PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin103

Lecture notes for Lecture 2
Chapter 3.17 of F&W

Introduction to the calculus of variations

1. Mathematical construction
2. Practical use
3. Examples

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	
4	Mon, 9/04/2023			

PHY 711 -- Assignment #2

Assigned: 8/30/2023 Due: 9/4/2023

Start reading Chapter 3, especially Section 17, in **Fetter & Walecka**.

1. Using calculus of variations, find the equation, $y(x)$, of the shortest length "curve" which passes through the points ($x=0$, $y=0$) and ($x=3$, $y=4$). What is the length of this "curve"?

Reminder --

Physics Colloquium Series

Welcome to Fall 2023

Introductions, Presentations, and Announcements

4 PM in Olin 101

(Pre-Colloquium Reception in Olin Lobby at 3:30 PM)

WAKE FOREST
UNIVERSITY

August 31, 2023

Your questions –
From David:

1. Could we discuss defining degrees of freedom in multiple dimensions during the next class? I am usually clumsily identifying them.
We will probably talk about that next time (please remind).
2. Since we are performing optimizations, shouldn't we also do ill-conditioning tests and compute the second derivative to evaluate if it's maximum or minimum? You are right about this; we are not always as rigorous about this as we should.

From Joe:

1. I was able to make it through Ch. 3(17) on calculus of variations but I had a question regarding the second derivation of the Euler-Lagrange equation, for tomorrow's class. What significance does this derivation have? It's made shorter by introducing the "little delta" notation, but I don't see how it is any more helpful to use that notation. At the beginning of the derivation it's stated its "frequently helpful" to use that notation (and I saw a similar statement online but can't find the source again). The notation is indeed confusing. We will try to think about this more when we get to this part.

From Mitch:

1. I'm aware that generally we use calculus of variations to determine minima and sometimes maxima, but are there any practical applications regarding inflection points. I am not aware of this idea, but it seems like a very nice suggestion.

From Athul:

1. In slide 6 of the curve describing the local minima and the global minima, how the following equations would change if the global minima is same as that of the local minima. Will it be applicable in such a condition? The answer may depend on the details, but the analysis may be the same.

From Thilini:

1. I got this question when referring to chapter 3 (17) in the Textbook and the lecture notes. It says in the book that we take an arbitrary function of x, n(X1) = n(x2) = 0 to solve the derivation. It's not clear to me how we get those terms equal to zero and why. Perhaps we can discuss this when we get to this section.

The “calculus of variation” as a mathematical construction.

