \frac{1}{2} \left(\phi(x) + \frac{1}{c} \int^x \psi(x') dx' \right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} (\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

Example with $\psi(x) = 0$ and $\phi(x) = \frac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

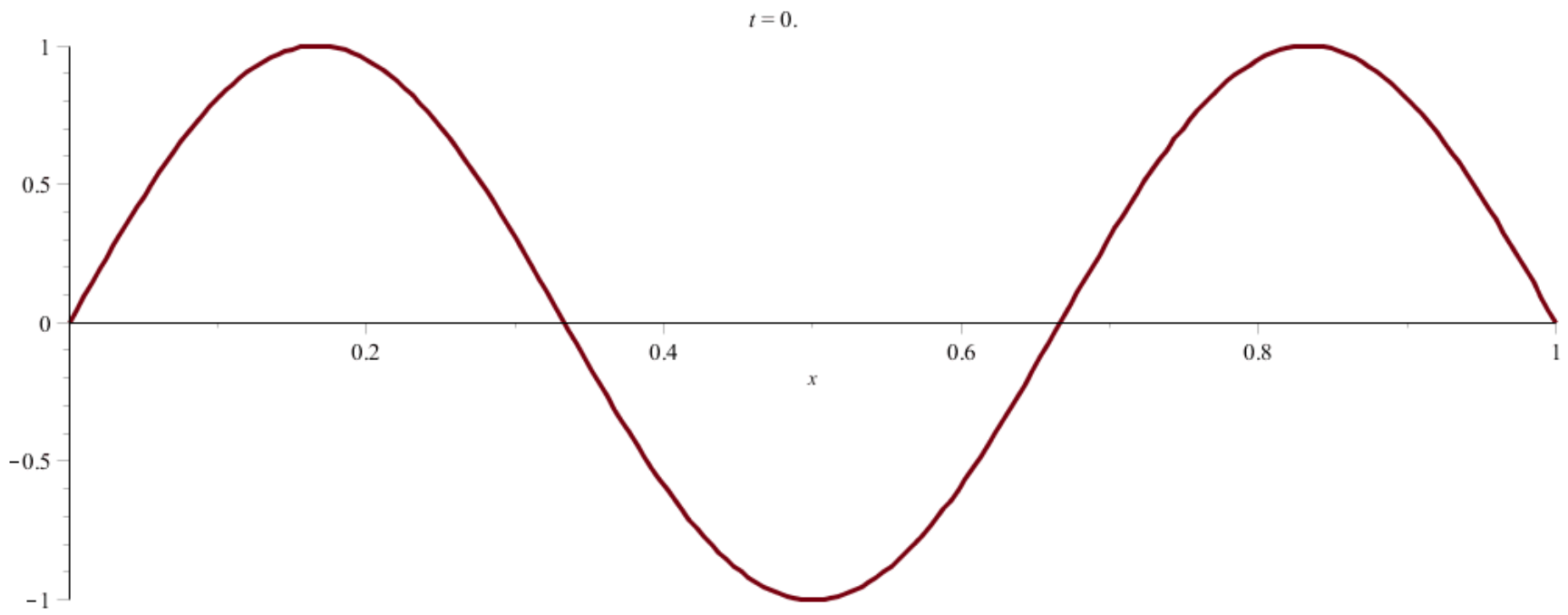
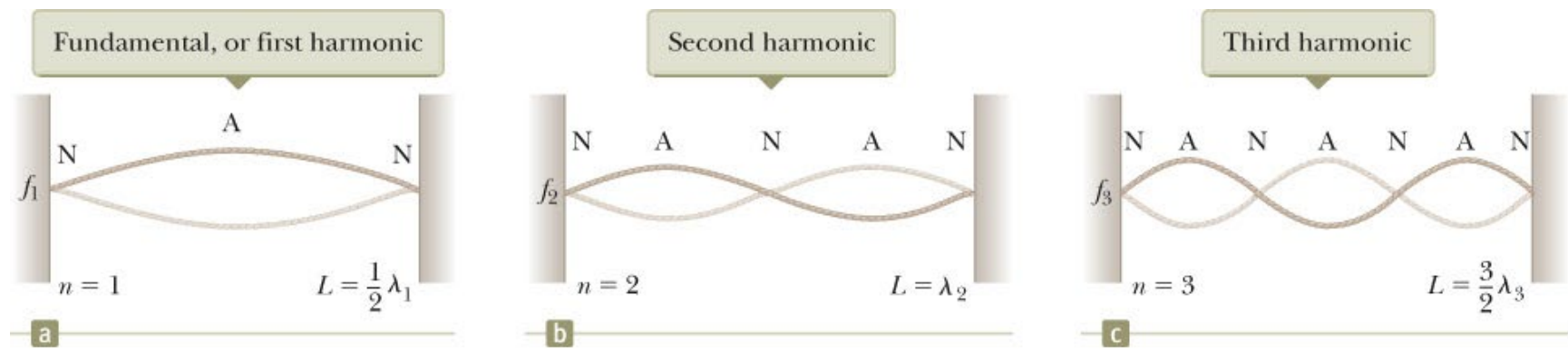
with $\mu(0, t) = \mu(L, t) = 0$.

Assume: $\mu(x, t) = \Re\left(e^{-i\omega t} \rho(x)\right)$

where $\frac{d^2 \rho(x)}{dx^2} + k^2 \rho(x) = 0$ $k = \frac{\omega}{c}$

$$\rho_\nu(x) = A \sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \quad \omega_\nu = ck_\nu$$



Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

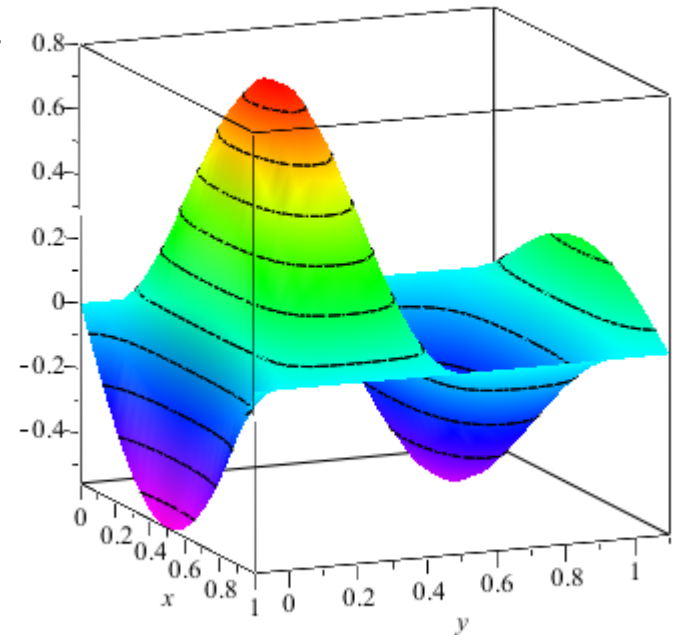
$$u(x, y, t) = \Re\left(e^{-i\omega t} \rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right) \rho(x, y) = 0$$

$$\text{where } k = \frac{\omega}{c}$$

$$\rho(x, y)$$

Note that here we are visualizing transverse waves. Longitudinal waves can also exist.



