PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 28 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Irrotational and incompressible fluids
3. Irrotational and isentropic fluids
4. Approximate solutions in the linear limit – next time

Physics Colloquium

Thursday

November 2nd, 2023

Physical Intelligence at the Interface of Biology and Engineering

Micro and nanopatterning of rigid materials such as semiconductors, metals, and dielectrics has revolutionized human life by enabling integrated circuits, microsensors, and smart phones. Imagine applying the same micro/nano patterning principles and multi-scale integration to biological and living materials. We could then create adaptable implants, living robots, microphysiological systems, and homeostatic materials. This talk will describe our efforts to apply micro and nanoscale patterning and heterogeneous integration to hydrogels and living cells/organoids. I will describe shell microelectrode array interfaces for brain organoids for organoid intelligence (O.I). I will describe shape-changing temperature responsive and DNA polymerization gels that display diverse functions such as gripping, locomotion, and complex programmable shape change. I will give examples wherein such hydrogel transformers can be applied widely, including for adaptive implants, soft-robots, digitally programmable materials, and automata). I will also describe the first demonstration of the patterning of live cells with nanolithographic arrays of gold dots and wires using a biocompatible biotransfer process. Integrating lithographic patterns on live cells offers the potential to create living material interfaces and incorporate electronic/optical tattoos. These studies indicate the potential for the design of a range of intelligent materials, robots, and integrated devices that have the touch and feel of biological matter.

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4 pm - Olin 101
Refreshments served prior to seminar beginning at 3:30 pm

21	Mon, 10/16/2023	Chap. 7	The wave and other partial differential equations	#15
22	Wed, 10/18/2023	Chap. 7	Sturm-Liouville equations	#16
23	Fri, 10/20/2023	Chap. 7	Sturm-Liouville equations	#17
24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#21
28	Wed, 11/01/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#22
29	Fri, 11/03/2023			
31	Mon, 11/06/2023			
32	Wed, 11/08/2023			
33	Fri, 11/10/2023			
34	Mon, 11/13/2023			
35	Wed, 11/15/2023			
36	Fri, 11/17/2023			
37	Mon, 11/20/2023			
	Wed, 11/22/2023	Thanksgiving		
	Fri, 11/24/2023	Thanksgiving		
38	Mon, 11/27/2023			
39	Wed, 11/29/2023			

PHY 711 -- Assignment #22

Assigned: 11/01/2023   Due: 11/06/2023

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider the example discussed in Lecture 28, concerning the flow of an incompressible fluid in the **z** direction in the presence of a stationary cylindrical log oriented in the **y** direction. For this homework problem, the log is replaced by a stationary sphere. Find the velocity potential for this case, using the center of the sphere as the origin of the coordinate system and spherical polar coordinates.

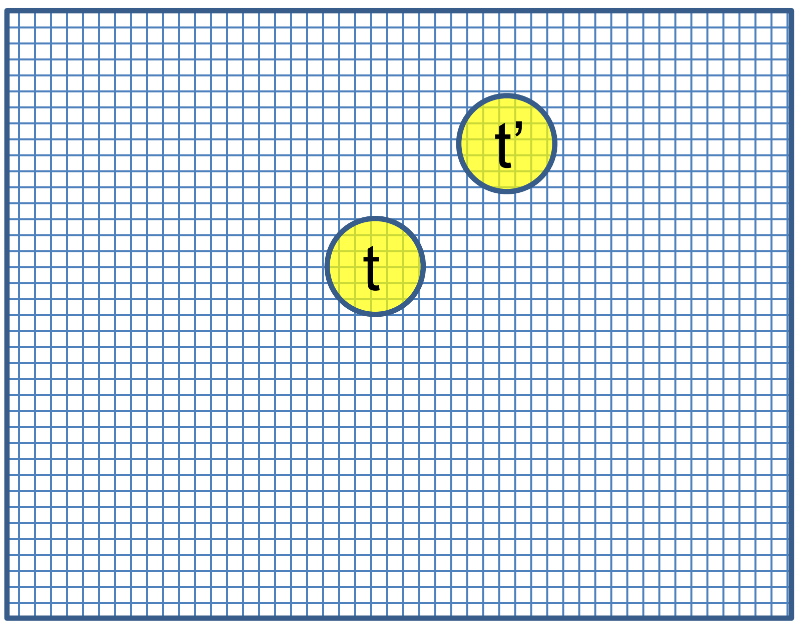
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$



