PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 29 -- Chap. 9 in F & W

More hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Approximate solutions in the linear limit
3. Linear sound waves

21	Mon, 10/16/2023	Chap. 7	The wave and other partial differential equations	#15
22	Wed, 10/18/2023	Chap. 7	Sturm-Liouville equations	#16
23	Fri, 10/20/2023	Chap. 7	Sturm-Liouville equations	#17
24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#21
28	Wed, 11/01/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#22
29	Fri, 11/03/2023	Chap. 9	Ideal gas fluids	#23
31	Mon, 11/06/2023			
32	Wed, 11/08/2023			
33	Fri, 11/10/2023			
34	Mon, 11/13/2023			
35	Wed, 11/15/2023			
36	Fri, 11/17/2023			
37	Mon, 11/20/2023			
	Wed, 11/22/2023	Thanksgiving		

PHY 711 -- Assignment #23

Assigned: 11/03/2023   Due: 11/06/2023

Continue reading Chapter 9 in **Fetter & Walecka**.

1. In class, we derived an expression for the linearized differential equation describing density fluctuations $\delta\rho$ within an ideal gas under isentropic conditions, finding a wave equation. Choose an ideal gas (such as air) and two different conditions of pressure, temperature, etc. to estimate the wave speeds. c_0.

Recall the basic equations of hydrodynamics

Basic variables: Density $\rho(\mathbf{r},t)$

Velocity $\mathbf{v}(\mathbf{r},t)$

Pressure $p(\mathbf{r},t)$

Newton-Euler equation of motion:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

+ relationships among the variables due to principles of thermodynamics due to the particular fluid (In fact, we will focus on an ideal gas.)

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Additional relationships among the variables apply, depending on the fluid material and on thermodynamics

At the moment we are interested in the case where there is no heat exchange.

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$dE_{\text{int}} = -dW = -pdV$ Here W == work

V == volume

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -p\,dV$$

In terms of mass density: $\rho = \dfrac{M}{V}$

For fixed M and variable V: $d\rho = -\dfrac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

Internal energy per unit mass

$$dE_{\text{int}} = M\,d\varepsilon = -dW = -p\,dV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Note: Under conditions of constant entropy, we assume e can be expressed in terms of the density alone.

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Note that here we are assuming that we can write ε as $\varepsilon(\rho,s)$.

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

if $\nabla\times\mathbf{v} = 0$ ➔ $\mathbf{v} = -\nabla\Phi$ $\mathbf{f}_{applied} = -\nabla U$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For isentropic and irrotational fluid.

Some details --

$$\left(\nabla\times\mathbf{v}\right)=0 \qquad \text{"irrotational flow"} \qquad \Rightarrow \mathbf{v}=-\nabla\Phi$$

Check: $\left(\nabla\times\mathbf{v}\right)=-\left(\nabla\times\nabla\Phi\right)=?$

$$\left(\nabla\times\nabla\Phi\right)\Big|_x=\frac{\partial^2\Phi}{\partial y\partial z}-\frac{\partial^2\Phi}{\partial z\partial x}$$

Summary: For isentropic and irrotational fluid with internal energy per unit mass ε:

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2} \frac{k}{M_0} T = \frac{f}{2} \frac{p}{\rho}$$

In terms of specific heat ratio: $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2} \frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2} \frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\frac{C_p}{C_V} \equiv \gamma = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Digression

Internal energy for ideal gas: $f \equiv$ "degrees of freedom"

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma-1} NkT \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1} \frac{k}{M_0} T = \frac{1}{\gamma - 1} \frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions:

$$d\varepsilon = -\frac{p}{M} dV = \frac{p}{\rho^2} d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1} \frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Back to analyzing the fluid mechanics equations

$$\nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For isentropic and irrotational fluid.