In this case, we have mapped the one dimensional elastic string to a two dimensional elastic membrane

$$\frac{\partial^2}{\partial x^2} \rightarrow \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (\text{in Cartesian coordinates})$$

Lagrangian density : $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dx dy$$

Hamilton's principle :

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial t)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial x)} - \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial y)} = 0$$



Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial t)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial x)} - \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t} \rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right) \rho(x, y) = 0 \quad \text{where } k = \frac{\omega}{c}$$

Consider a rectangular boundary:



Clamped boundary conditions :

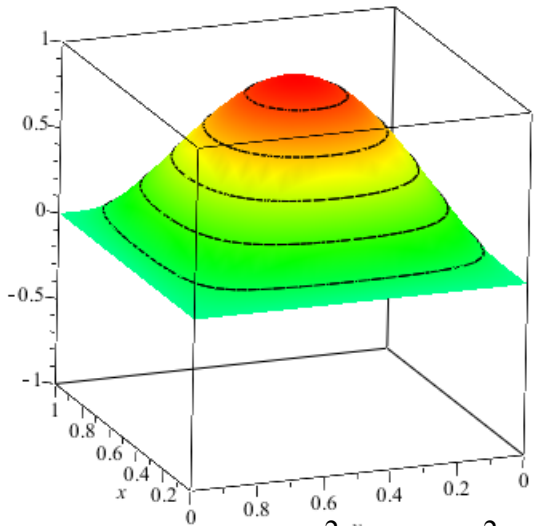
$$\rho(0, y) = \rho(a, y) = \rho(x, 0) = \rho(x, b) = 0$$

$$\Rightarrow \rho_{mn}(x, y) = A \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

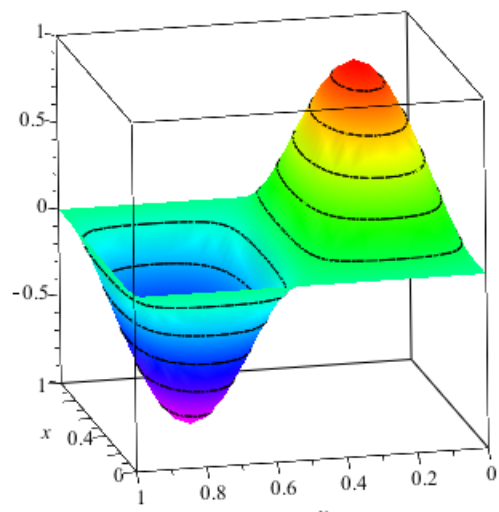
$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \omega_{mn} = ck_{mn}$$

$$(\nabla^2 + k^2)\rho(x, y) = 0$$

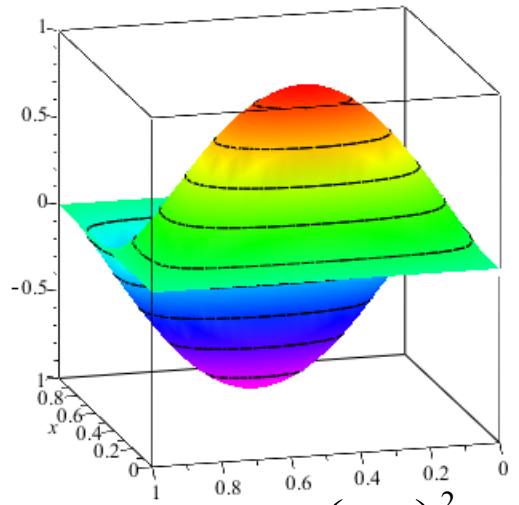
$$\text{where } k = \frac{\omega}{c}$$



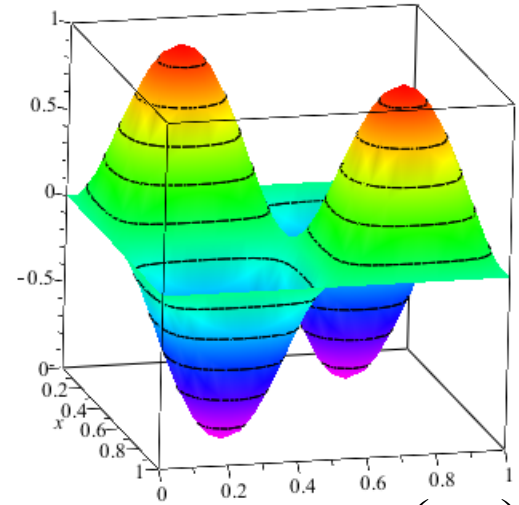
$$k_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{12}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$



$$k_{21}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{22}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

More general boundary conditions:

$\tau \nabla u|_b = \kappa u|_b$ represents bounded side constrained with spring

$\tau \nabla u|_b = 0$ represents "free" side

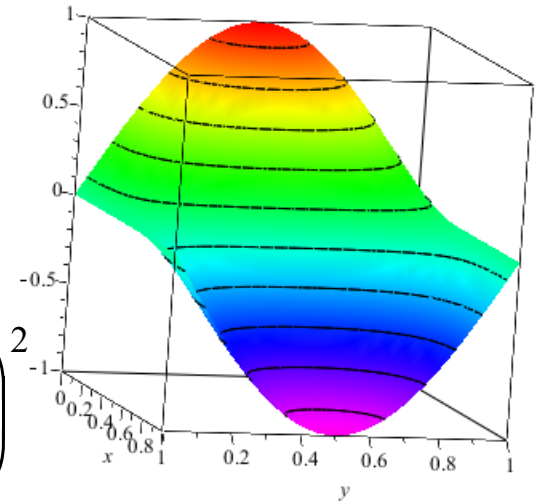
Mixed boundary conditions :

$$\rho(x,0) = \rho(x,b) = \frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \omega_{mn} = ck_{mn}$$