$$\text{Particle at } t: \quad \mathbf{r}, t$$

$$\text{Particle at } t': \quad \mathbf{r} + \mathbf{v}\,\delta t, t'$$

$$t' = t + \delta t$$

Euler analysis -- continued

Particle at t: $\quad \mathbf{r},t$

Particle at t': $\quad \mathbf{r}+\mathbf{v}\delta t, t' \qquad$ where $\quad \delta t = t' - t$

For $f(\mathbf{r},t)$:

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{f(\mathbf{r},t') - f(\mathbf{r},t)}{\delta t} + \frac{f(\mathbf{r}+\mathbf{v}\delta t,t) - f(\mathbf{r},t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left(\mathbf{v}\cdot\nabla\right) f$$

It can be shown that: $\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \nabla\left(\frac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left(\mathbf{v}\cdot\nabla\right) f$$

For $f \rightarrow v_x$ $\quad \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \left(\mathbf{v}\cdot\nabla\right) v_x$

For $f \rightarrow v_y$ $\quad \frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + \left(\mathbf{v}\cdot\nabla\right) v_y$

For $f \rightarrow v_z$ $\quad \frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + \left(\mathbf{v}\cdot\nabla\right) v_z$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v}$

Note that $\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)\left(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}\right)$

$$\mathbf{v} = \mathbf{v}\left(x, y, z, t\right)$$

In vector form $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v}$

Note that $\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\left(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}\right)$

$$= \frac{1}{2}\nabla\left|\mathbf{v}\right|^2 - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right)$$

For example, applying this analysis to Newton's equation of motion for fluids:

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$m = \rho dV$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\, \rho dV - \left(\nabla p\right) dV$$

$$\mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\mathbf{F}_{\text{pressure}} = -\nabla p dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

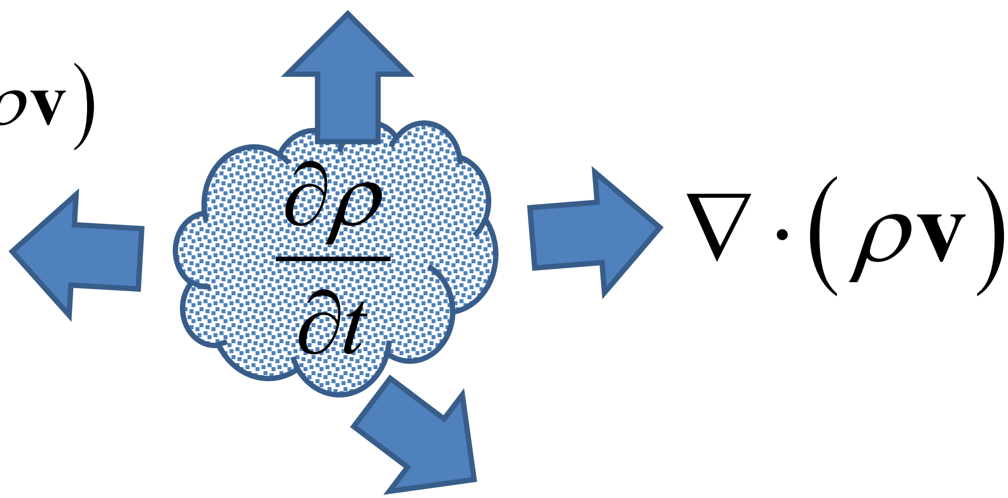
Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

The notion of the continuity is a common feature of continuous closed systems. Here we assume that there are no mechanisms for creation or destruction of the fluid.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \left(\nabla \cdot \mathbf{v} \right) + \left(\nabla \rho \right) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = \left(\text{constant} \right)$

velocity potential

$\Rightarrow \nabla \cdot \mathbf{v} = 0$

Irrotational flow: $\nabla \times \mathbf{v} = 0$ $\Rightarrow \mathbf{v} = -\nabla \Phi$