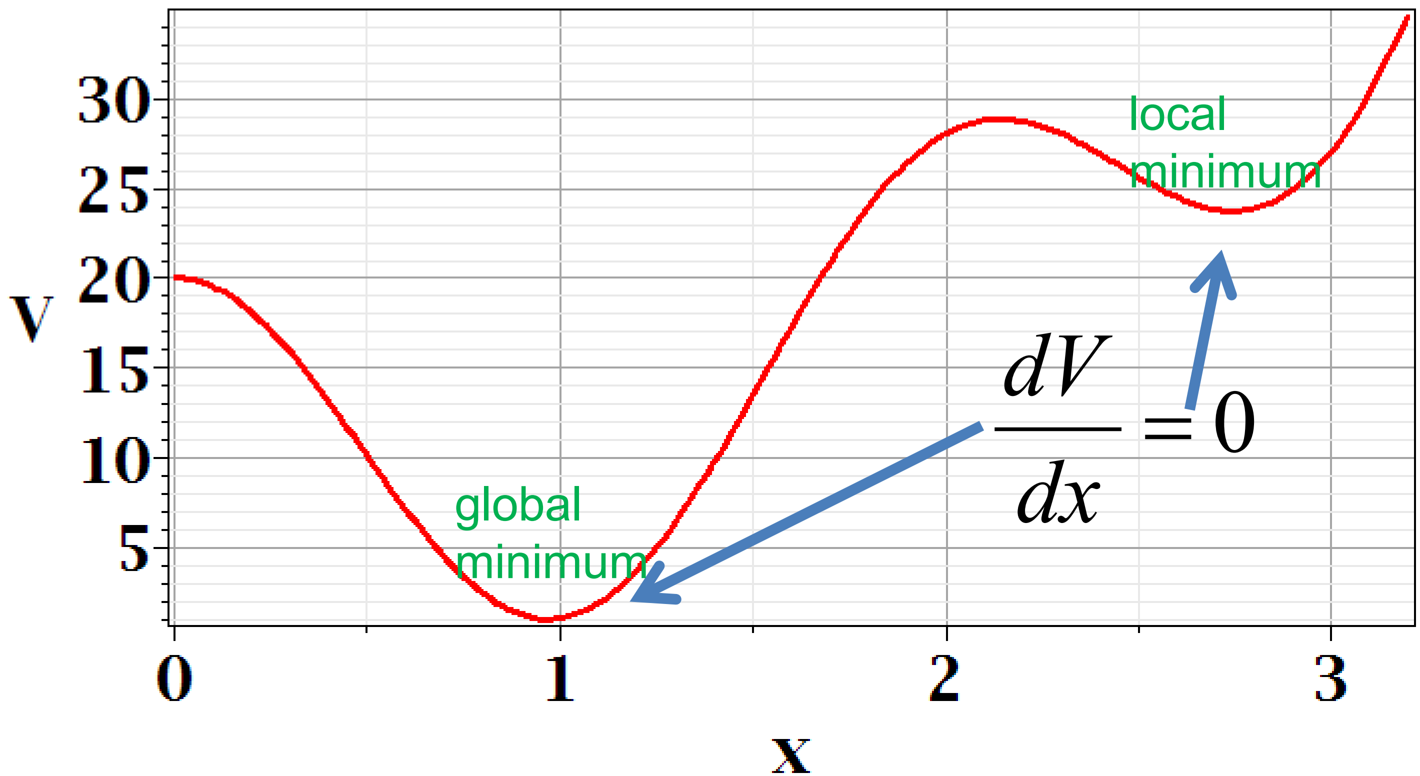
According wikipedia –
Joseph-Louis Lagrange (born **Giuseppe Luigi Lagrangia** or **Giuseppe Ludovico De la Grange Tournier**; 25 January 1736 – 10 April 1813), also reported as **Giuseppe Luigi Lagrange** or **Lagrangia**, was an Italian mathematician and astronomer, later naturalized French. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

According to Wikipedia –

Leonard Euler (April 7, 1707-September 18, 1783) Swiss mathematician, physicist, astronomer, geographer, logician and engineer who founded the studies of graph theory and topology and made pioneering and influential discoveries in many other branches of mathematics such as analytic number theory, complex analysis, and infinitesimal calculus. He introduced much of modern mathematical terminology and notation, including the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory.

