PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin103

Lecture notes for Lecture 3
Chapter 3.17 of F&W

More about the calculus of variations

1. Review examples – Area of lamp shade
2. Brachistochrone problem
3. Calculus of variation with constraints

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023			
5	Wed, 9/06/2023			

Note that Monday is not a holiday for us…

PHY 711 – Assignment #3

Assigned: 09/01/2023 Due: 09/04/2023

This exercise is designed to illustrate the differences between partial and total derivatives.

1. Consider an arbitrary function of the form $f = f(q, \dot{q}, t)$, where it is assumed that $q = q(t)$ and $\dot{q} \equiv dq/dt$.

   (a) Write a formal expression for $\frac{df}{dt}$ in terms of an arbitrary form of $f = f(q, \dot{q}, t)$ and an arbitrary function $q(t)$.

   (b) Now suppose that
   $$f(q, \dot{q}, t) = q\dot{q}^2 t, \quad \text{where} \quad q(t) = \mathrm{e}^{-t/\tau}.$$
   Here τ is a constant. Evaluate df/dt as a function of t using the expression you derived in part (a)..

   (c) Now find the expression for f as an explicit function of t ($f(t)$) and then take its time derivative directly to check your previous result.

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta L = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Example: Find minimum curve between points -- $y(0)=0; y(1)=1$

$$L=\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

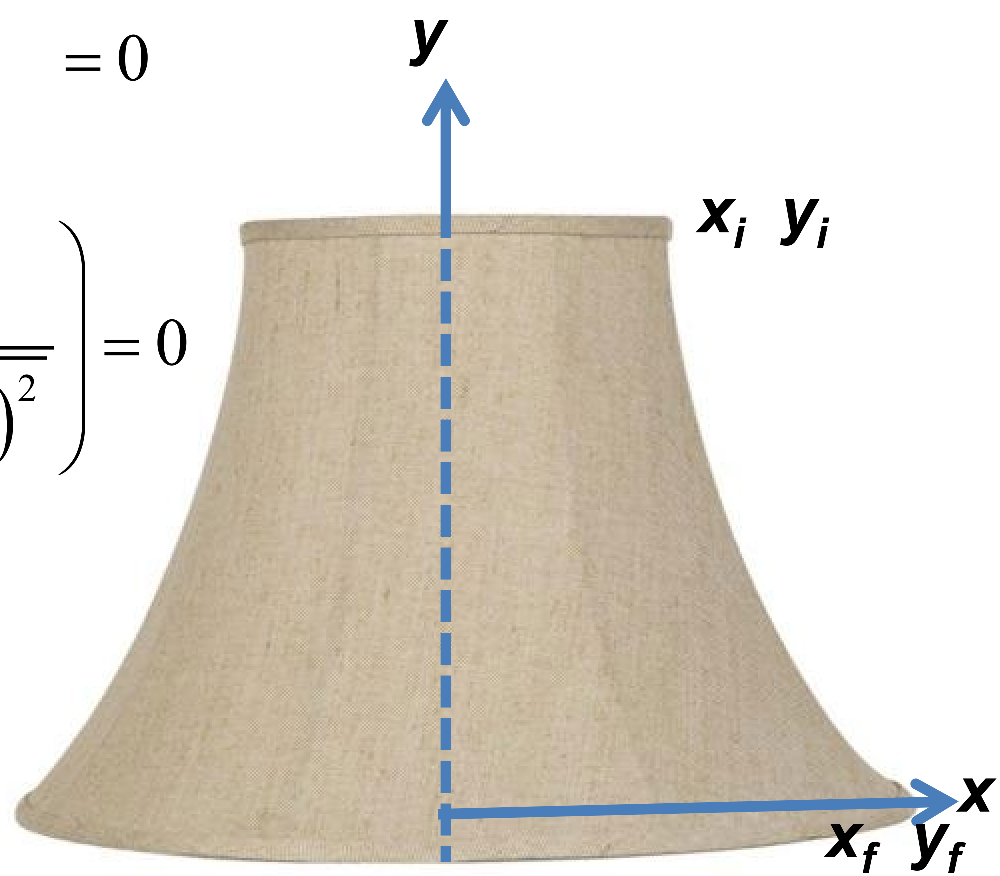
$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