For irrotational flow of an incompressible fluid: $\nabla^2 \Phi = 0$

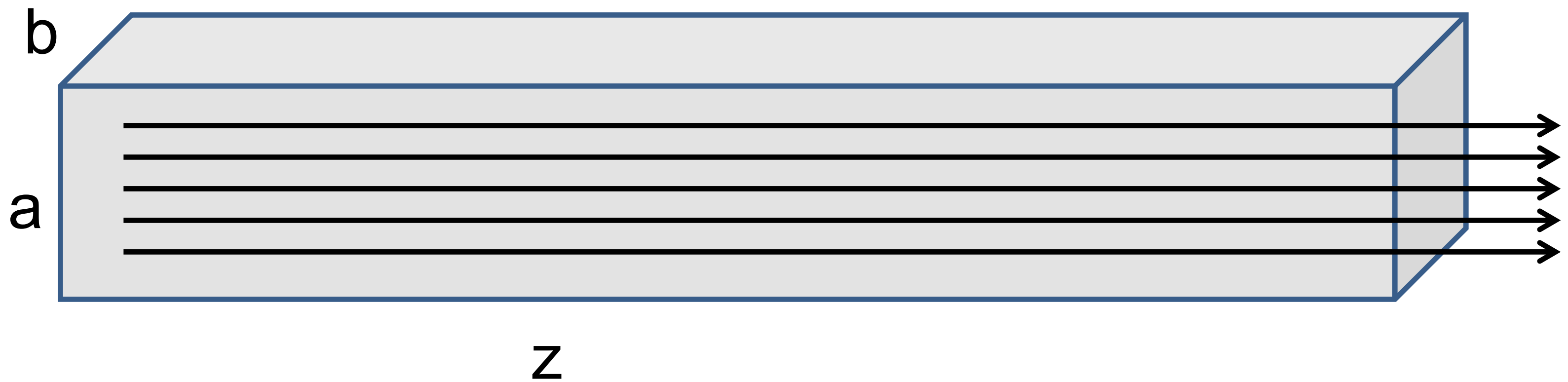
Checking --

Why does $\nabla \times \mathbf{v} = 0$ imply that $\mathbf{v} = -\nabla\Phi$?

Consider: $$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla \times \left(\nabla\Phi\right)\Big|_x = \frac{\partial^2\Phi}{\partial y \partial z} - \frac{\partial^2\Phi}{\partial z \partial y} = 0$$ Similar results for other directions.

Example of irrotational flow of an incompressible fluid – uniform flow



$$\nabla^2 \Phi = 0$$

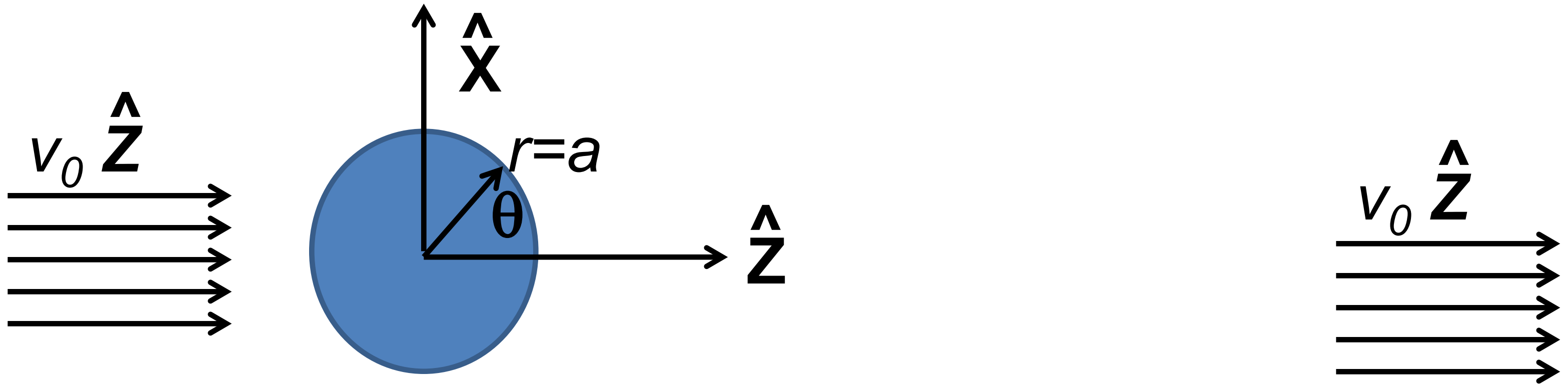
$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the *Y* direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r, \theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r\to\infty) = v_0$