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

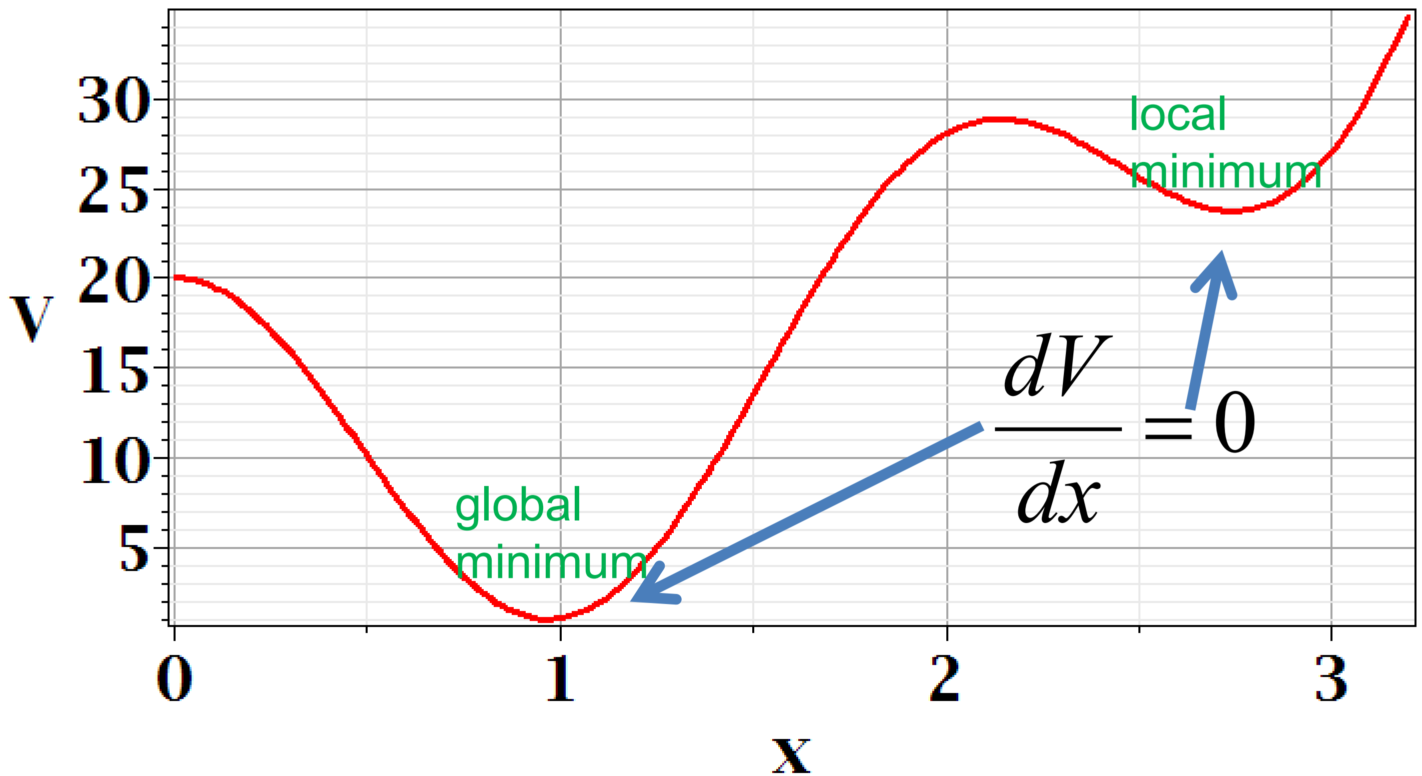
Minimization of a simple function



Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx} = 0$



Functional minimization of an integral relationship

Consider a family of functions $y(x)$, with fixed end points

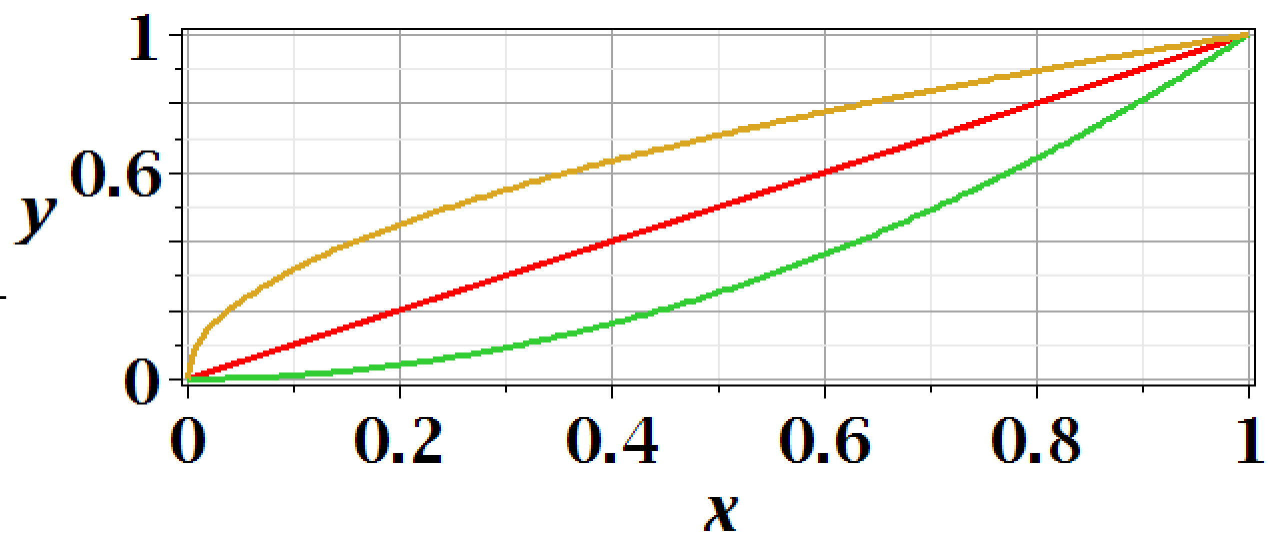
$y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral form $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Difference between minimization of a function *V(x)* and the minimization in the calculus of variation.

