PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 30 -- Chap. 9 of F&W

Wave equation for sound in linear approximation

1. Wave equations for sound
2. Plane wave solutions
3. Standing wave solutions
4. Coupling of resonances to audible sound.

	Fri, 10/13/2023	Fall Break		
21	Mon, 10/16/2023	Chap. 7	The wave and other partial differential equations	#15
22	Wed, 10/18/2023	Chap. 7	Sturm-Liouville equations	#16
23	Fri, 10/20/2023	Chap. 7	Sturm-Liouville equations	#17
24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#21
28	Wed, 11/01/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#22
29	Fri, 11/03/2023	Chap. 9	Ideal gas fluids	#23
31	Mon, 11/06/2023	Chap. 9	Traveling and standing waves in the linear approximation	#24
32	Wed, 11/08/2023	Chap. 9	Non-linear and other wave properties	
33	Fri, 11/10/2023			
34	Mon, 11/13/2023			
35	Wed, 11/15/2023			
36	Fri, 11/17/2023			
37	Mon, 11/20/2023			
	Wed, 11/22/2023	Thanksgiving		

PHY 711 -- Assignment #24

Assigned: 11/06/2023 Due: 11/13/2023

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider a cylindrical pipe of length 0.5 m and radius 0.05 m, open at both ends. For air at 300 K and atmospheric pressure in this pipe, find several of the lowest frequency resonances, including at least one that has non-trivial radial dependence.

Review –

Hydrodynamic equations for isentropic air + linearization about equilibrium → wave equation for air (sound waves)

Which of the following things correctly describe the wave equation for sound in air and the wave equation for elastic media?

a. The wave velocity is different for sound in air and waves in elastic media.
b. The wave motion in elastic media can be either transverse or longitudinal.
c. The wave motion for sound in air can be either transverse or longitudinal.

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}_{applied}-\frac{\nabla p}{\rho} \qquad \Rightarrow \qquad \frac{\partial \delta\mathbf{v}}{\partial t}=-\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \qquad \Rightarrow \qquad \frac{\partial \delta\rho}{\partial t}+\rho_0\nabla\cdot\delta\mathbf{v}=0$$

In terms of the velocity potential:

$$\delta\mathbf{v}=-\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t}=-\frac{\nabla \delta p}{\rho_0} \qquad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t}+\frac{\delta p}{\rho_0}\right)=0$$

$$\frac{\partial \delta\rho}{\partial t}+\rho_0\nabla\cdot\delta\mathbf{v}=0 \qquad \Rightarrow \frac{\partial \delta\rho}{\partial t}-\rho_0\nabla^2\Phi=0$$

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho_0} \delta\rho \equiv c_0^2 \ \delta\rho$ (strictly keeping to the linear approximation)

$$\nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \qquad \Rightarrow -\frac{\partial\Phi}{\partial t} + c_0^2\frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2\Phi}{\partial t^2} + \frac{c_0^2}{\rho_0}\frac{\partial\delta\rho}{\partial t} = 0$$ (assuming we can adjust Φ accordingly)

$$\frac{\partial\delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0 \qquad \Rightarrow \frac{\partial^2\Phi}{\partial t^2} - c_0^2\nabla^2\Phi = 0$$

$$\Rightarrow \delta\rho = \frac{\rho_0}{c_0^2}\frac{\partial\Phi}{\partial t} \qquad \delta p = \rho_0\frac{\partial\Phi}{\partial t}$$

Wave equation for air:

$$\frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0$$

Here, $c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho_0}$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have:

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_0^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c_0^2 \nabla^2 \delta p = 0$$

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution:

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

$$\delta\mathbf{v} = -\nabla\Phi = -i\mathbf{k}Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$

$\Leftarrow$ Note this is a pure longitudinal wave

$$\delta\rho = \frac{\rho_0}{c_0^2}\frac{\partial\Phi}{\partial t} = -i\omega\frac{\rho_0}{c_0^2}Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\delta p = \rho_0\frac{\partial\Phi}{\partial t} = -i\omega\rho_0 Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

Boundary values of wave equation

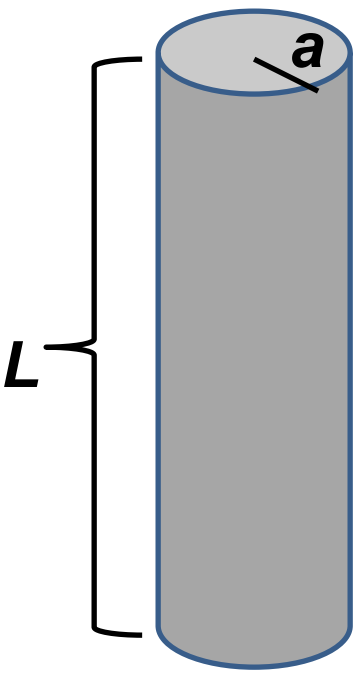
Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta\mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla\Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla\Phi = 0$