Internal energy for ideal gas:

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

$$\nabla\left(\frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Also need to include continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$$

Now consider the fluid to be air near equilibrium

Near equilibrium:

$$\rho = \rho_0 + \delta\rho$$

$$p = p_0 + \delta p$$

$$\mathbf{v} = 0 + \delta\mathbf{v} = -\nabla\delta\Phi$$

$$\mathbf{f}_{applied} = 0$$

ρ_0 represents the average air density

p_0 represents the average air pressure (usually $\approx$ 1 atmosphere)

$\mathbf{v}_0 = 0$ average velocity

=0 =0

$$\nabla\left(\frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Linearized equations near equilibrium

$$\left(\frac{1}{\gamma-1}+1\right)\left(\nabla\left(\frac{p}{\rho}\right)\right)+\frac{\partial\delta\mathbf{v}}{\partial t}=0 \quad \Rightarrow \quad \left(\frac{\gamma}{\gamma-1}\right)\left(\nabla\left(\frac{p}{\rho}\right)\right)+\frac{\partial\delta\mathbf{v}}{\partial t}=0$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{v}\right)=0$$

Further relationships for isentropic ideal gas

$$p=p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad \Rightarrow \quad \frac{\gamma}{\gamma-1}\nabla\left(\frac{p}{\rho}\right)=\frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0^{\gamma}}(\gamma-1)\rho^{\gamma-2}\nabla\rho$$

$$=\frac{\gamma p_0}{\rho_0^{\gamma}}\rho^{\gamma-2}\nabla\rho$$

Complete linearization

$$\frac{\gamma p_0}{\rho_0^2}\nabla\delta\rho+\frac{\partial\delta\mathbf{v}}{\partial t}=0 \qquad \frac{\partial\delta\rho}{\partial t}+\rho_0\nabla\cdot\left(\delta\mathbf{v}\right)=0$$

Decoupling linearized equations --

$$\frac{\gamma p_0}{\rho_0^2}\nabla\delta\rho + \frac{\partial\delta\mathbf{v}}{\partial t} = 0 \qquad \frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$$

$$\frac{\gamma p_0}{\rho_0}\nabla^2\delta\rho - \frac{\partial^2\delta\rho}{\partial t^2} = 0$$

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\nabla\left(\frac{\gamma p_0}{\rho_0^2}\delta\rho - \frac{\partial\Phi}{\partial t}\right) = 0 \qquad \Rightarrow \frac{\gamma p_0}{\rho_0^2}\delta\rho - \frac{\partial\Phi}{\partial t} = \text{constant}$$

$$\Rightarrow \frac{\gamma p_0}{\rho_0^2}\frac{\partial\delta\rho}{\partial t} - \frac{\partial^2\Phi}{\partial t^2} = 0$$

$$\frac{\gamma p_0}{\rho_0}\nabla^2\Phi - \frac{\partial^2\Phi}{\partial t^2} = 0$$

Have we seen these equations before?

$$\frac{\gamma p_0}{\rho_0}\nabla^2\delta\rho-\frac{\partial^2\delta\rho}{\partial t^2}=0$$

$$\frac{\gamma p_0}{\rho_0}\nabla^2\Phi-\frac{\partial^2\Phi}{\partial t^2}=0$$

It is also possible to show that

$$\frac{\gamma p_0}{\rho_0}\nabla^2\delta p-\frac{\partial^2\delta p}{\partial t^2}=0$$

For an ideal gas under isentropic conditions with irrotational flow, close to equilibrium, the linear fluctuations in density, pressure, and velocity are characterized by a wave equation with velocity

$$c_0^2 \equiv \frac{\gamma p_0}{\rho_0}.$$

More general case -- Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Density dependence of speed of sound for ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0\gamma}{\rho_0}\frac{p/p_0}{\rho/\rho_0} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \quad \text{for } c_0^2 \equiv \frac{p_0\gamma}{\rho_0}$$

Summary of linearized hydrodynamic equations for isentropic fluid

In terms of the velocity potential:

$$\delta \mathbf{v} = -\nabla \Phi \qquad \frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0 \qquad c_0^2 = \left(\frac{\partial p}{\partial \rho} \right)_{s,\rho_0}$$

In term of density fluctuation: $$\frac{\partial^2 \delta\rho}{\partial t^2} - c_0^2 \nabla^2 \delta\rho = 0$$

In term of pressure fluctuation: $$\frac{\partial^2 \delta p}{\partial t^2} - c_0^2 \nabla^2 \delta p = 0$$

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$