Minimization of a function – *V(x)*

➔Know *V(x)* ➔Find x_0 such that $V(x_0)$ is a minimum.

Calculus of variation

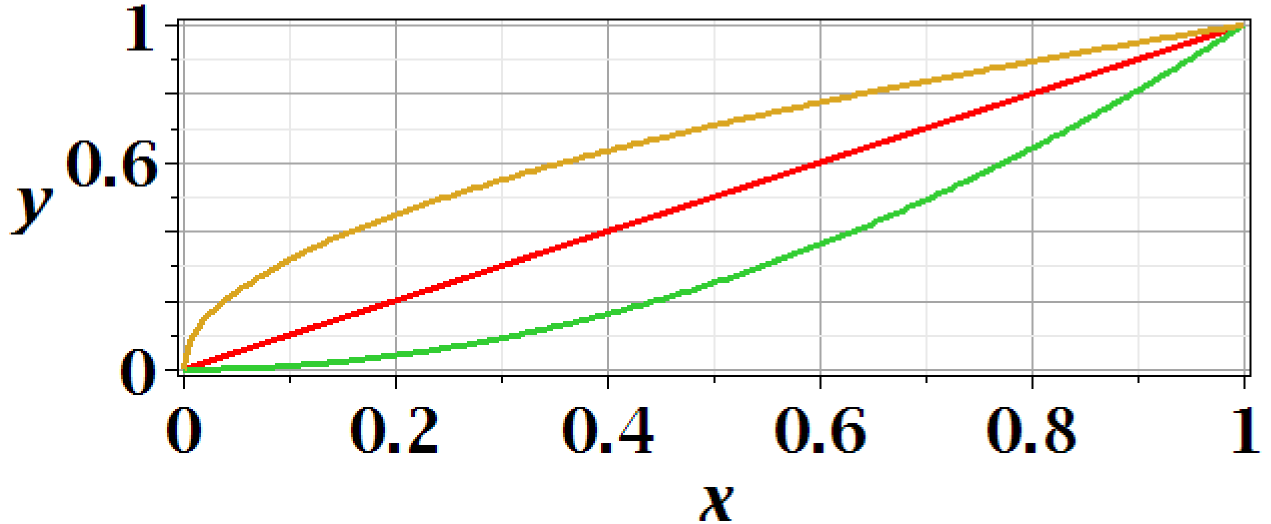
For $x_i \leq x \leq x_f$ want to find a function $y(x)$ that minimizes an integral that depends on $y(x)$.

The analysis involves deriving and solving a differential equation for the function $y(x)$.

Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Sample functions:

$$y_1(x) = \sqrt{x} \qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$$

$$y_2(x) = x \qquad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$$

$$y_3(x) = x^2 \qquad L = \int_0^1 \sqrt{1+4x^2}\, dx = 1.4789$$

Calculus of variation example for a pure integral function

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

where $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$

Necessary condition : $\delta L = 0$

At any x, let $y(x) \rightarrow y(x) + \delta y(x)$

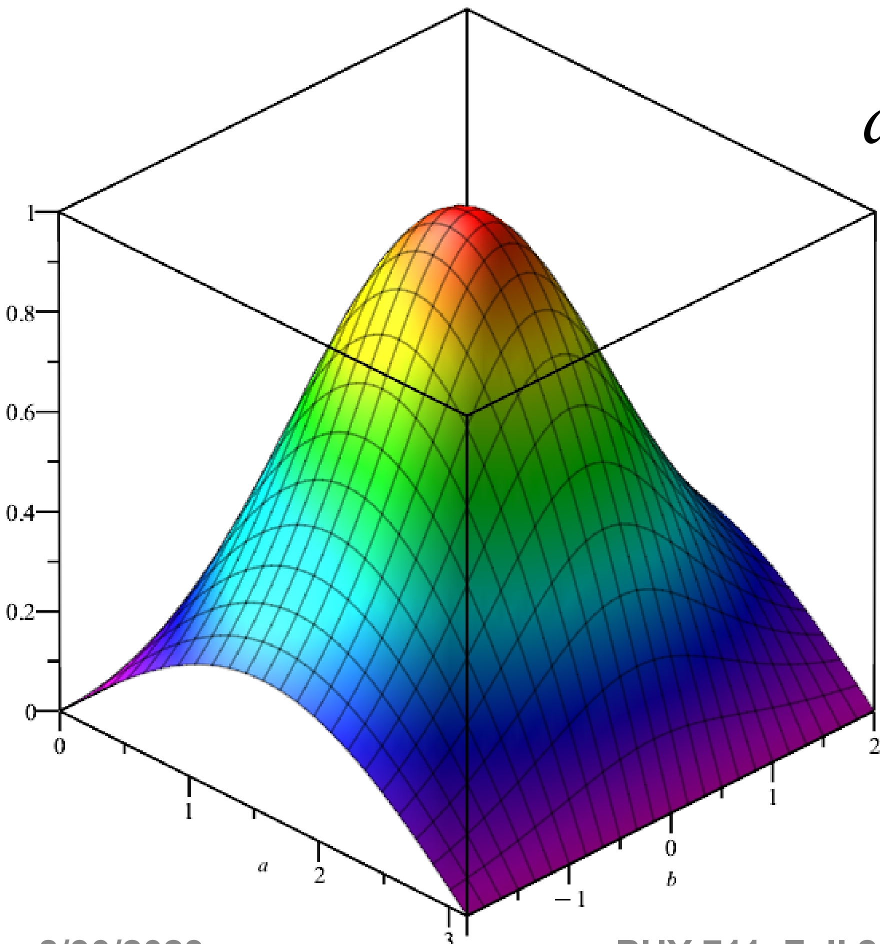
$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx.$$

Comment on partial derivatives -- function $f(a,b)$

$$\frac{\partial f}{\partial a} \equiv \lim_{da \to 0}\left(\frac{f(a+da,b)-f(a,b)}{da}\right) \equiv \left.\frac{\partial f}{\partial a}\right|_b$$



$$df = \left(\frac{\partial f}{\partial a}\right)_b da + \left(\frac{\partial f}{\partial b}\right)_a db$$

Comment about notation concerning functional dependence and partial derivatives

Suppose x, y, z represent independent variables that determine a function f :

We write $f(x, y, z)$. A partial derivative with respect to x implies that we hold y, z fixed and infinitessimally change x

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}\right)$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta\left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y \, dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Note that this is a
“total” derivative

“Some” derivations --

Consider the term

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx:$$

If $y(x)$ is a well-defined function, then $\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}\delta y$ *

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx=\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\frac{d}{dx}\delta y\right]dx$$

$$=\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Note that the δy notation is meant to imply a general infinitessimal variation of the function $y(x)$

*

Clarification -- what is the meaning of the following statement:

$$\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}\delta y$$

Up to now, the operator δ is not well defined and meant to represent a general infinitessimal difference. Suppose that $\delta y\equiv\frac{dy}{da}$, where a appears in the functional form somehow. For most functional forms that one can think of, $\frac{d^2y(x,a)}{dxda}=\frac{d^2y(x,a)}{dadx}$. One can show this to be the case even for $y(x,a)=x^a$ where $\frac{d^2y(x,a)}{dxda}=\frac{d^2y(x,a)}{dadx}=x^{a-1}\left(1+a\ln\left(x\right)\right)$.