At free surface: $\frac{\partial \Phi}{\partial t} = 0$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \text{Define:} \quad k \equiv \frac{\omega}{c}$$

In cylindrical coordinates:

$$\Phi(r,\phi,z,t) = R(r)F(\phi)Z(z)e^{-i\omega t} \equiv R(r)F(\phi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\phi,z,t) = 0$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi(r,\varphi,z,t)=0$$

$$\Phi(r,\varphi,z,t)=R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi)=e^{im\varphi};\ \ F(\varphi)=F(\varphi+2\pi N)\Rightarrow m=\text{integer}$$

$$Z(z)=e^{i\alpha z};\ \ \alpha=\text{real}\ \left(+\text{ other restrictions}\right)$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

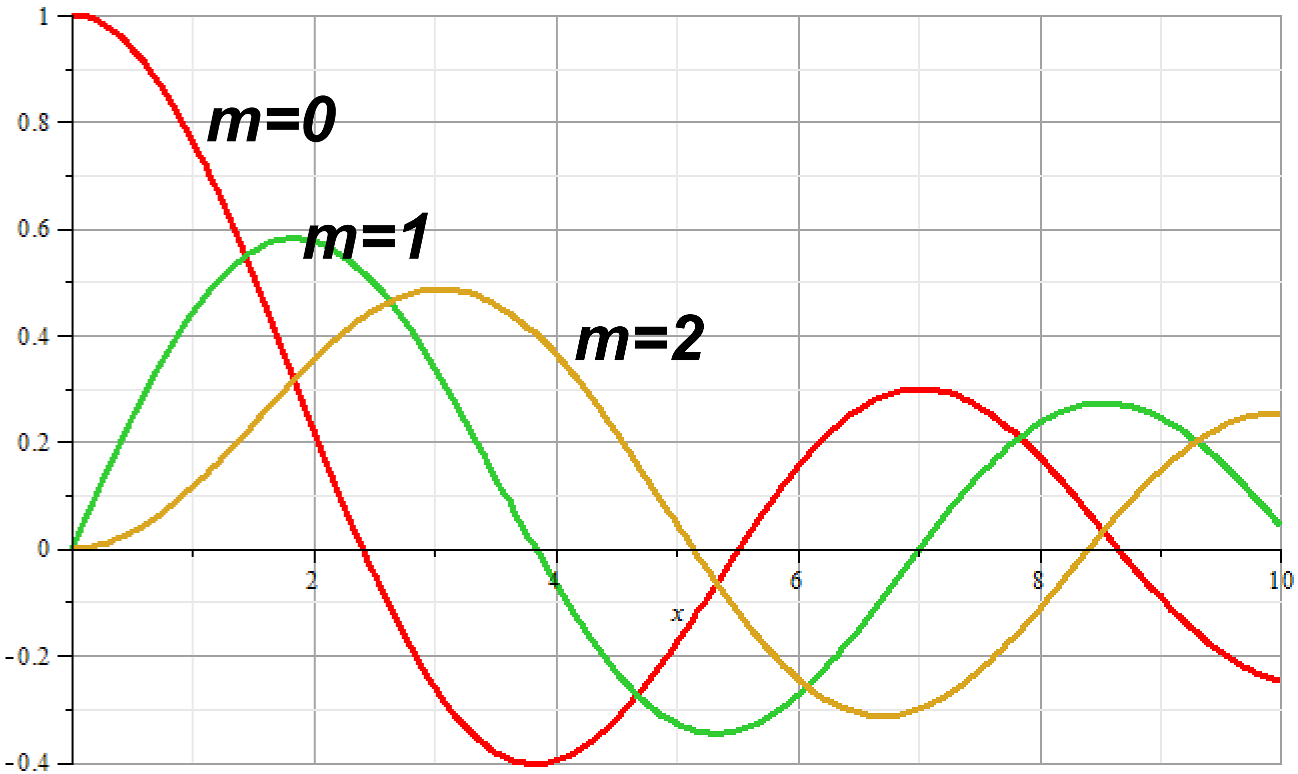
For $k^2 \geq \alpha^2$ define $\kappa^2 \equiv k^2-\alpha^2$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+\kappa^2\right)R(r)=0$$

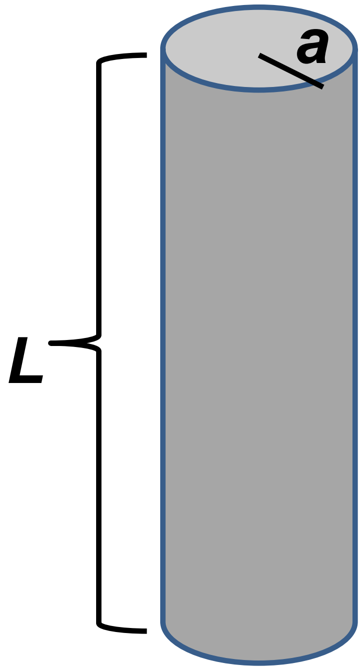
Cylinder surface boundary conditions: $\left.\frac{dR}{dr}\right|_{r=a}=0$

$\Rightarrow R(r)=J_m(\kappa r)$ where for $\frac{dJ_m(x'_{mn})}{dx}=0,\quad \kappa_{mn}=\frac{x'_{mn}}{a}$

Bessel functions : $J_m(x)$



Now recall the boundary conditions



Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

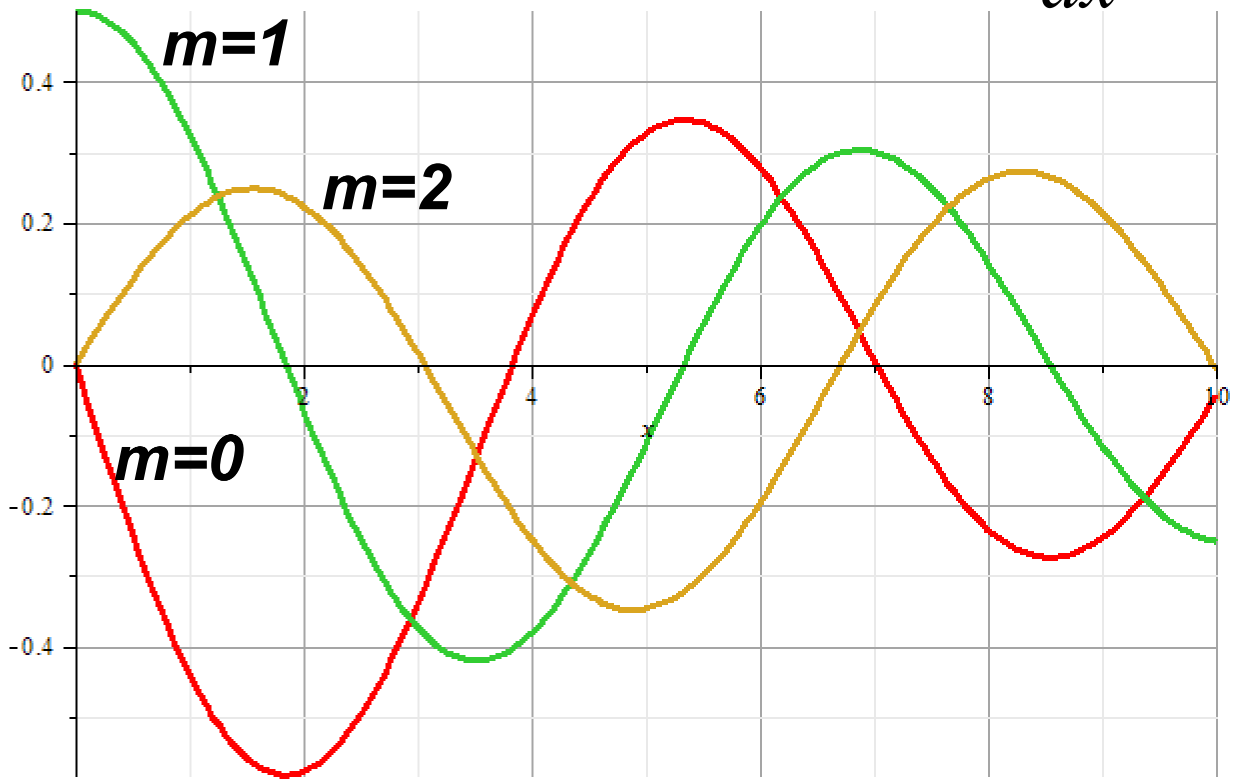
At free surface: $\dfrac{\partial \Phi}{\partial t} = 0$

$$\Phi(r, \varphi, z, t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$= J_m(\kappa r)e^{im\varphi}e^{i\alpha z}e^{-i\omega t}$$

$$\text{For} \quad r = a, \quad \left. \frac{\partial \Phi(r, \varphi, z, t)}{\partial r} \right|_{r=a} = 0$$

Bessel function derviatives : $\frac{dJ_m(x)}{dx}$



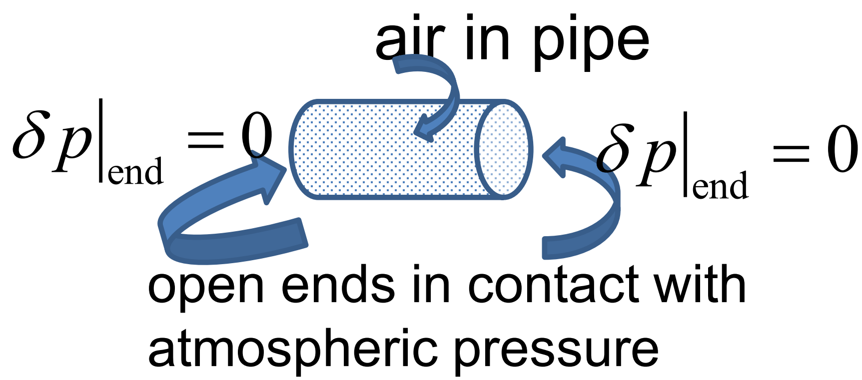
Note error on pg. 552 of F&W (thanks to David for noticing)

Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 0.00000, 3.05424, 6.70613

Some details on the open pipe boundary conditions --

Comment --

1. Open pipe boundary condition



Boundary condition for z=0, z=L:

For open - open pipe :

$$Z(0) = Z(L) = 0 \qquad \Rightarrow \; Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \qquad p = 1,2,3\ldots$$

Resonant frequencies:

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2 \equiv k_{mnp}^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

Resonant frequencies: $\omega = ck_{mnp}$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

Comment about units of frequency --

$$\Phi(\mathbf{r},t) = f(\mathbf{r})e^{-i\omega t} = f(\mathbf{r})e^{-2\pi i\nu t}$$

Note that ω has units of radians/sec

ν has units of cycles/sec (Hz)

$$\nu = \frac{\omega}{2\pi}$$

Example of open pipe of length L and radius a:

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2 \left(1 + \left(\frac{L}{a}\right)^2 \left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$

$$x'_{mn} = 0.00, 1.84, 3.05.... \quad \text{for } x'_{00}, x'_{10}, x'_{20}$$

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$= J_m\left(\frac{x'_{mn}}{a} r\right) e^{im\varphi} \sin\left(\frac{p\pi}{L} z\right) e^{-i\omega t}$$

Alternate boundary condition for z=0, z=L:

For open-closed pipe:

$$\frac{dZ(0)}{dz} = Z(L) = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \qquad p = 0,1,2,3\ldots$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

The above analysis pertains to resonant air waves within a cylindrical pipe. As previously mentioned, you can hear these resonances if you put your ear close to such a pipe. The same phenomenon is the basis of several musical instruments such as organ pipes, recorders, flutes, clarinets, oboes, etc.

Question – what about a trumpet, trombone, French horn, etc?

a. Same idea?
b. Totally different?

But for musical instruments, you do not want to put your ear next to the device – additional considerations must apply. Basically, you want to couple these standing waves to produce traveling waves.

Modifications needed for the pandemic --

Image from the Winston-Salem Journal 11/1/2020

For other instruments, the resonance is initiated by another resonant device which couples to air --

In order to understand how audible sound couples to sound wave resonances, consider the following simple model of a sound amplifier --

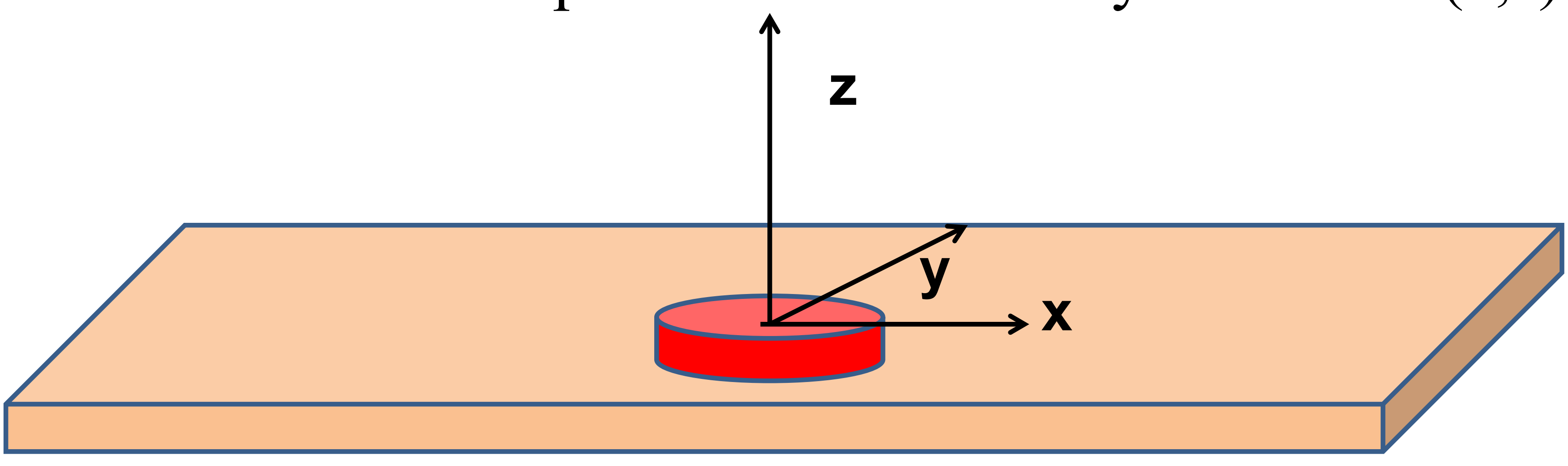
Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$

can be represented as boundary value of $\Phi(\mathbf{r},t)$



Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function:

$$\Phi(\mathbf{r},t) = \Phi_0(\mathbf{r},t) + \int d^3r' \int dt' G(\mathbf{r}-\mathbf{r}',t-t') f(\mathbf{r}',t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi_0(\mathbf{r},t) = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}',t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega$$

Green's theorem

Consider two functions *h(***r***)* and *g(***r***)*

Note that: $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}}\, d^2r$

$$\nabla^2\tilde{\Phi} + k^2\tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot \hat{\mathbf{n}}\, d^2r$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}d^2r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}',\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega)\right) d^3r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

If the integration volume V includes the point $\mathbf{r} = \mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

➔extra contributions from boundary

Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega)=\int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r'+$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0}=\begin{cases}0 & \text{for } x^2+y^2>a^2\\ i\omega\varepsilon a & \text{for } x^2+y^2<a^2\end{cases}$$

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}+\frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}'=-z'; \quad z>0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$

Some more details --

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}+\frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}'=-z'; \quad z>0$$

Note that $|\mathbf{r}-\mathbf{r}'|\equiv\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

$$|\mathbf{r}-\bar{\mathbf{r}}'|\equiv\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}$$

Fourier transform of velocity potential:

$$\tilde{\Phi}(\mathbf{r},\omega)=\int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r}',\omega)d^3r'+$$

$$\oint_S\left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}'d^2r'$$

Need this term to vanish at z'=0

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$= -i\omega\varepsilon a\int_0^a r'\,dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain: $x' = r'\cos\varphi'$

$$y' = r'\sin\varphi'$$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

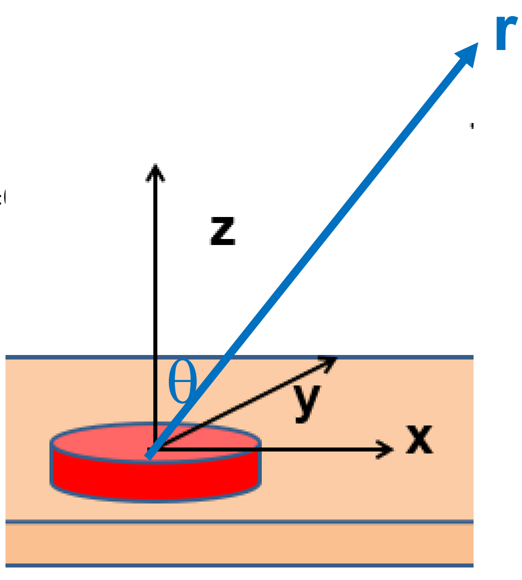
$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

More details

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z'}dx'dy'$$

$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\varphi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$



Integration domain: $x' = r'\cos\varphi'$

$y' = r'\sin\varphi'$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'dr'\int_0^{2\pi} d\phi' e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\frac{1}{2\pi}\int_0^{2\pi} d\phi' e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'dr' J_0(kr'\sin\theta)$$

$$\int_0^w udu J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux : $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\langle \mathbf{j}_e \rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

$$= \tfrac{1}{2}\rho_0 \Re\left(\left(-\nabla\Phi\right)\left(-i\omega\Phi\right)^*\right)$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin\theta)}{ka \sin\theta} \right|^2$$