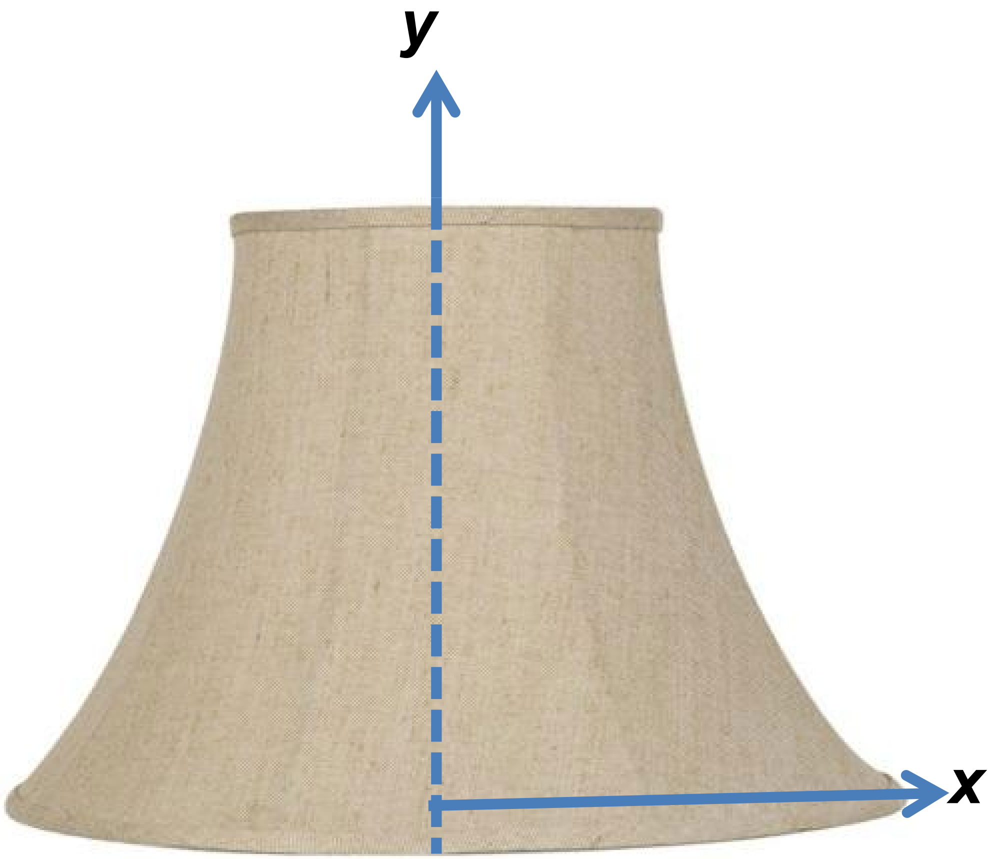
Another example: Lamp shade shape *y(x)*

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

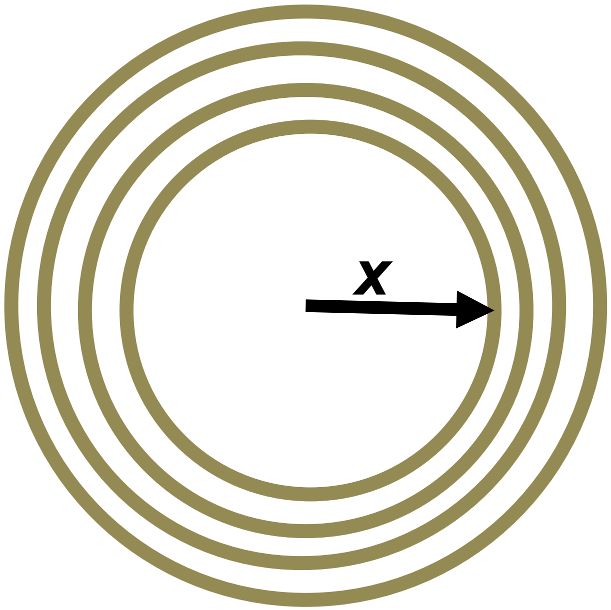
$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$





Top view



$$A = 2\pi \int_{x_i y_i}^{x_f y_f} x\sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi \int_{x_i}^{x_f} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Lamp shade area