“Some” derivations (continued)--

$$\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]_{x_i}^{x_f}-\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$= \quad 0 \quad -\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Euler-Lagrange equation:

$$\Rightarrow\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0 \quad \text{for all } x_i\le x\le x_f$$

Clarfication – Why does this term go to zero?

$$\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]_{x_i}^{x_f}-\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\qquad 0 \qquad -\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Answer --

By construction $\delta y(x_i)=\delta y(x_f)=0$

Recap --

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x,\frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x,\frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x,\frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Here we conclude that the integrand has to vanish at every argument in order for the integral to be zero

a. Necessary?
b. Overkill?

Example: End points -- $y(0)=0;\, y(1)=1$

$$L=\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K' \equiv \frac{K}{\sqrt{1-K^2}}$$

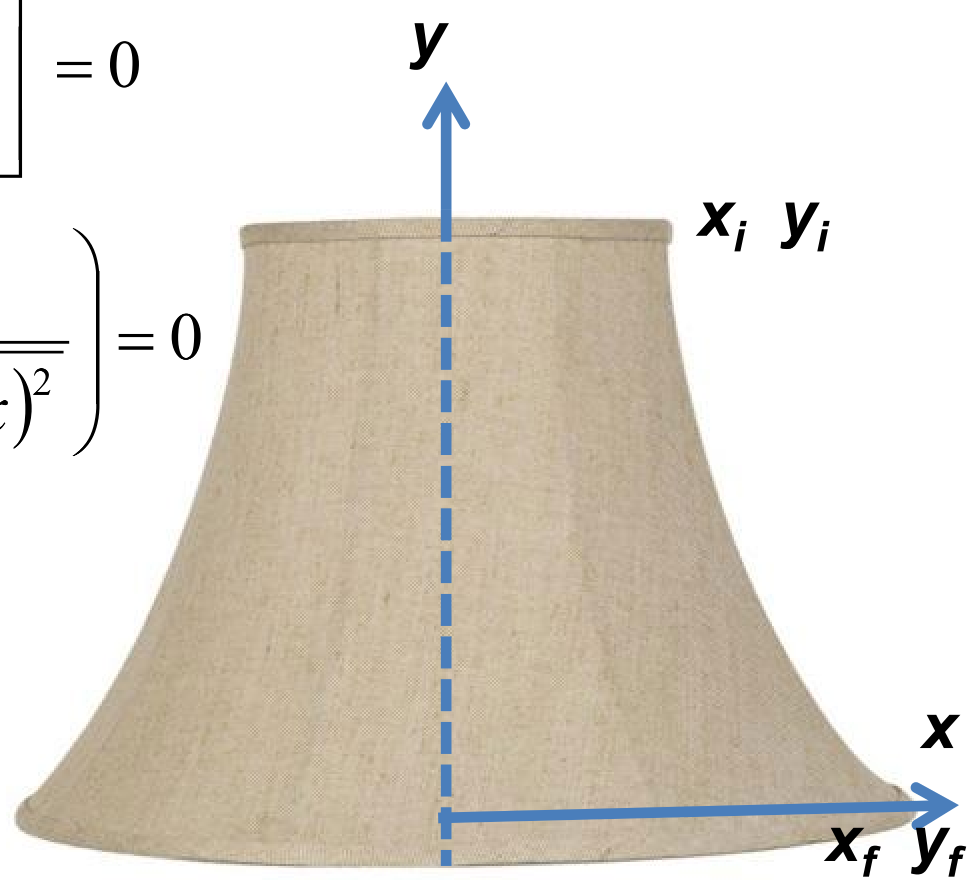
$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

Example: Lamp shade shape $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$