$$\frac{\partial\Phi}{\partial z}(r\to\infty) = -v_0 \qquad \Rightarrow \Phi(r\to\infty,\theta) = -v_0 r\cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial\theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r}$$ where A, B are constants

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition as $r \to \infty$: $\Rightarrow A = -v_0$

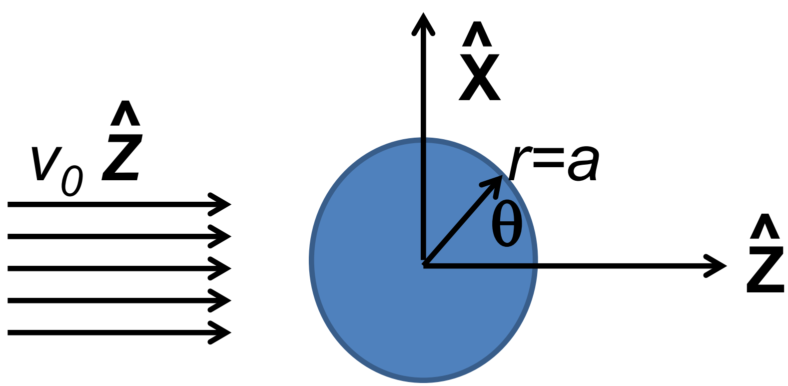
$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

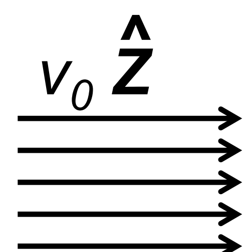
$$v_r = -\frac{\partial\Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For $r \to \infty$

$$\mathbf{v} \to v_0\cos\theta\,\hat{\mathbf{r}} - v_0\sin\theta\,\hat{\boldsymbol{\theta}} = v_0\hat{\mathbf{z}}$$





Now consider the case of your homework problem --

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

In our case:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Phi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f = 0$$

(Continue analysis for homework)

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right)=0$ "irrotational flow"

$$\Rightarrow \mathbf{v}=-\nabla\Phi$$

2. $\mathbf{f}_{applied}=-\nabla U$ conservative applied force

3. $\rho=(\text{constant})$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)=-\nabla U-\frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

For incompressible fluid

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\quad \mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\left(\Phi(\mathbf{r},t)+C(t)\right)$

It is convenient to modify $\Phi(\mathbf{r},t)\rightarrow\Phi(\mathbf{r},t)+\int^{t}C(t')dt'$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Extension of these ideas to some compressible fluids – now assuming conditions of constant entropy (no heat transfer).

Under what circumstances can there be no heat transfer?

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla\Phi$$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho \neq$ (constant) isentropic fluid

A little thermodynamics

Comment about sign convention

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$$dE_{\text{int}} = -dW = -pdV$$

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -pdV$$

In terms of mass density: $\rho = \dfrac{M}{V}$

For fixed M and variable V: $\quad d\rho = -\dfrac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider :} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging :} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Is this useful?

a. Yes

b. No

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2} v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla \times \mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2} v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2} v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Summary of Bernoulli's results

For incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^{2}-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. In order to continue, we need to know the form of

$$\varepsilon(\rho,s)$$

which is different for each fluid….

Example – ideal gas fluid

Equation of state for ideal gas:

$$pV = NkT \qquad\qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$$k = 1.38\times 10^{-23} J/K$$

$M_0 =$ average mass of each molecule

Internal energy for ideal gas: in terms of f (degrees of freedom)

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio: $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Digression

Internal energy for ideal gas: $f \equiv$ "degrees of freedom"

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma-1} NkT \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions:

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$