$2\pi x dL$ where $dL = \sqrt{(dx)^2 + (dy)^2}$

$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}=K_1$$

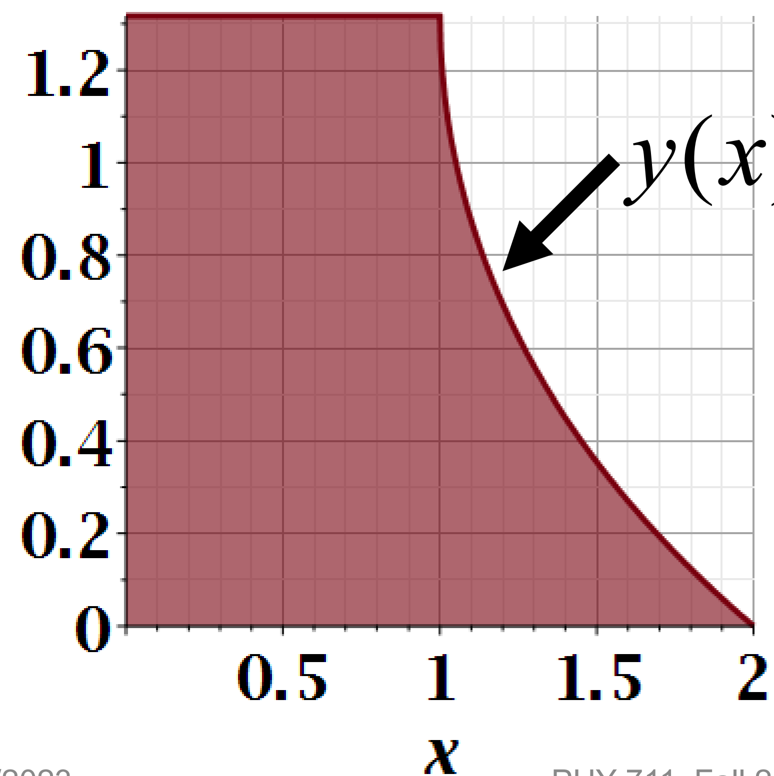
$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

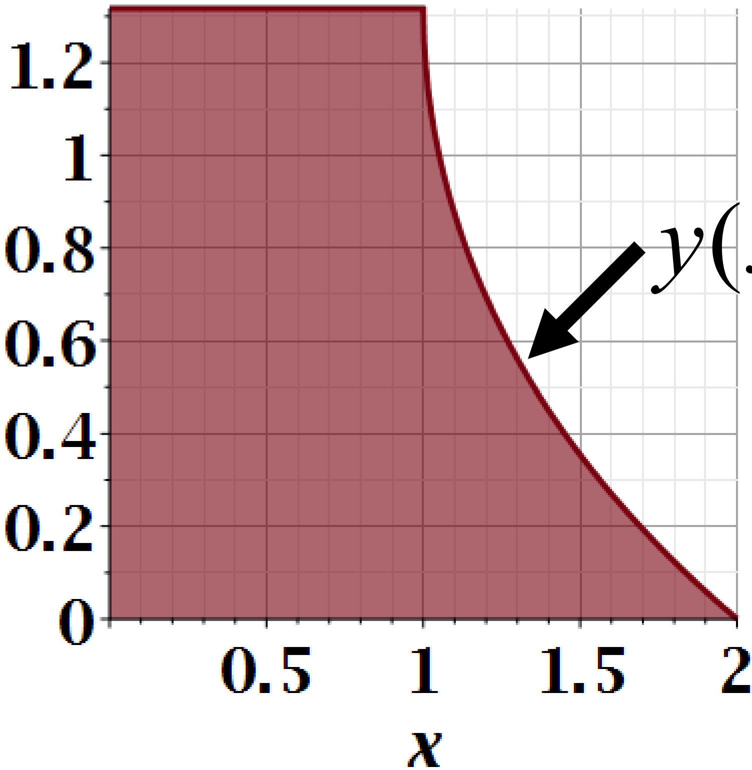
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose $K_1 = 1$ and $K_2 = \ln\left(2 + \sqrt{3}\right)$



$$y(x) = \ln\left(\frac{2 + \sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Review: for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

⇦ Euler-Lagrange equation

Note that for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial (dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial (dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

⇦ Alternate Euler-Lagrange equation

A few more steps --

Note that for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

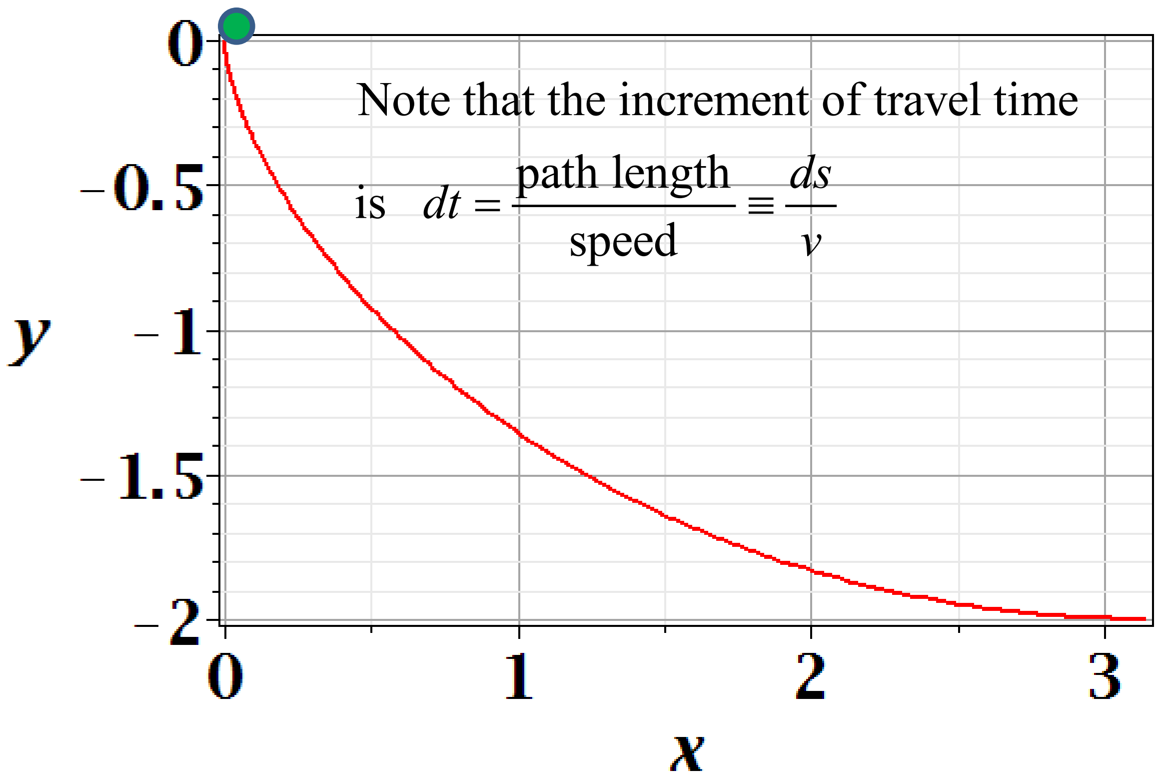
$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right)\right) + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

$$E = \frac{1}{2}mv^2 + mgy \qquad \text{with } y(t=0)=0 \text{ and } \dot{y}(t=0)=0$$

With this choice of initial conditions, $E = 0$

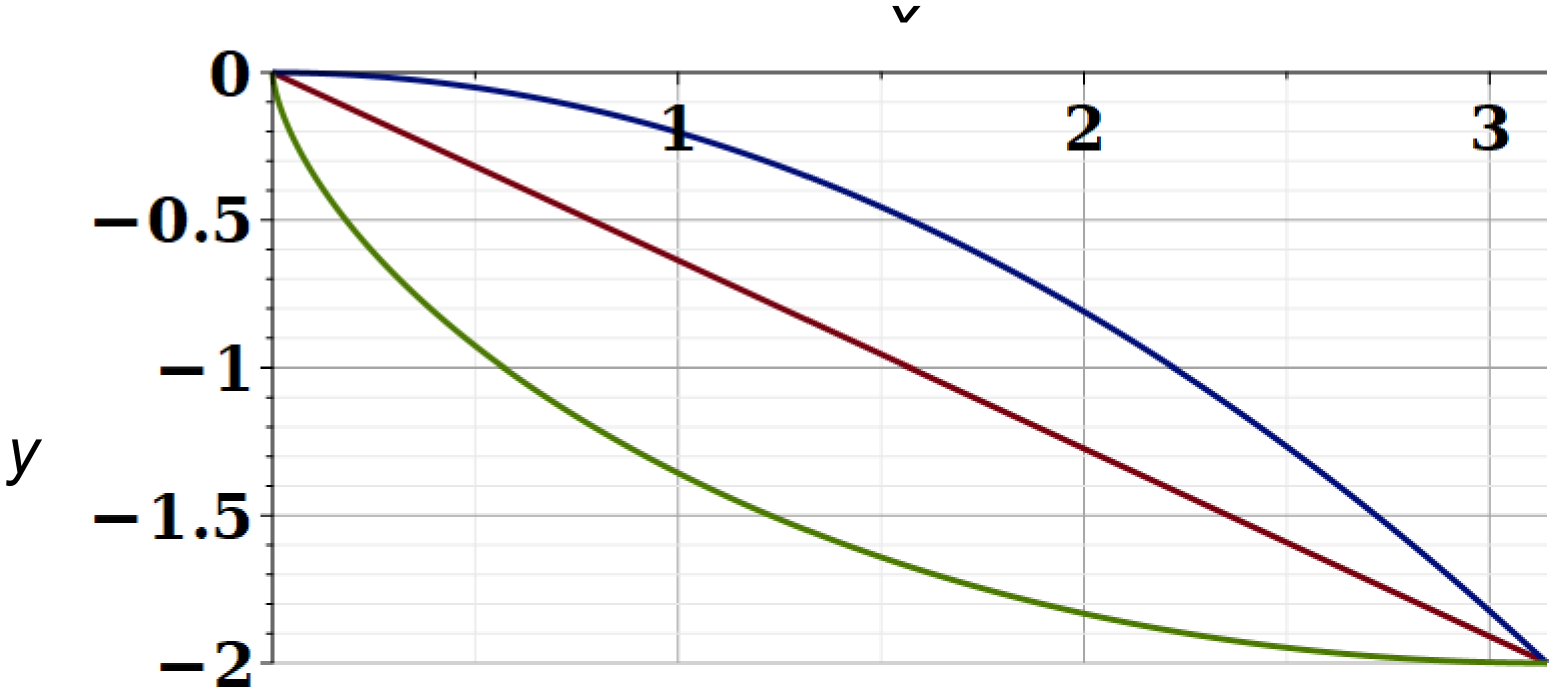
Note that the increment of travel time

is $dt = \dfrac{\text{path length}}{\text{speed}} \equiv \dfrac{ds}{v}$

Alternatively --

$v = \dfrac{ds}{dt} \qquad\qquad \Rightarrow dt = \dfrac{ds}{v}$

Vote for your favorite path



Which gives the shortest time?

a. Green
b. Red
c. Blue

$$T=\int_{x_i y_i}^{x_f y_f}\frac{ds}{v}=\int_{x_i}^{x_f}\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}dx \quad \text{because} \quad \tfrac{1}{2}mv^2=-mgy$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}}-\frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

Question – why this choice? Answer – because the answer will be more beautiful. (Be sure that was not my cleverness.)

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

Let $y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

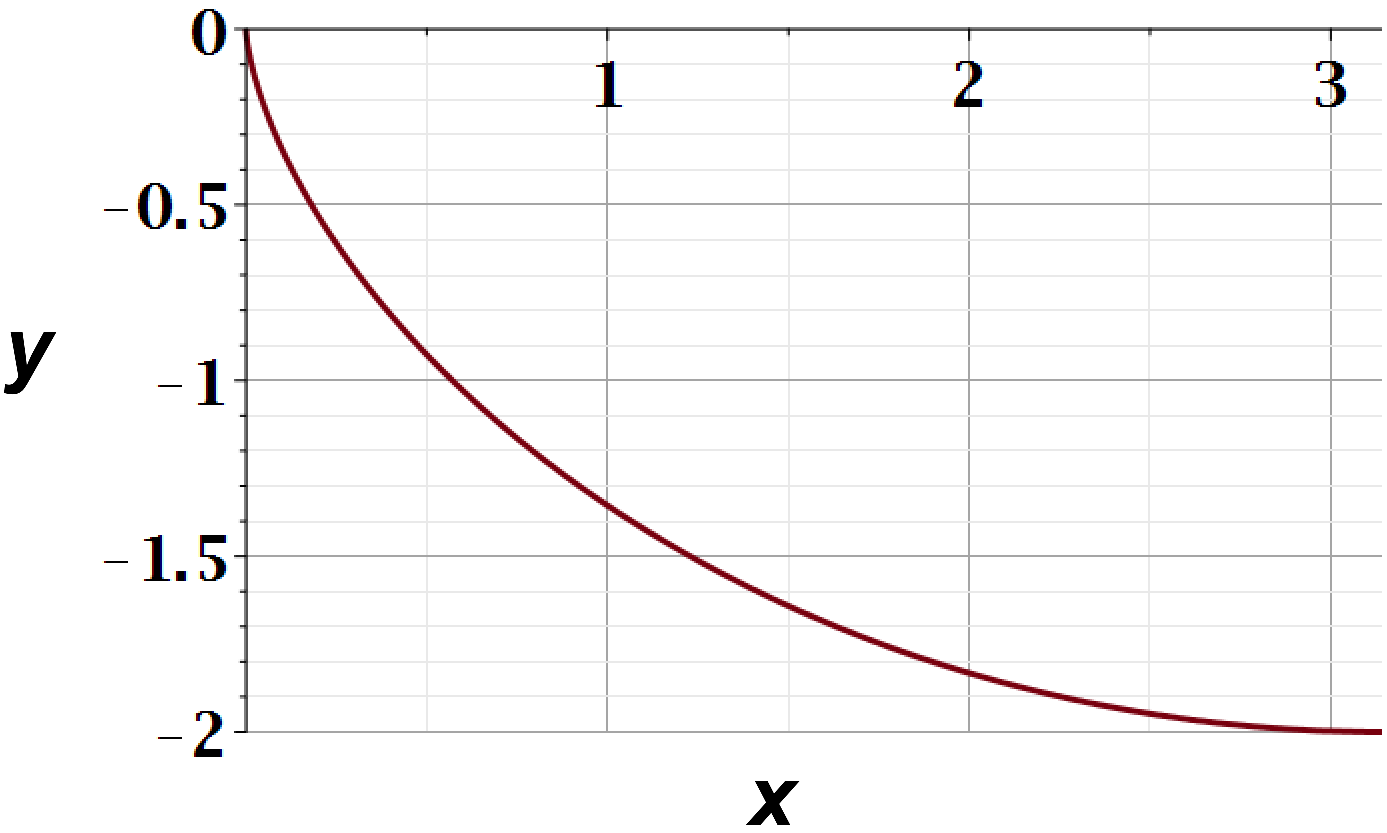
$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$

Parametric plot --

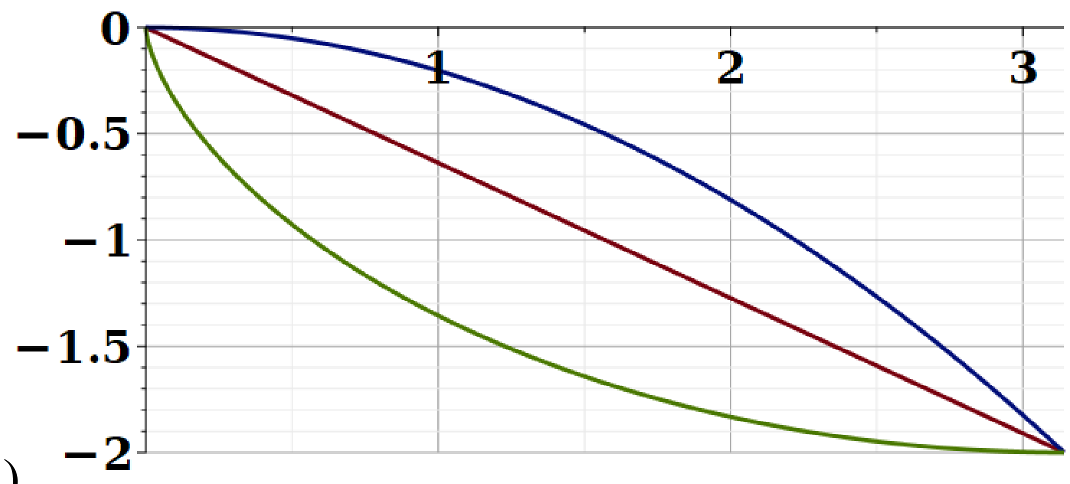
plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



Checking the results

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx$$

units of $\frac{1}{\sqrt{(2g)}}$; $(0,0) \rightarrow (\pi, -2)$



T=4.4429 $x = \theta - \sin\theta \quad y = \cos\theta - 1$

T=5.2668 $y(x) = -2x / \pi$

T=infinite $y(x) = -2x^2 / \pi^2$

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Euler-Lagrange equation:

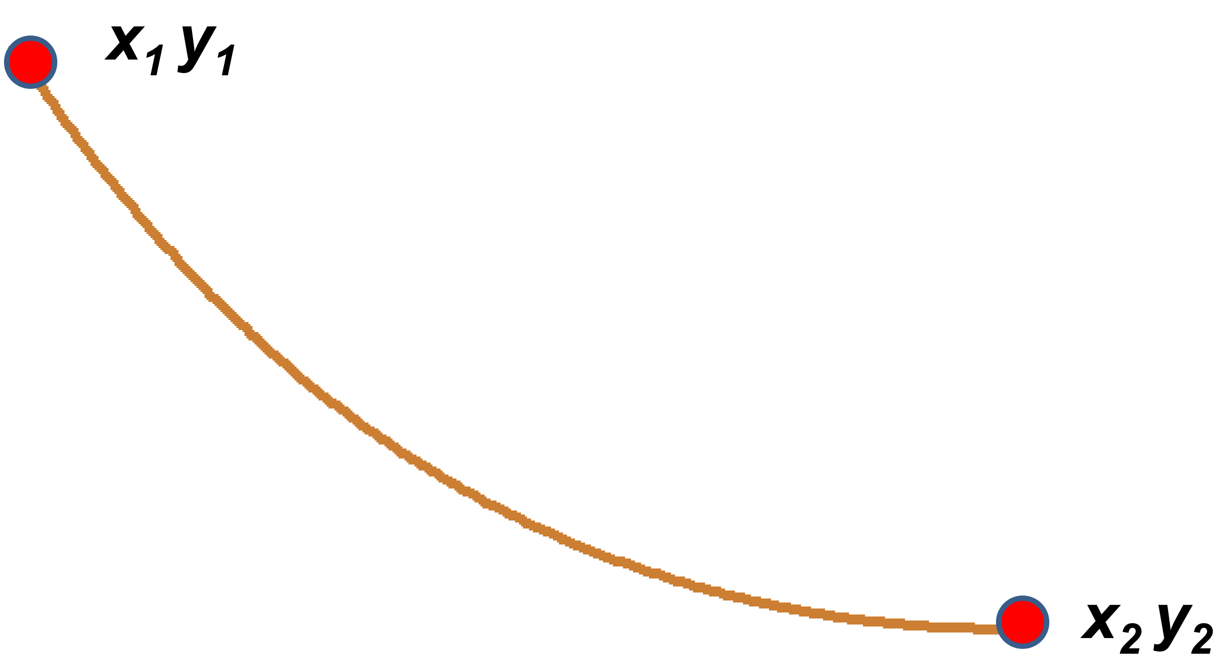
$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density ρ hanging between two points



Example from internet --

Potential energy of hanging rope:

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Length of rope:

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Define a composite function to minimize:

$$W \equiv E + \lambda L$$

Lagrange multiplier

$$\delta W = 0 = \delta E + \lambda \delta L \text{ for fixed } \lambda$$

is a very clever mathematical trick to help solve the minimization and constraint at the same time.

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow (\rho g y + \lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$(\rho g y+\lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$(\rho g y+\lambda)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$y(x)=-\frac{1}{\rho g}\left(\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants: K, a, λ

Constraints: $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = L$$

Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx \qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)\right] = 0$$

or $$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$