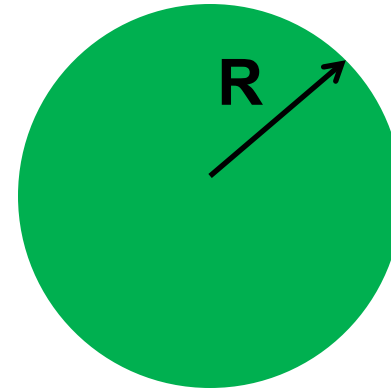
$$k_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



Consider a circular boundary:

Clamped boundary conditions for $\rho(r, \varphi)$:

$$\rho(R, \varphi) = 0$$



$$(\nabla^2 + k^2)\rho(r, \varphi) = 0 \quad \text{where } k = \frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r, \varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$$\Rightarrow m = \text{integer}$$



Consider circular boundary -- continued

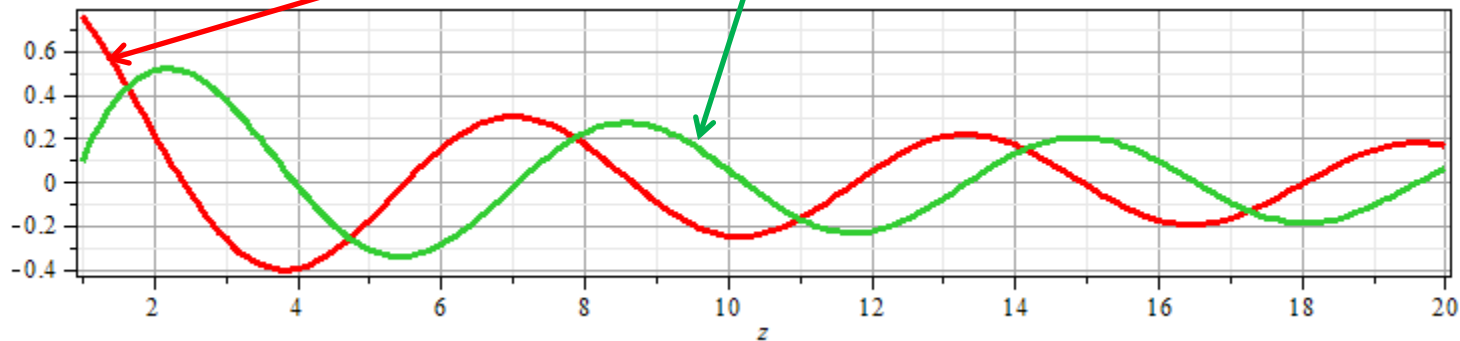
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

⇒ Bessel equation of integer order with transcendental solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



Some properties of Bessel functions

Ascending series :
$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!} \left(\frac{z}{2}\right)^{2j}$$

Recursion relations :
$$J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$$

$$J_{m-1}(z) - J_{m+1}(z) = 2 \frac{dJ_m(z)}{dz}$$

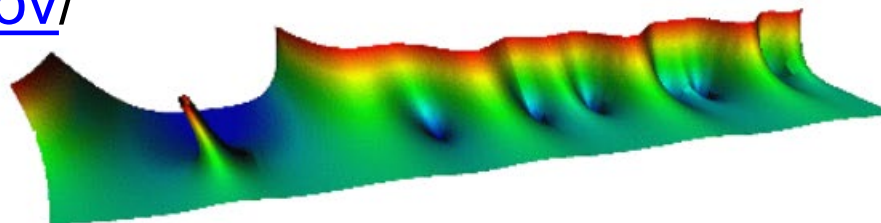
Asymptotic form :
$$J_m(z) \xrightarrow{z \gg 1} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

Zeros of Bessel functions $J_m(z_{mn}) = 0$

$m = 0$: $z_{0n} = 2.406, 5.520, 8.654, \dots$

$m = 1$: $z_{1n} = 3.832, 7.016, 10.173, \dots$

$m = 2$: $z_{2n} = 5.136, 8.417, 11.620, \dots$



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 [DLMF Update; Version 1.0.9](#)

2014-04-25 [DLMF Update; Version 1.0.8; errata & improved MathML](#)

2014-03-21 [DLMF Update; Version 1.0.7; New Features improve Math & 3D Graphics](#)

2013-08-16 [Bille C. Carlson, DLMF Author, dies at age 89](#)

[More news](#)

Foreword

Preface

Mathematical Introduction

1 Algebraic and Analytic Methods

2 Asymptotic Approximations

3 Numerical Methods

4 Elementary Functions

5 Gamma Function

6 Exponential, Logarithmic, Sine, and Cosine Integrals

7 Error Functions, Dawson's and Fresnel Integrals

8 Incomplete Gamma and Related Functions

9 Airy and Related Functions

10 Bessel Functions

11 Struve and Related Functions

12 Parabolic Cylinder Functions

19 Elliptic Integrals

20 Theta Functions

21 Multidimensional Theta Functions

22 Jacobian Elliptic Functions

23 Weierstrass Elliptic and Modular Functions

24 Bernoulli and Euler Polynomials

25 Zeta and Related Functions

26 Combinatorial Analysis

27 Functions of Number Theory

28 Mathieu Functions and Hill's Equation

29 Lamé Functions

30 Spheroidal Wave Functions

31 Heun Functions

32 Painlevé Transcendents

33 Coulomb Functions

Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(\nu + k + 1)}.$$

$$Y_n(z) = -\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi} \ln\left(\frac{1}{2}z\right) J_n(z) \\ - \frac{\left(\frac{1}{2}z\right)^n}{\pi} \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!},$$

Some properties of Bessel functions -- continued

Note: It is possible to prove the following

identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

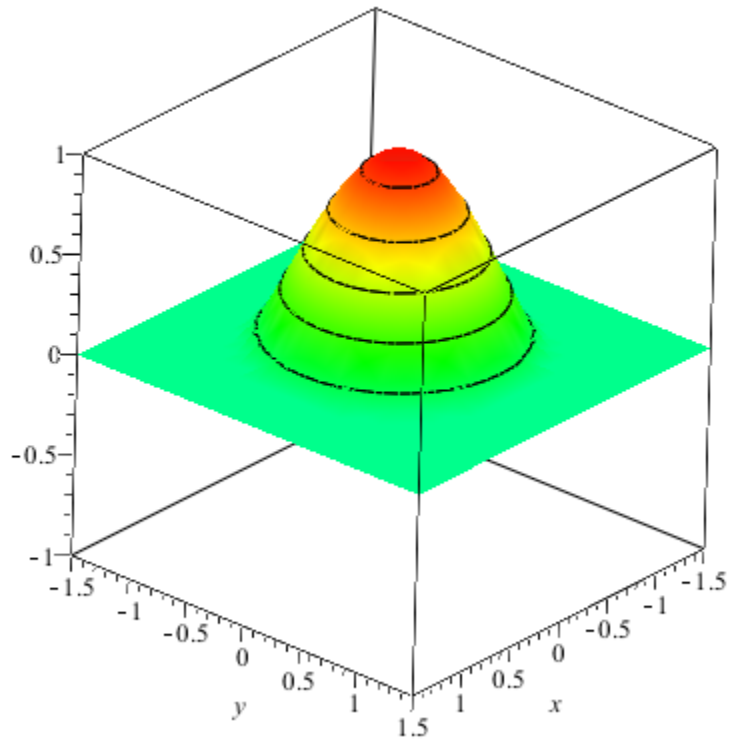
$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right) J_m\left(\frac{z_{mn'}}{R}r\right) r dr = \frac{R^2}{2} (J_{m+1}(z_{mn}))^2 \delta_{nn'}$$

Returning to differential equation for radial function:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

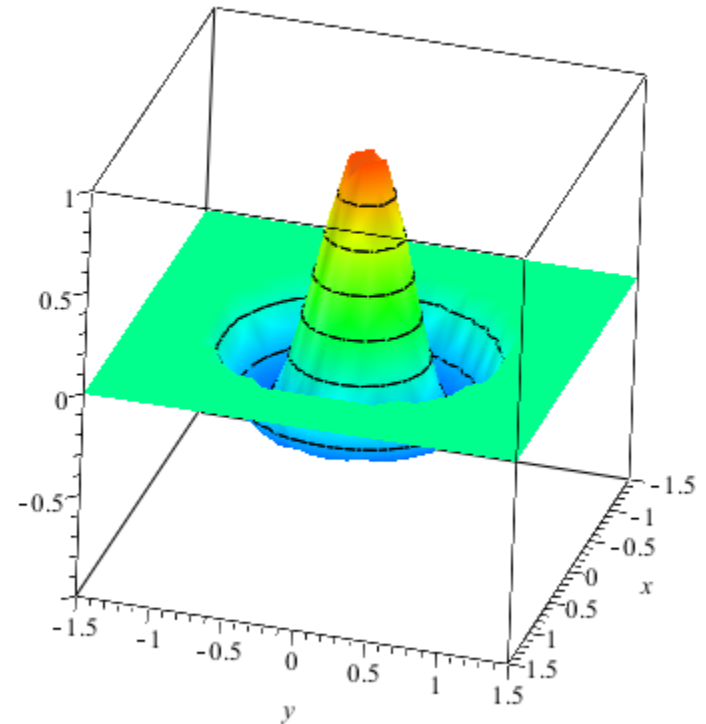
$$\Rightarrow f_{mn}(r) = A J_m\left(\frac{z_{mn}}{R}r\right); \quad k_{mn} = \frac{z_{mn}}{R} \quad \text{where } J(z_{mn}) = 0$$

$$\rho_{01}(r, \varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right)$$



$$k_{01} = \frac{2.406}{R}$$

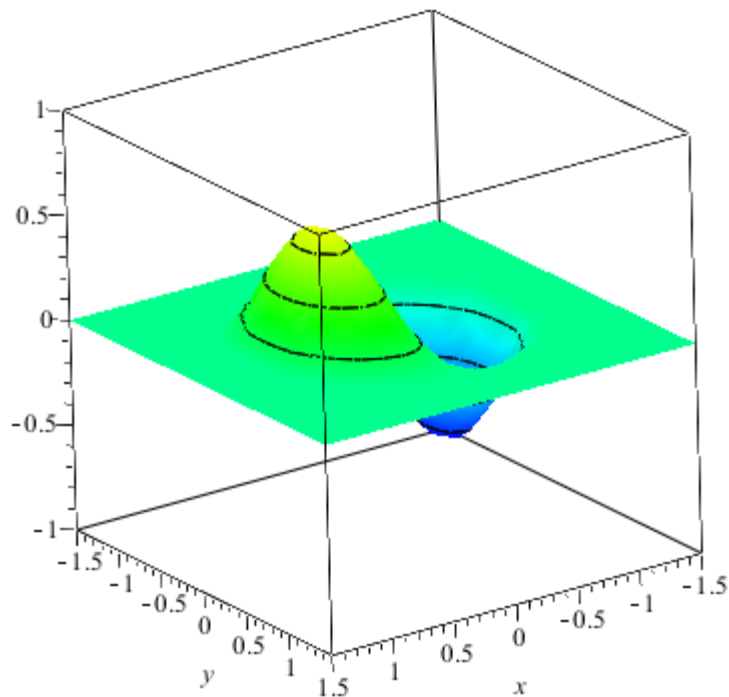
$$\rho_{02}(r, \varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$



$$k_{02} = \frac{5.520}{R}$$

$$\rho_{11}(r, \varphi) = f_{11}(r) \cos(\varphi)$$

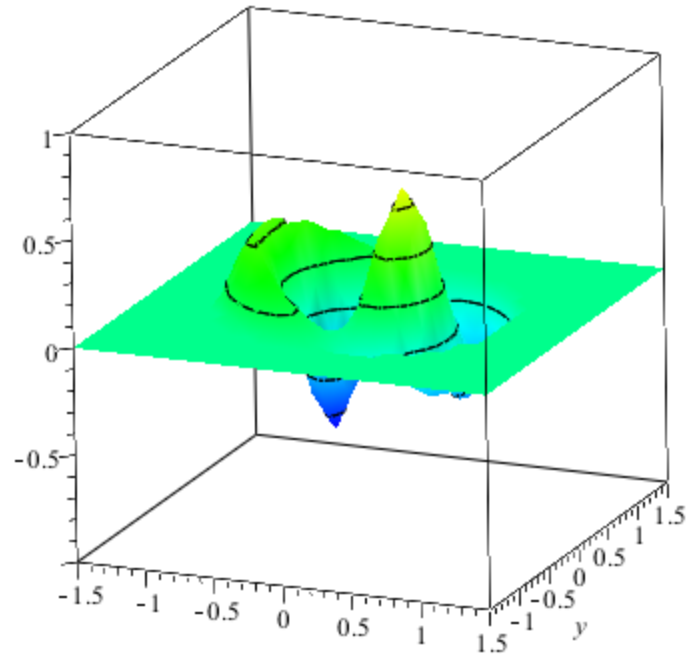
$$= AJ_1\left(\frac{z_{11}}{R} r\right) \cos(\varphi)$$



$$k_{11} = \frac{3.832}{R}$$

$$\rho_{12}(r, \varphi) = f_{12}(r) \cos(\varphi)$$

$$= AJ_1\left(\frac{z_{12}}{R} r\right) \cos(\varphi)$$



$$k_{12} = \frac{7.016}{R}$$

Ernst Chladni



Ernst Chladni

Born 30 November 1756
Wittenberg, Electorate of Saxony
in the Holy Roman Empire

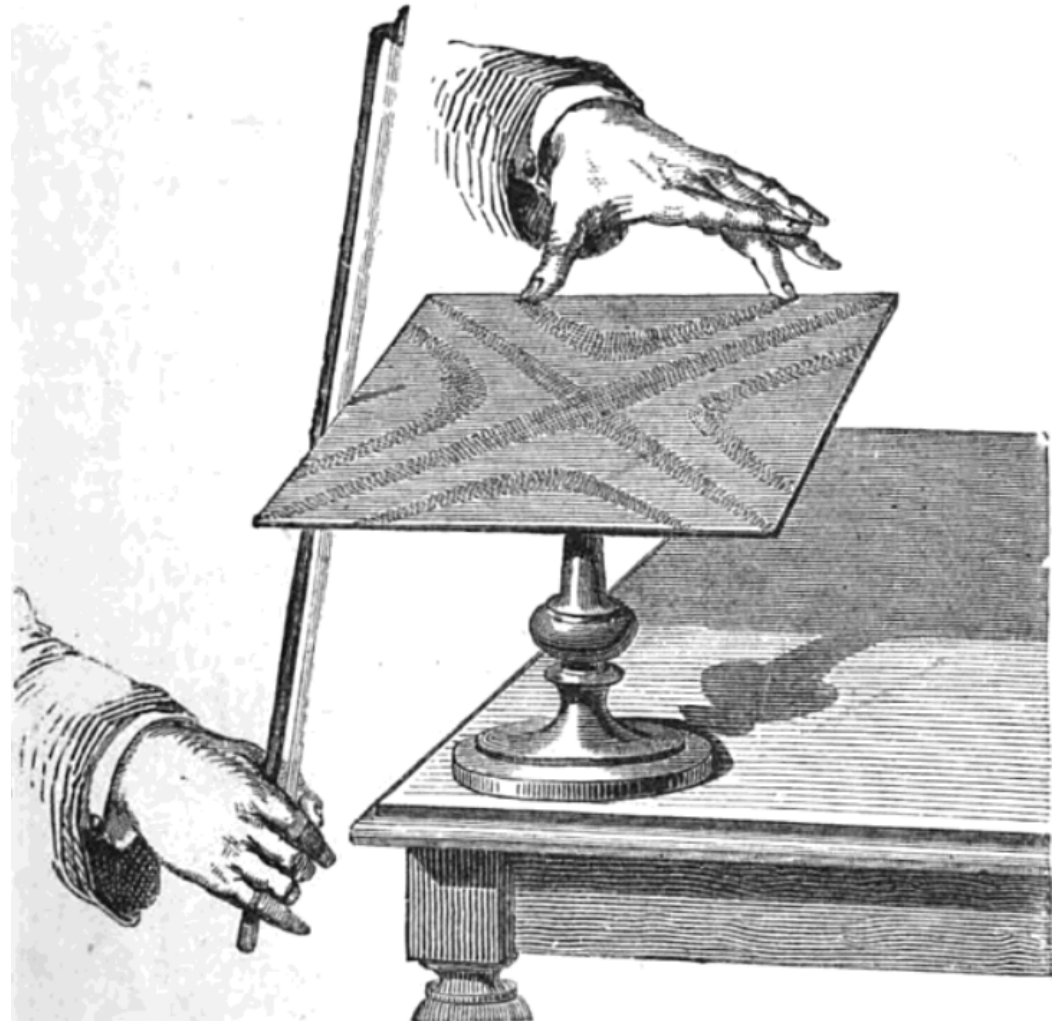
Died 3 April 1827 (aged 70)
Breslau, Province of Silesia in the
Kingdom of Prussia, a part of the
German Confederation

Nationality German

Known for Study of acoustics
Chladni plates and figures
Estimating the speed of sound
Chladni's law
Theory of meteorites' origins

Scientific career

Fields Physics



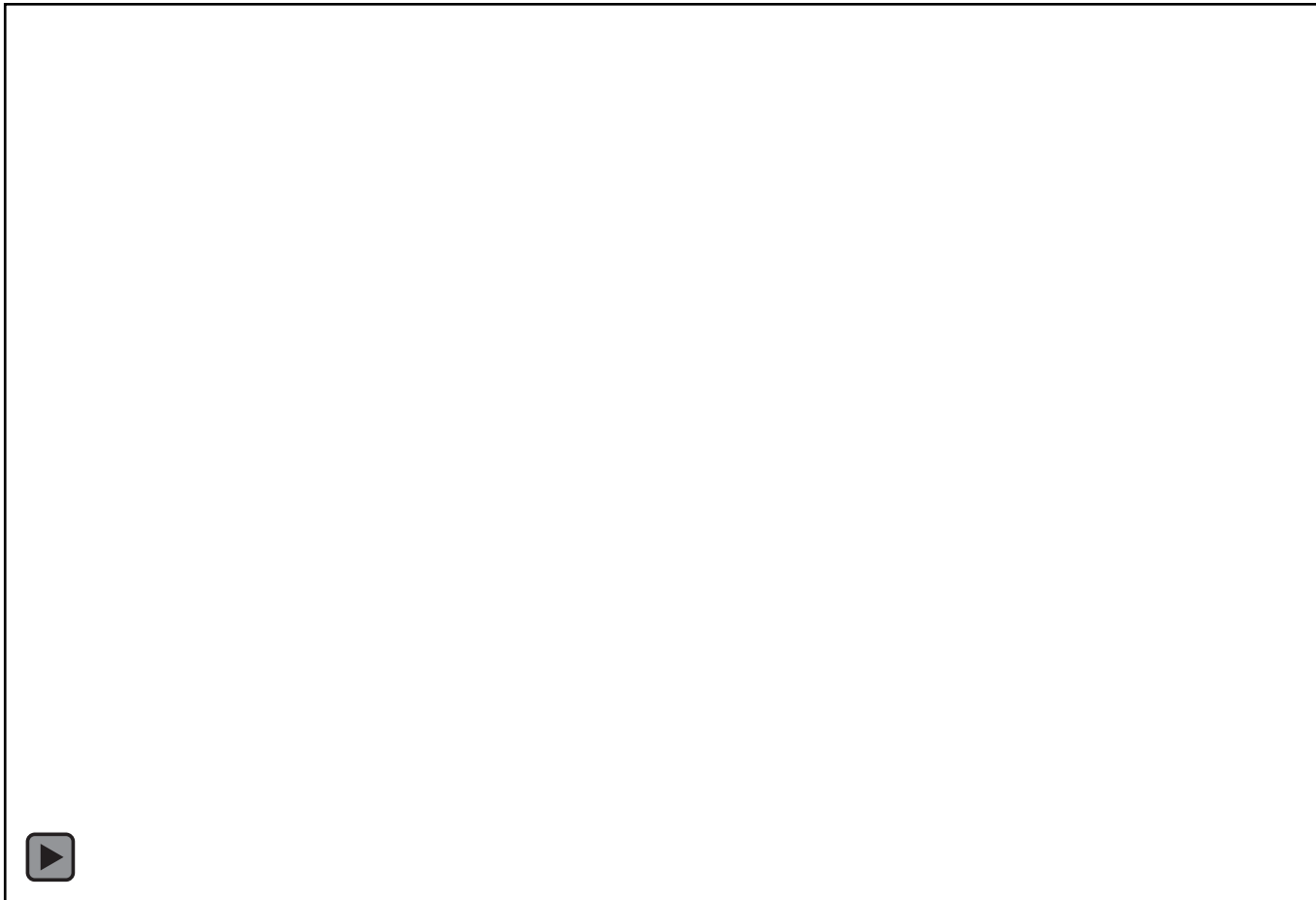


Demonstration with motor in the middle – (PASCO)

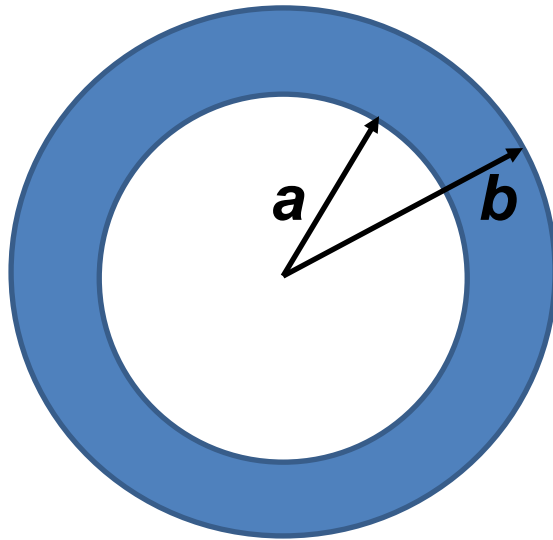




<http://www.physics.wfu.edu/resources/education-resources/demo-videos/waves/>



More complicated geometry – annular membrane



In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r, \varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\Rightarrow m = \text{integer}$

Consider circular boundary -- continued

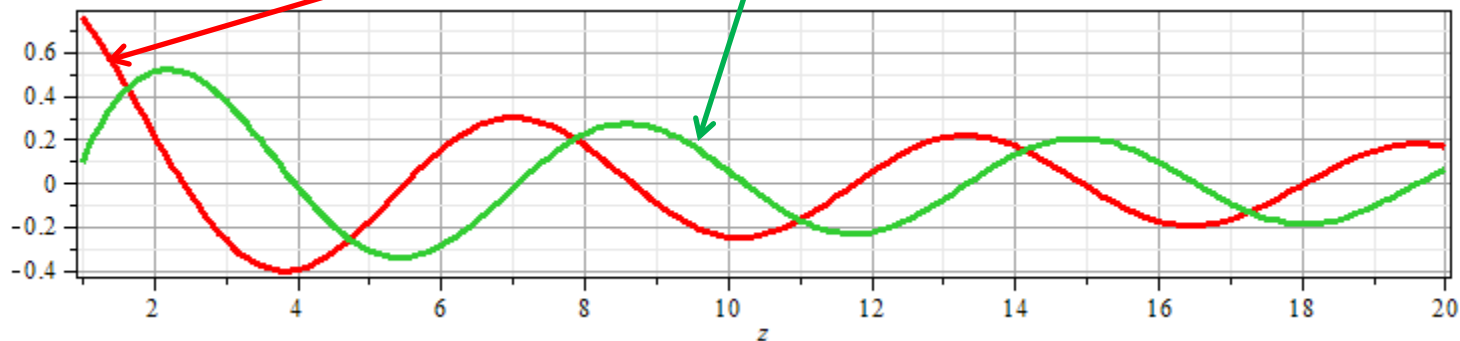
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

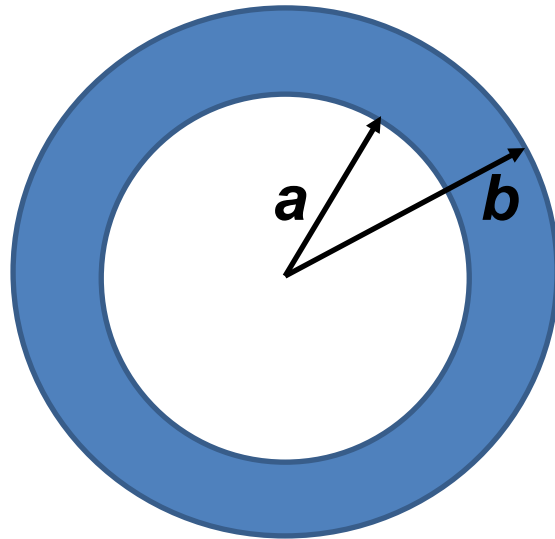
⇒ Bessel equation of integer order with transcendental solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



Normal modes of an annular membrane -- continued

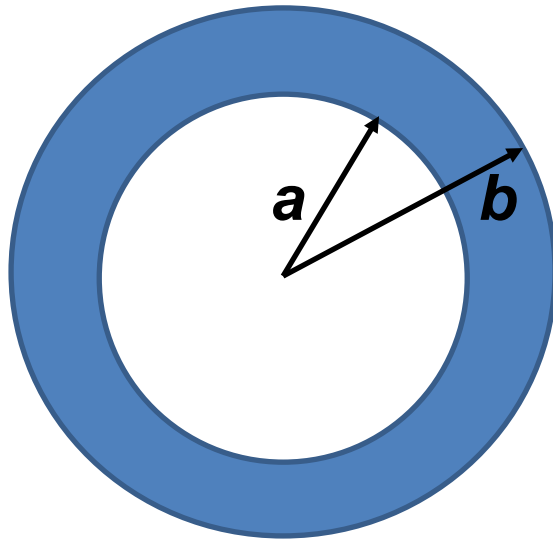


Differential equation for radial function:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

General form of radial function: $f(r) = AJ_m(kr) + BN_m(kr)$

Normal modes of an annular membrane -- continued



Boundary conditions:

$$f(a) = 0 \quad f(b) = 0$$

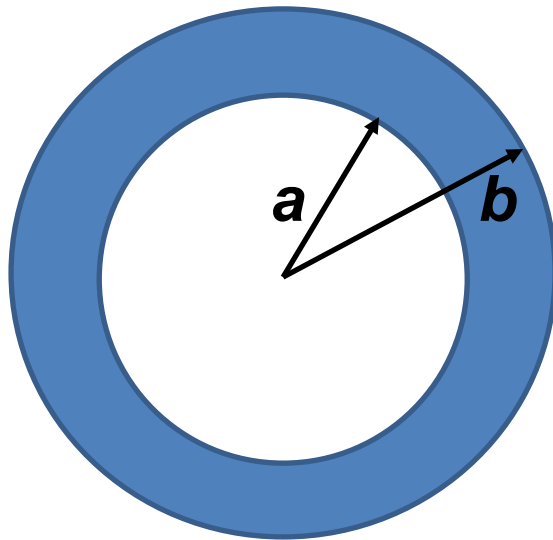
$$AJ_m(ka) + BN_m(ka) = 0$$

$$AJ_m(kb) + BN_m(kb) = 0$$

\Rightarrow 2 equations and 2 unknowns -- k and $\frac{B}{A}$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)} \quad (\text{transcendental equation for } k)$$

Normal modes of an annular membrane -- continued



Boundary conditions:

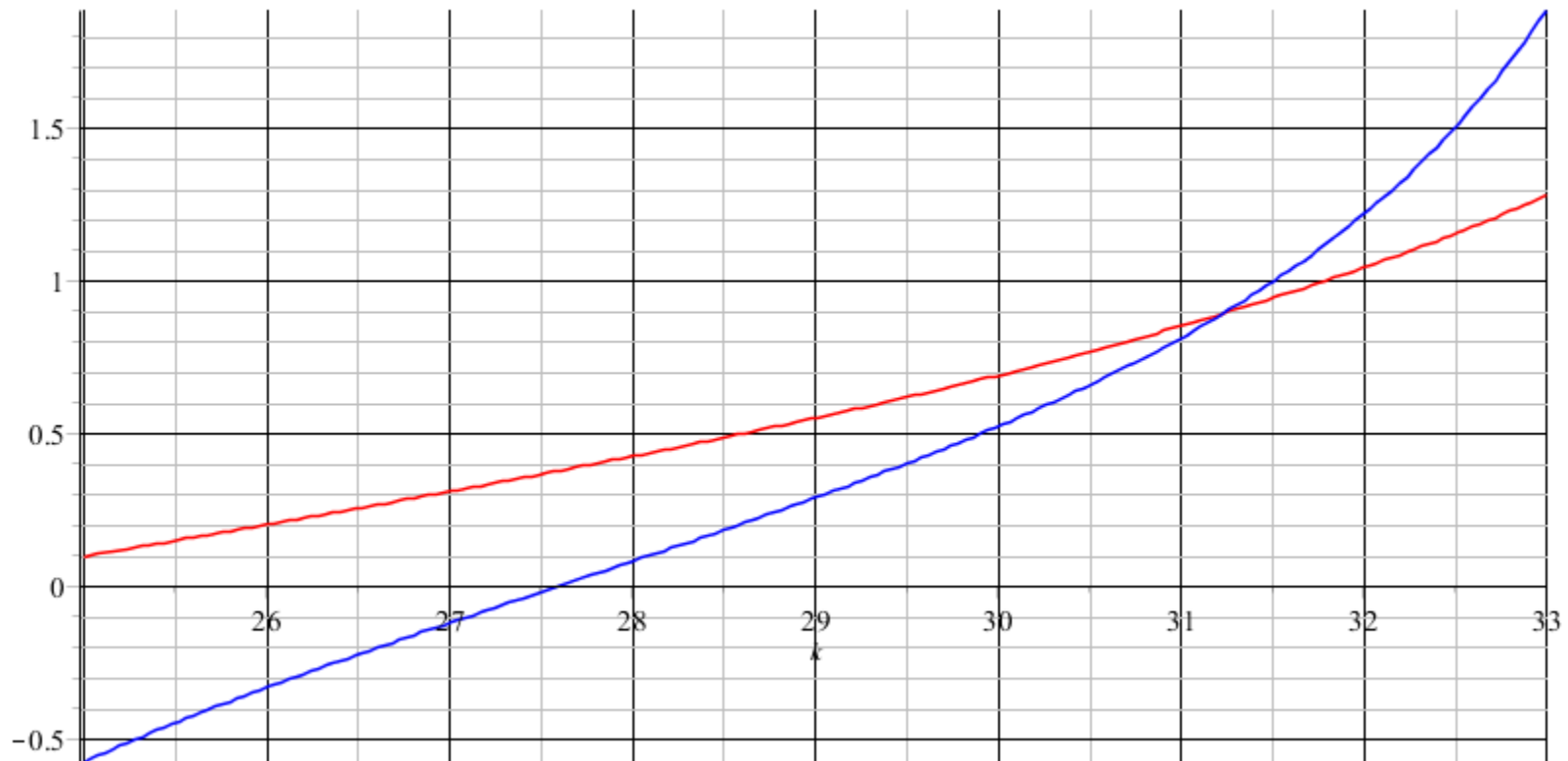
$$f(a) = 0 \quad f(b) = 0$$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)} \quad \text{-- in terms of solution } k_{mn} :$$

$$f(r) = A \left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right)$$

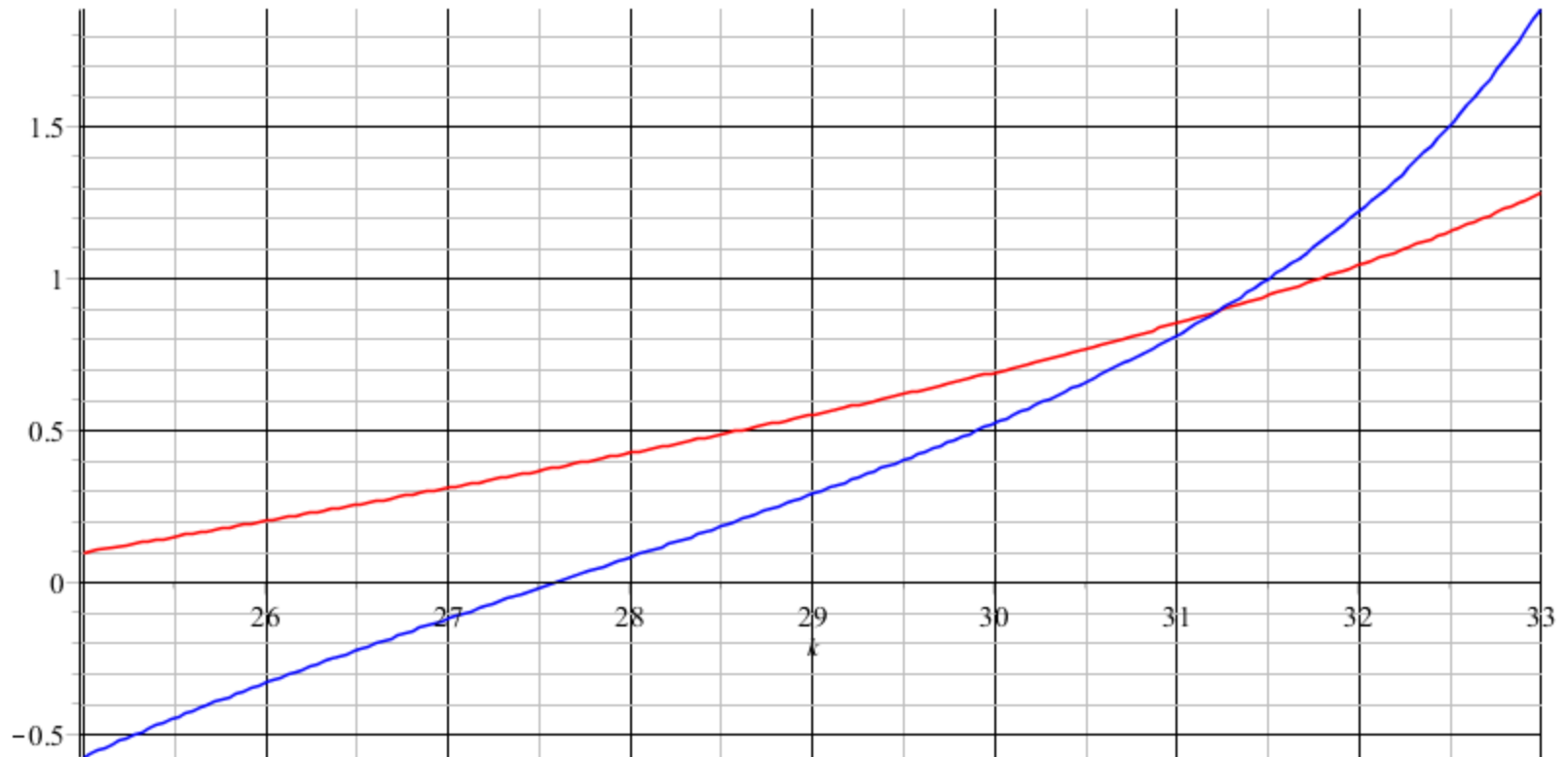
Analysis for $m=0$ and $a=0.1$, $b=0.2$:

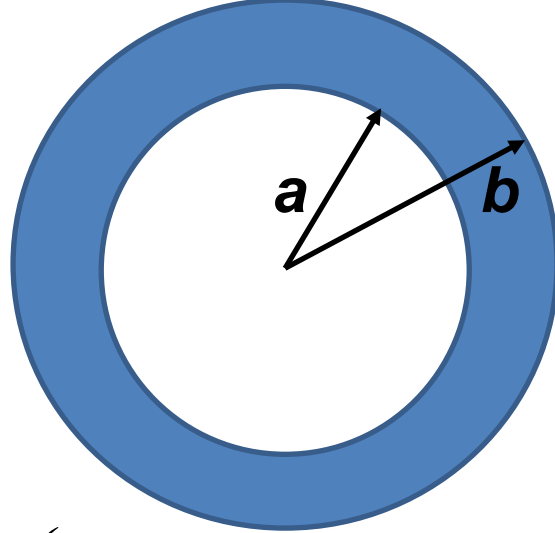
```
=  
> plot( {  $\frac{-\text{BesselJ}(0, 0.1 \cdot k)}{\text{BesselY}(0, 0.1 \cdot k)}$ ,  $\frac{-\text{BesselJ}(0, 0.2 \cdot k)}{\text{BesselY}(0, 0.2 \cdot k)}$  }, k = 25 .. 33, color = [red, blue] );
```



```
=  
> fsolve( ( -BesselJ(0, 0.1*k) / BesselY(0, 0.1*k) = -BesselJ(0, 0.2*k) / BesselY(0, 0.2*k) , k, 30 ..33 );  
=
```

31.23030920





$$f(r) = A \left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right) \quad k_{01} = 31.230309$$