$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}=K_1$$

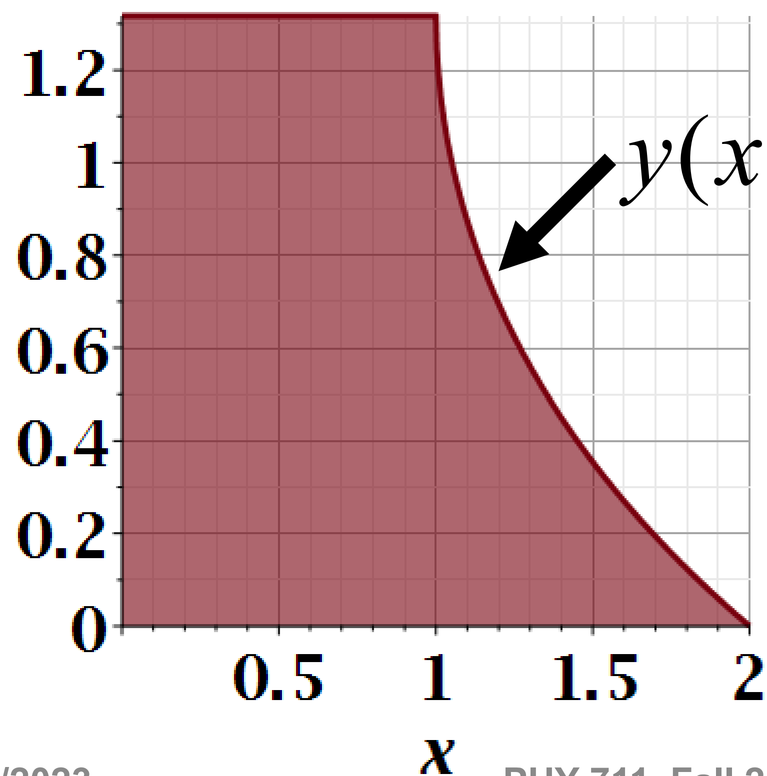
$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

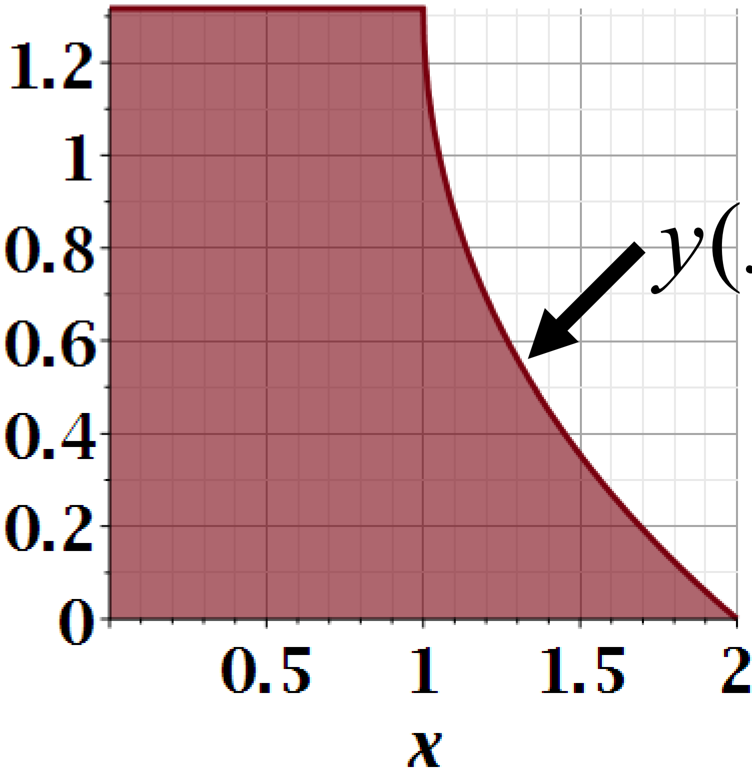
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose $K_1 = 1$ and $K_2 = 2 + \sqrt{3}$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Another example:

(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0) = 0$ and $y(1) = 1$

that minimize the integral :

$$I = \int_0^1 \left(\left(\frac{dy}{dx} \right)^2 - ay^2 \right) dx \quad \text{for constant } a > 0$$

Euler - Lagrange equation :

$$\frac{d^2 y}{dx^2} + ay = 0$$

$$\Rightarrow y(x) = \frac{\sin\left(\sqrt{a}\, x\right)}{\sin\left(\sqrt{a}\right)}$$

Review : for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation