PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 31: Chap. 9 of F&W

Linear and non-linear sound waves

1. Comment on HW #21
2. Summary of linear sound phenomena
3. Introduction to non-linear effects
4. Analysis of instability – shock phenomena

Physics Colloquium

Thursday

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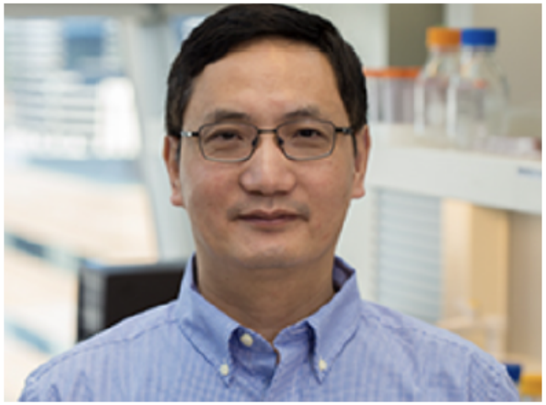
November 9th, 2023

CryoEM - what it is, and how we are using it to reveal the chemical and physical basis of biological macromolecules that duplicate the genomes?

The great physicist Richard Feynman once observed during a lecture "It is very easy to answer many of these fundamental biological questions; you just look at the thing! … Unfortunately the present microscope sees at a scale which is just a bit too crude. Make the microscope one hundred times more powerful and many problems of biology would be made very much easier…. the biologists would surely be very thankful to you - and they would prefer that to the criticism that they should use more mathematics". The modern-day cryogenic electron microscope (cryoEM) is exactly the microscope Feynman had dreamed of over six decades ago. Cryo-EM not only won its developers a Nobel prize in 2017, but has also revolutionized molecular biology around the world. In my talk I will introduce the cryoEM technology, and describe how my lab has been using cryo-EM to reveal the DNA replication mechanism.



Huilin Li, PhD
Professor
Department of Structural Biology
Van Andel Institute

4 pm - Olin 101

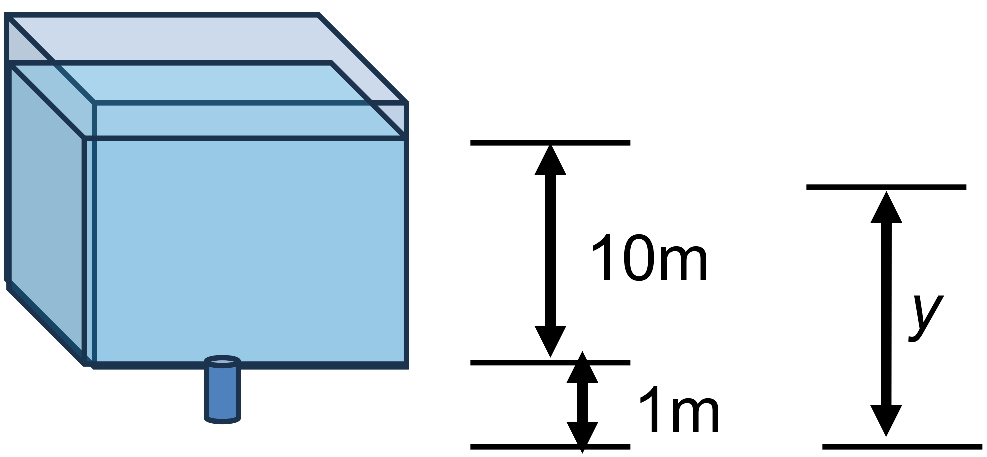
Refreshments will be served in Olin Lobby beginning at 3:30pm.

PHY 711 -- Assignment #21

Assigned: 10/30/2023 Due: 11/06/2023

Read Chapter 9 in **Fetter & Walecka**.

1. A tank having an area of 100 m^2 is open to the atmosphere and contains 1000 m^3 of water. It has a spigot at it bottom which has a height of 1 m above a drain in the floor. A hose having a diameter of 1 cm is used to empty the water from the tank when the spigot is opened. Using Bernoulli's analysis, estimate the time it takes to empty the tank via the floor drain.



Note that the speed of the water changes as the water flows out.

21	Mon, 10/16/2023	Chap. 7	The wave and other partial differential equations	#15
22	Wed, 10/18/2023	Chap. 7	Sturm-Liouville equations	#16
23	Fri, 10/20/2023	Chap. 7	Sturm-Liouville equations	#17
24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#21
28	Wed, 11/01/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#22
29	Fri, 11/03/2023	Chap. 9	Ideal gas fluids	#23
31	Mon, 11/06/2023	Chap. 9	Traveling and standing waves in the linear approximation	#24
32	Wed, 11/08/2023	Chap. 9	Non-linear and other wave properties	
33	Fri, 11/10/2023			
34	Mon, 11/13/2023			
35	Wed, 11/15/2023			
36	Fri, 11/17/2023			
37	Mon, 11/20/2023			
	Wed, 11/22/2023	Thanksgiving		
	Fri, 11/24/2023	Thanksgiving		
38	Mon, 11/27/2023			

No explicit homework, but please send me your presentation topic by Mon. 10/13/2023

Driven sound wave in linear approximation

In order to understand how audible sound couples to sound wave resonances, consider the following simple model of a sound amplifier --

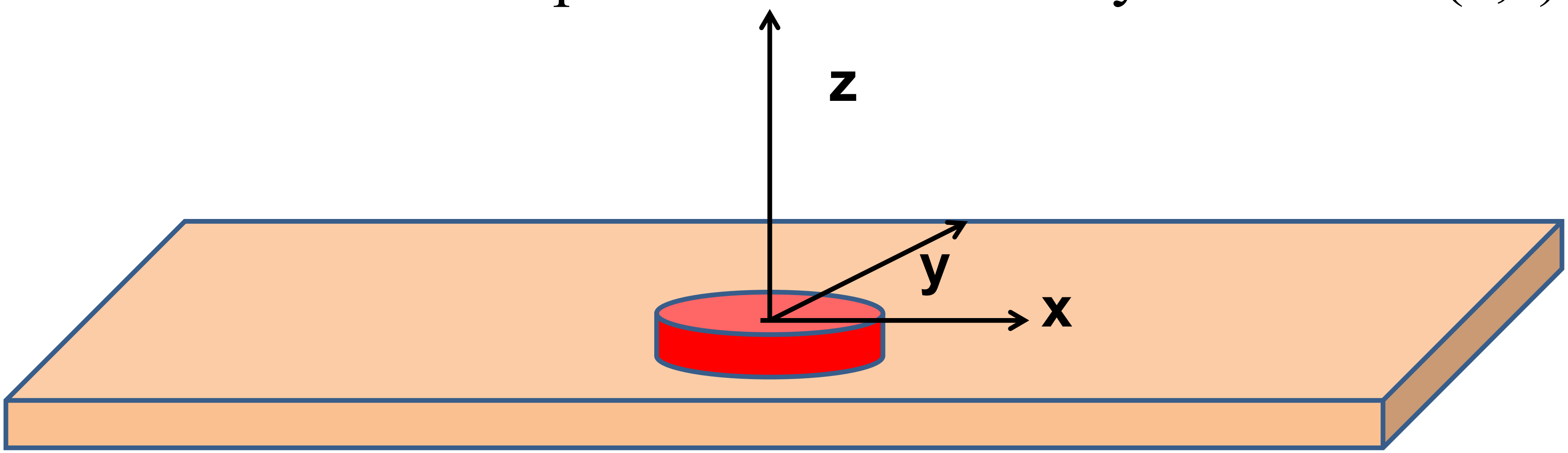
Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$

can be represented as boundary value of $\Phi(\mathbf{r},t)$



Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function:

$$\Phi(\mathbf{r},t) = \Phi_0(\mathbf{r},t) + \int d^3r' \int dt' \, G(\mathbf{r}-\mathbf{r}', t-t') f(\mathbf{r}',t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi_0(\mathbf{r},t) = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

In our case, we will be interested in a time harmonic force --

$$f(\mathbf{r},t)=\tilde{f}(\mathbf{r},\omega)e^{-i\omega t}$$

Similarly, $\Phi(\mathbf{r},t)=\tilde{\Phi}(\mathbf{r},\omega)e^{-i\omega t}$

$$G(\mathbf{r}-\mathbf{r}',t-t')=\tilde{G}(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t')}$$

$$\nabla^2\tilde{\Phi}+k^2\tilde{\Phi}=-\tilde{f}(\mathbf{r},\omega) \qquad \text{for } k=\frac{\omega}{c}$$

$$\left(\nabla^2+k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega)=-\delta(\mathbf{r}-\mathbf{r}')$$

In general, the Green's function can be designed to solve the differential equations with the correct boundary values.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that: $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}}\, d^2r$

$$\nabla^2\tilde{\Phi} + k^2\tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\tilde{f}(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot \hat{\mathbf{n}}\, d^2r$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}\,d^2r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}',\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega)\right) d^3r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\,d^2r'$$

If the integration volume V includes the point $\mathbf{r}=\mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega)\, d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\,d^2r'$$

➔extra contributions from boundary

Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

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$$\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$

Some more details --

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}+\frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}'=-z'; \quad z>0$$

Note that $|\mathbf{r}-\mathbf{r}'|\equiv\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

$$|\mathbf{r}-\bar{\mathbf{r}}'|\equiv\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}$$

Fourier transform of velocity potential:

$$\tilde{\Phi}(\mathbf{r},\omega)=\int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\,f(\mathbf{r}',\omega)\,d^3r'+$$

$$\oint_S\left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}'d^2r'$$

Need this term to vanish at z'=0

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$= -i\omega\varepsilon a\int_0^a r'\,dr'\int_0^{2\pi} d\phi'\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\Bigg|_{z'=0}$$

Integration domain: $x' = r'\cos\varphi'$

$$y' = r'\sin\varphi'$$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

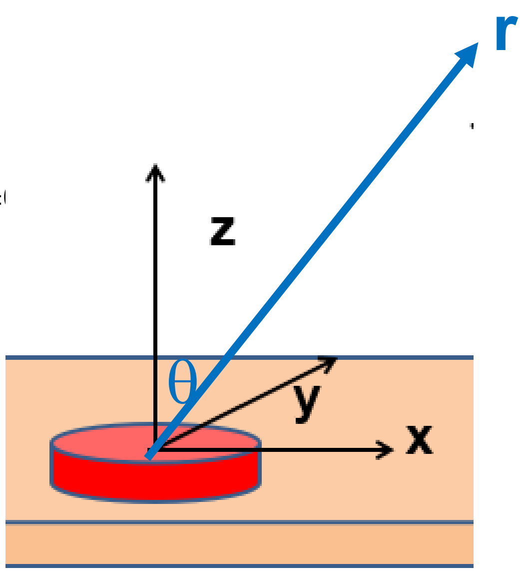
$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

More details

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:z'=0} \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z'}dx'dy'$$

$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\varphi' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\Bigg|_{z'=0}$$



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$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'dr'\int_0^{2\pi} d\phi' e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\frac{1}{2\pi}\int_0^{2\pi} d\phi' e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'dr' J_0(kr'\sin\theta)$$

$$\int_0^w u du J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3 \frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux : $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\left\langle \mathbf{j}_e \right\rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

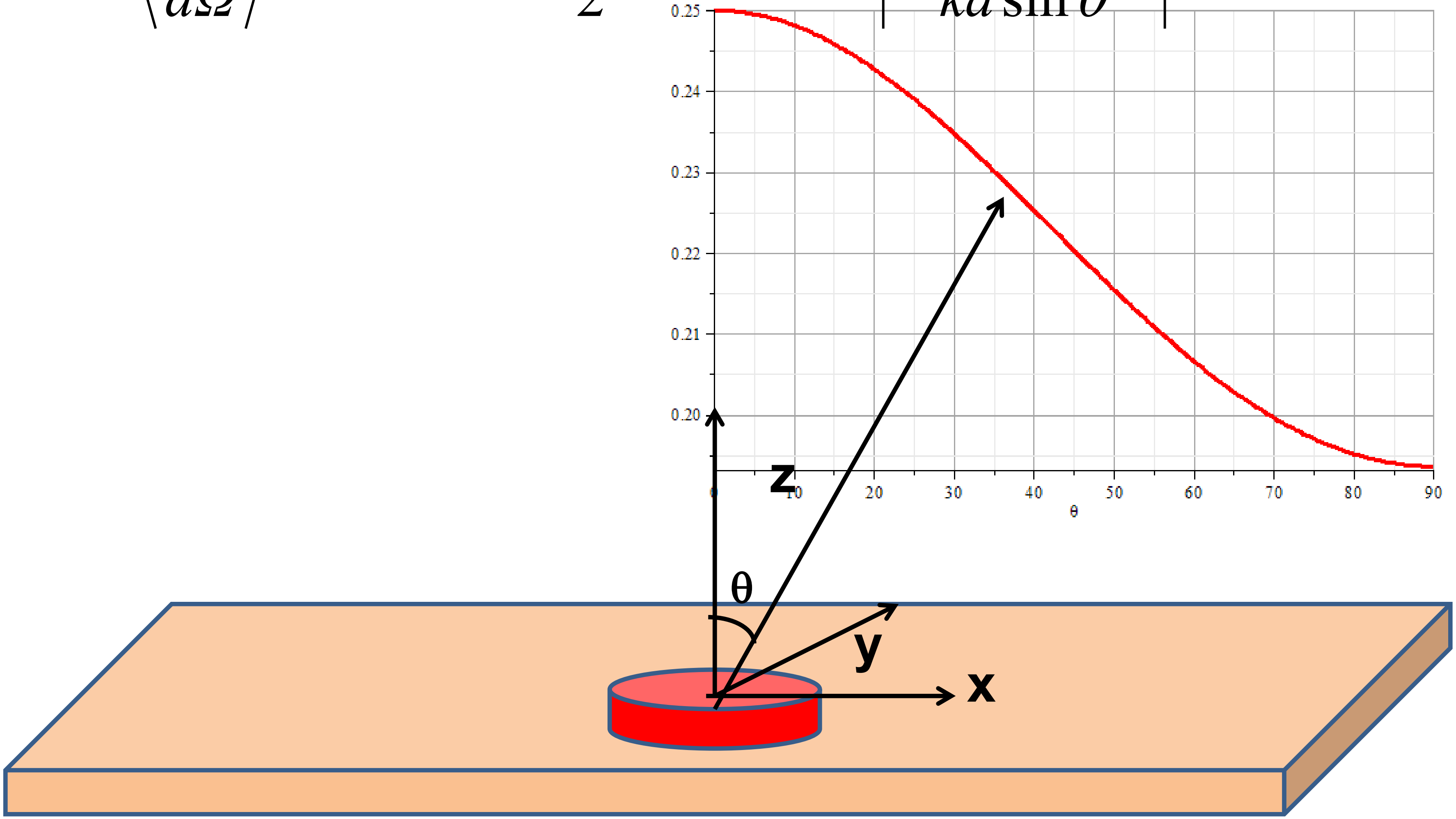
$$= \tfrac{1}{2}\rho_0\Re\left((-\nabla\Phi)(-i\omega\Phi)^*\right)$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \left\langle \mathbf{j}_e \right\rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

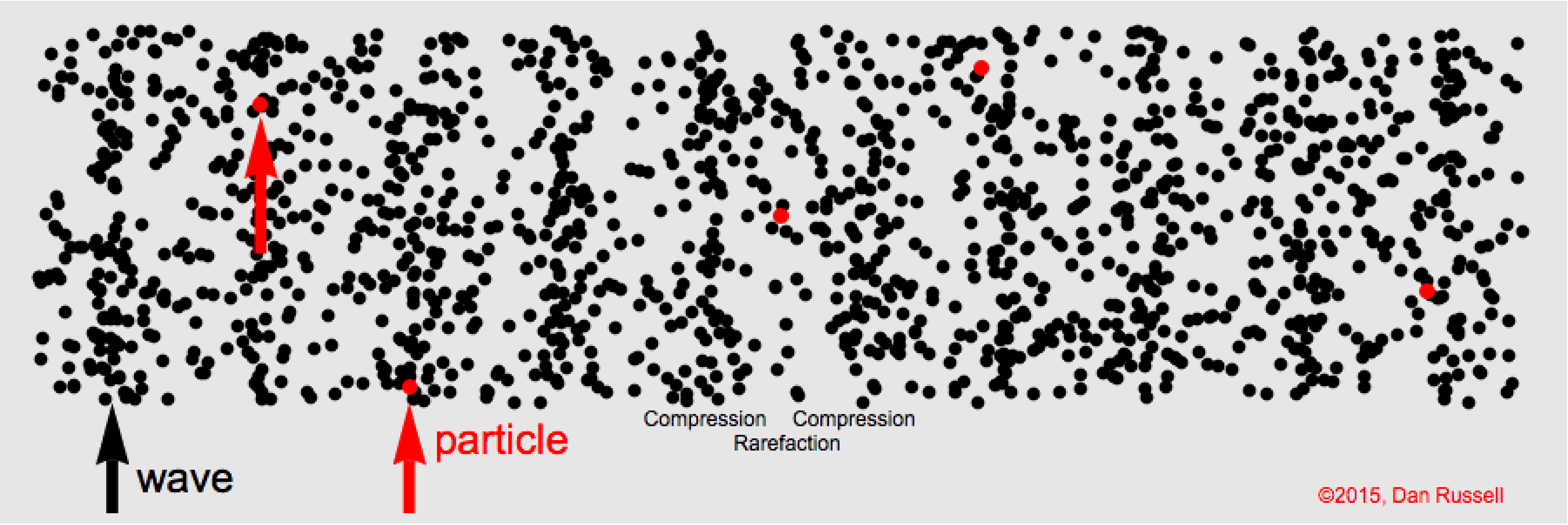
Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$



Visualization of longitudinal wave motion

From the website:
https://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html



Continuing to discuss linear sound waves –
Scattering from a rigid cylinder

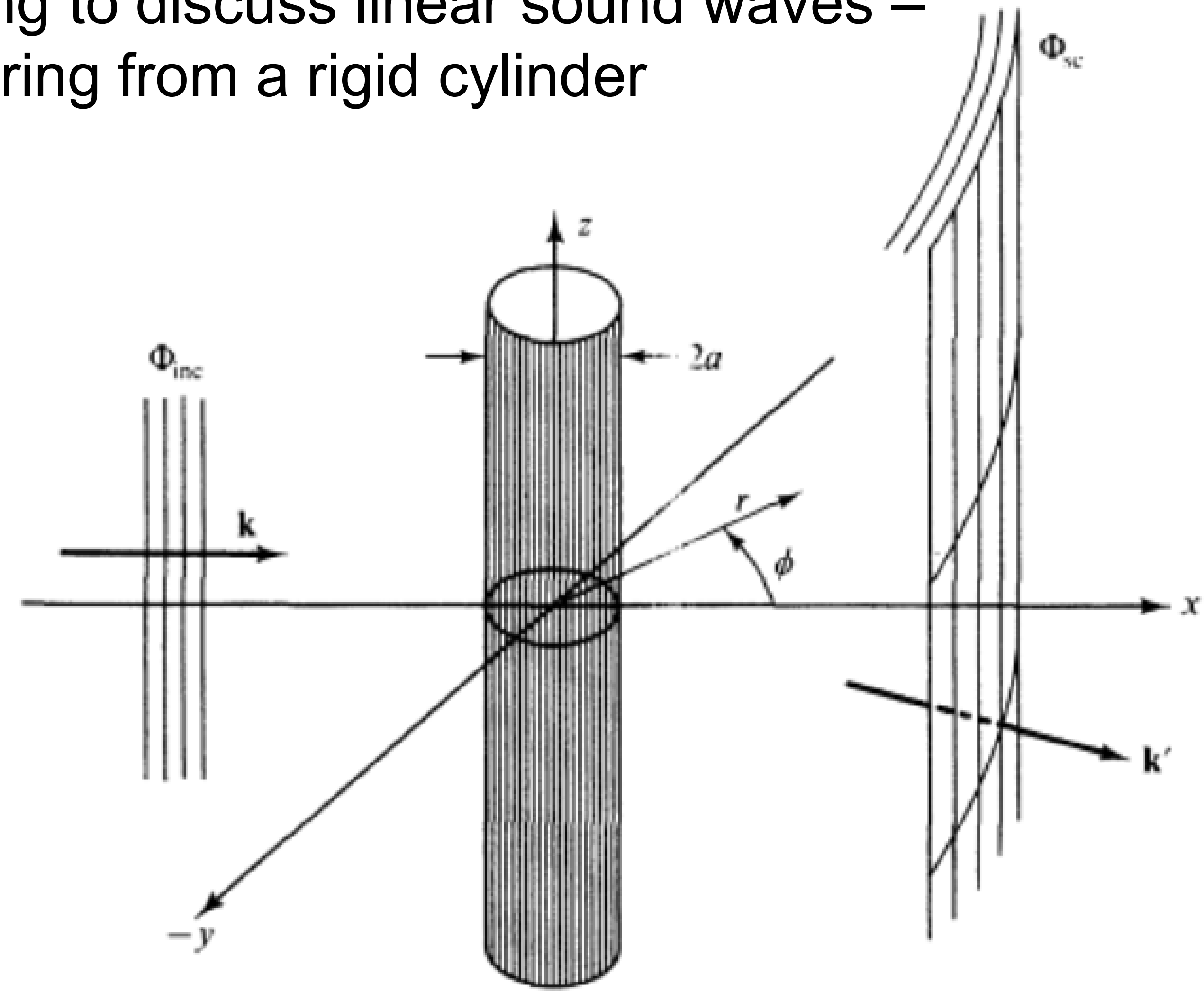


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Example of cylindrical scattering objects --

Suppose a trumpeter is playing near the columns. Maximal scattering occurs when

a. Facing toward the column

b. Facing away from the column.

Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right) R_m(r) = 0$

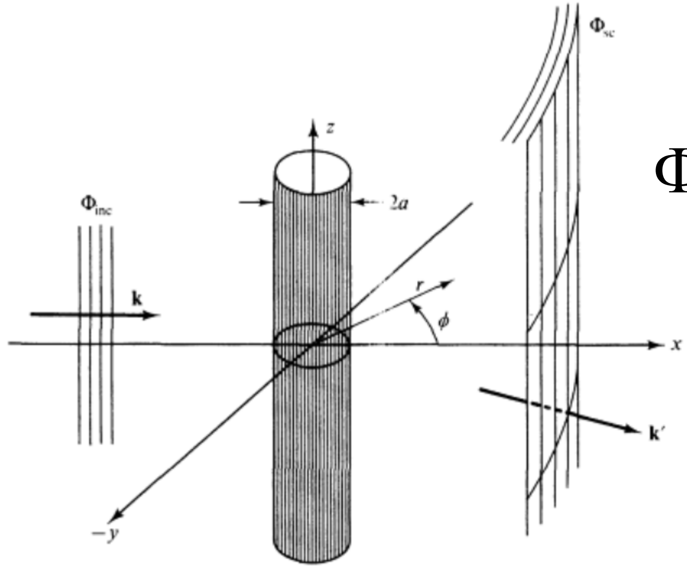


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr) \quad \text{where Hankel function}$$

represents an outgoing wave: $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

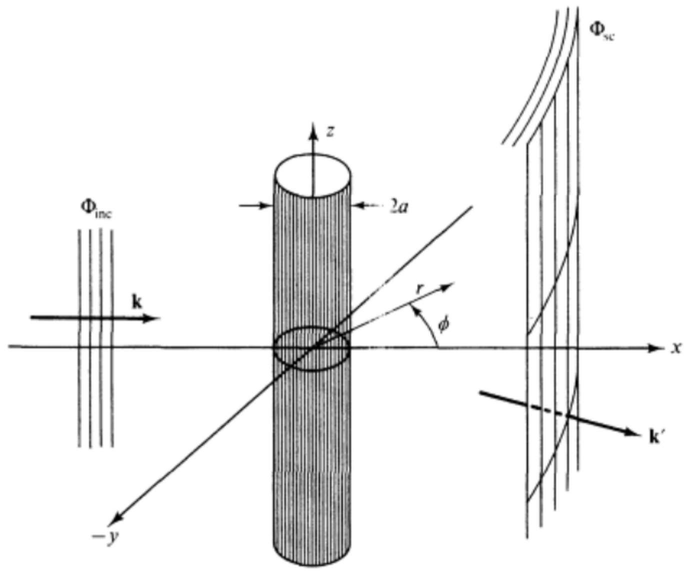


Figure 51.8 Scattering from a rigid cylinder.

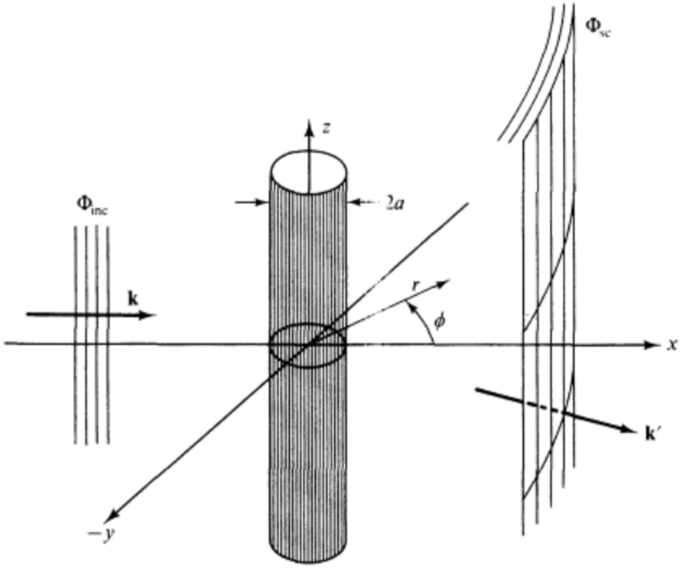
$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Rightarrow f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$



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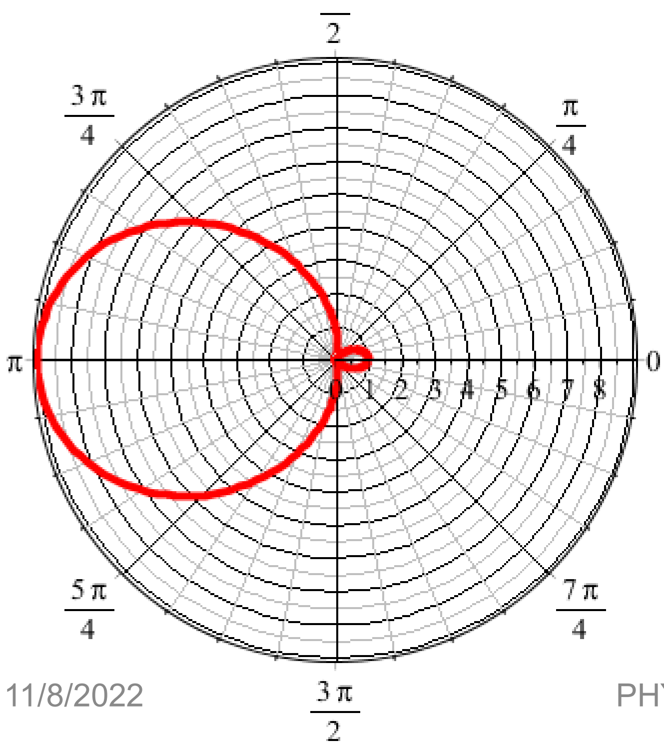


$$\frac{d\sigma}{d\phi} = \left| f(\phi) \right|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For *ka* << *1*

$$\frac{d\sigma}{d\phi} = \left| f(\phi) \right|^2 \approx \frac{1}{8} \pi k^3 a^4 \left(1 - 2\cos\phi\right)^2$$

Revisiting the trumpeter question --

Conclusion – be careful when choosing a place to play your trumpet --

Now consider some non-linear effects in sound

Examples?

We will consider the simple case –

1. One dimension for motion
2. Fluid is assumed to be an ideal gas
3. Adiabatic conditions
4. All variables will be expressed in terms of the density $\rho(x,t)$

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume spatial variation confined to x direction ;

assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0.$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ: $\quad p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where} \quad \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas: $\quad \frac{dp}{p} = \gamma\frac{d\rho}{\rho} \qquad p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where} \quad c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$$

Digression – What is gamma?

Internal energy for ideal gas: $pV = Nk_BT$

$E_{\text{int}} = \frac{f}{2} Nk_BT$ $\quad f \equiv$ degrees of freedom; 3 for atom, 5 for diatomic molecule

In terms of specific heat ratio: $\gamma \equiv \frac{C_p}{C_V}$

$$dE_{\text{int}} = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2} Nk_B$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2} Nk_B + Nk_B$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} = 1 + \frac{2}{f} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1} \qquad E_{\text{int}} = \frac{1}{\gamma - 1} Nk_BT$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation of v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

Some more algebra :

From Euler equation : $$\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

From continuity equation : $$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$$

Combined equation : $$\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\Rightarrow\left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow\frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process: $c^2 = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\Rightarrow \; c = c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm\frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm\frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Summary :

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm \frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Traveling wave solution:

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assume: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Solution in linear approxiation:

$$u = v + c \approx v_0 + c_0 = c_0\left(\frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1}\right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f\left(x - c_0 t\right)$$

Some details

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

Note that for $u = v + c$ (choice of + solution)

$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$ is satisfied by a function of the form

$\rho(x,t) = \rho_0 + f(x - u(\rho(x,t))t)$

Let $w \equiv x - u(\rho(x,t))t$

$$\frac{df}{dw}\frac{\partial w}{\partial t} + u\frac{df}{dw}\frac{\partial w}{\partial x} = \frac{df}{dw}(-u + u) = 0$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_{w})$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

$$u = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(1 + s(x-ut) \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w,t)$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations:

plot $s(w)$ vs $x(w,t)$ for range of w at each t

Summary

$$\frac{\partial \rho}{\partial t} + u(\rho)\frac{\partial \rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0\left(1 + s(x - u(\rho)t)\right)$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x - ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$

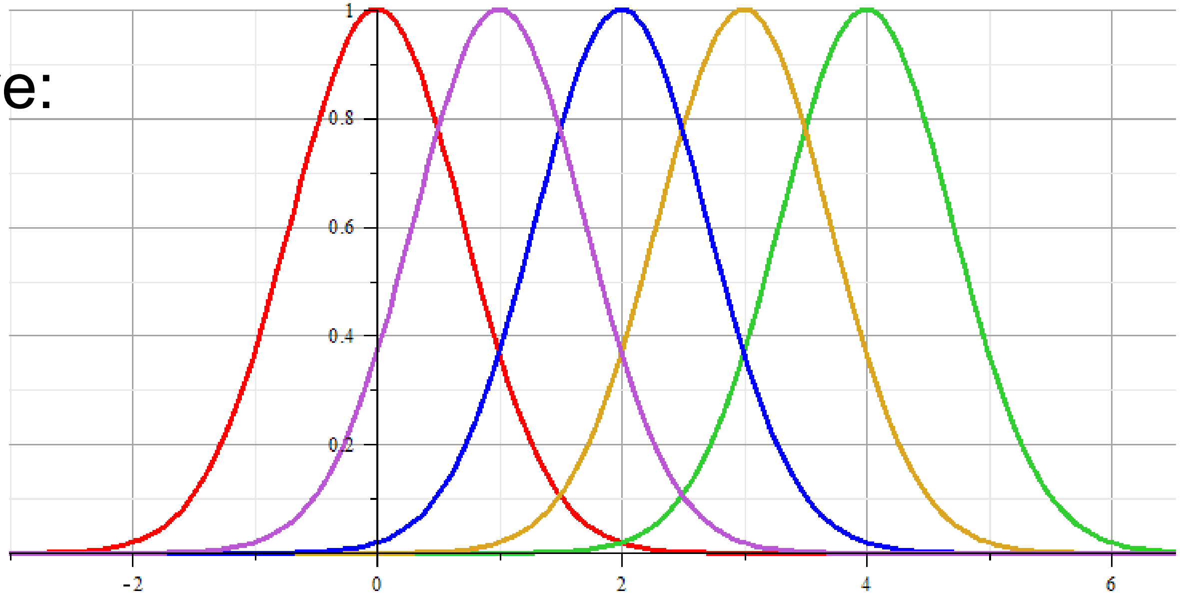
Plot $s(x - ut)$ for fixed t, as a function of x:

Let $w = x - ut \quad \Rightarrow \quad x = w + ut = w + u(w)t \equiv x(w,t)$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

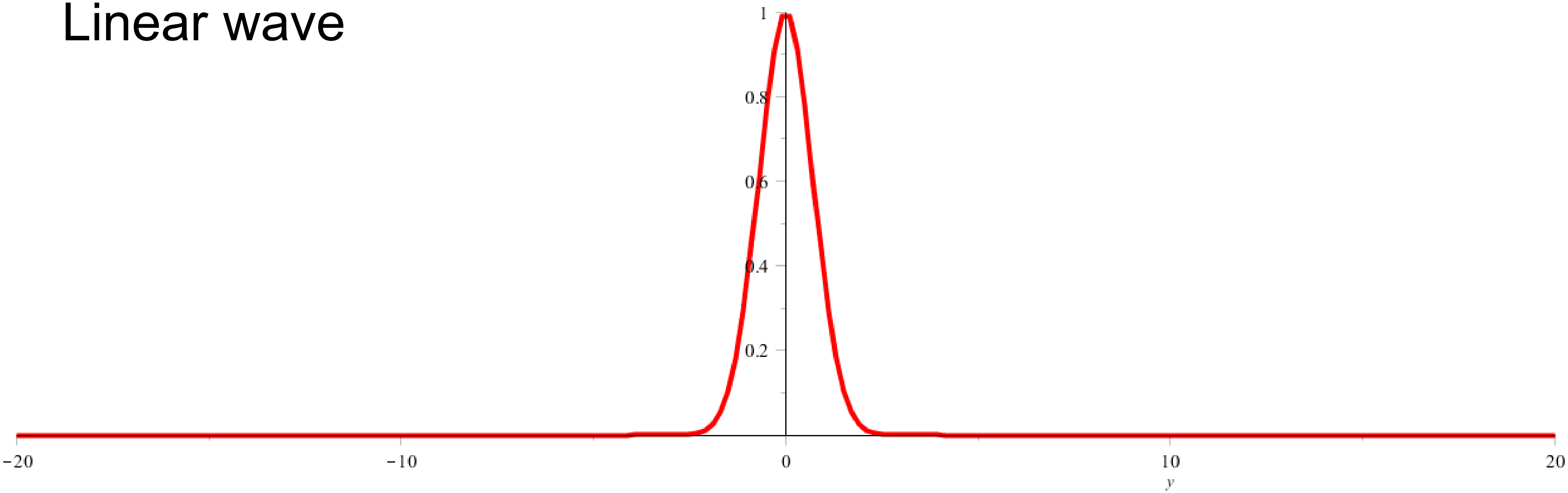
Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

Linear wave:

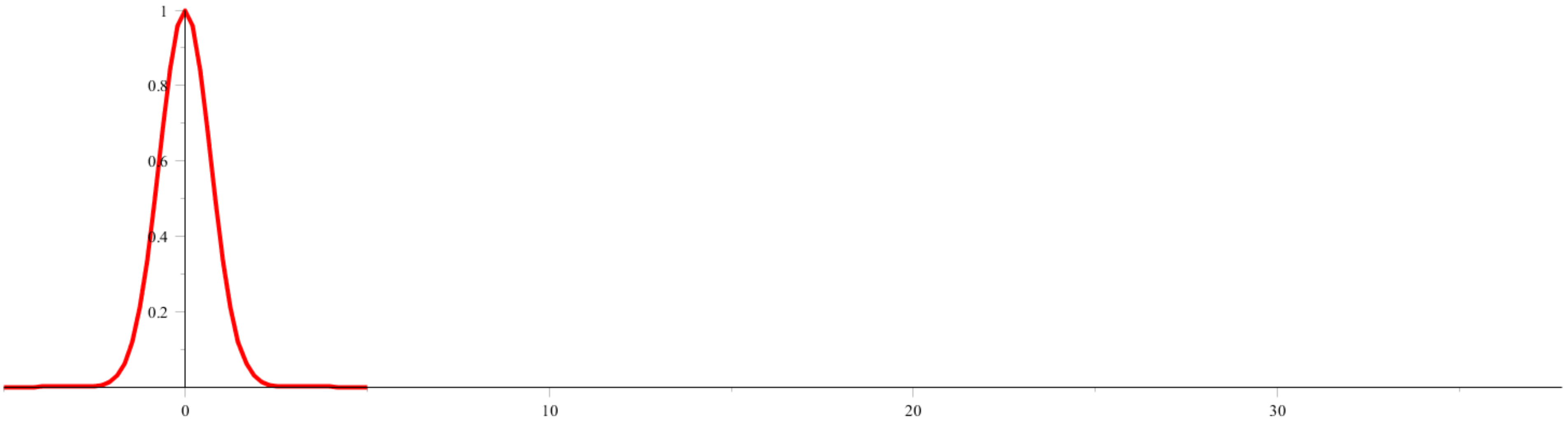


Non-linear wave:

Linear wave

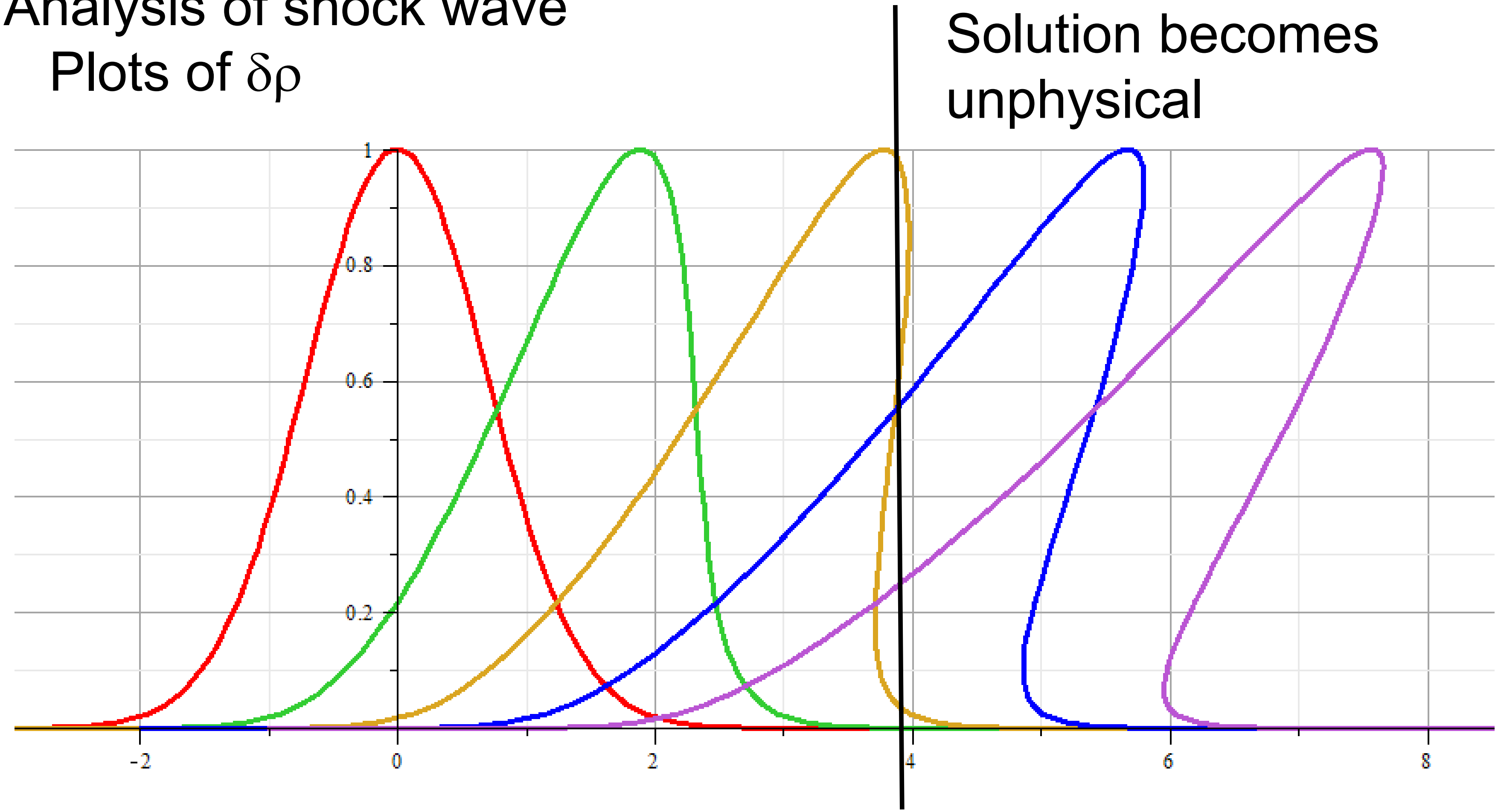


Non-linear wave



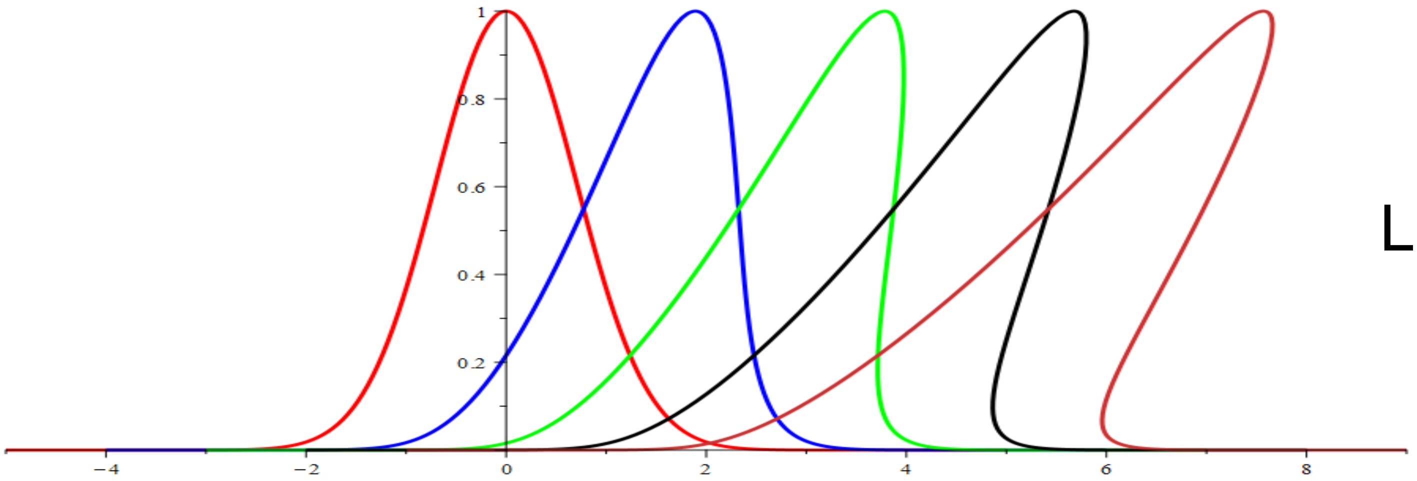
Analysis of shock wave
Plots of $\delta\rho$

Solution becomes unphysical

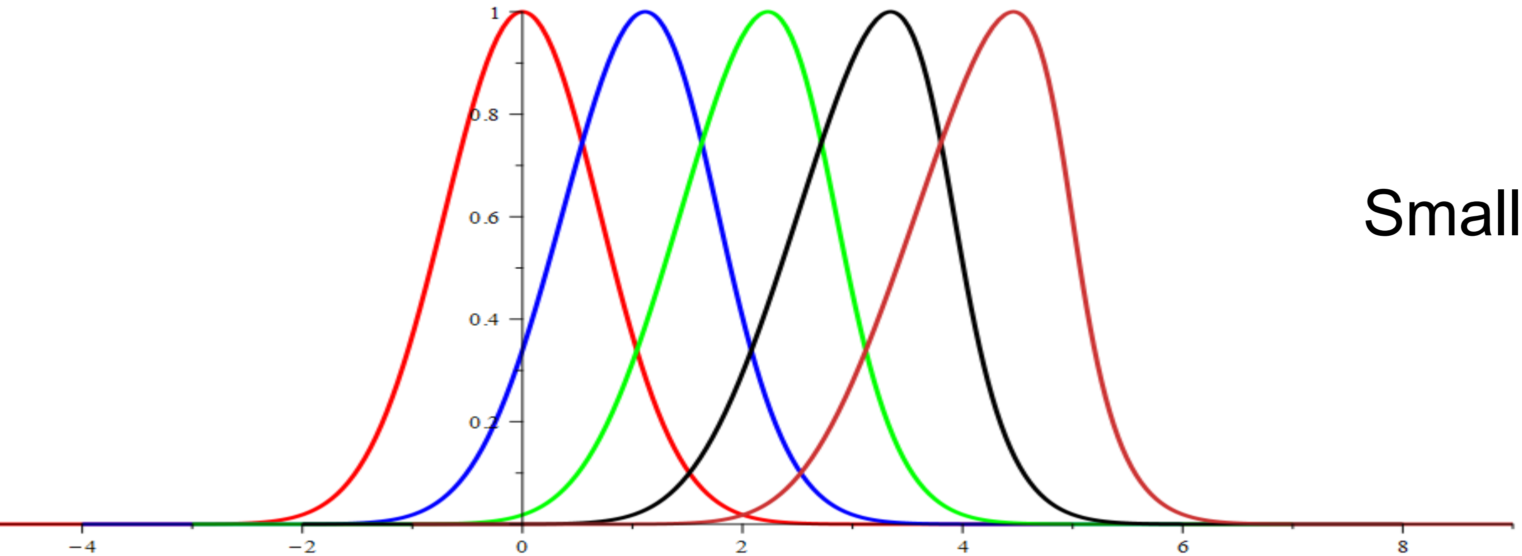


shock

Effects of amplitude of $\delta\rho$

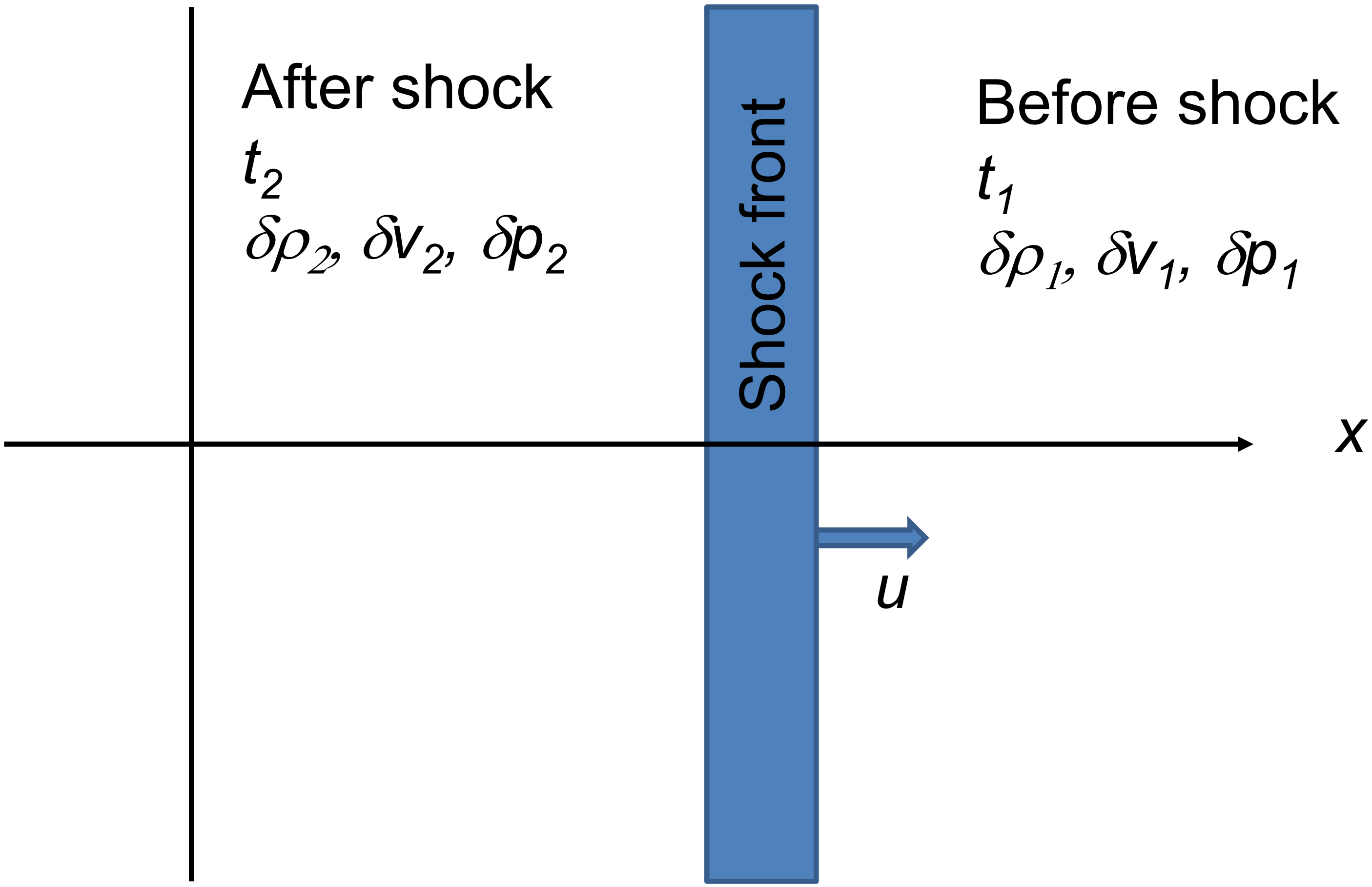


Large amplitude



Smaller amplitude

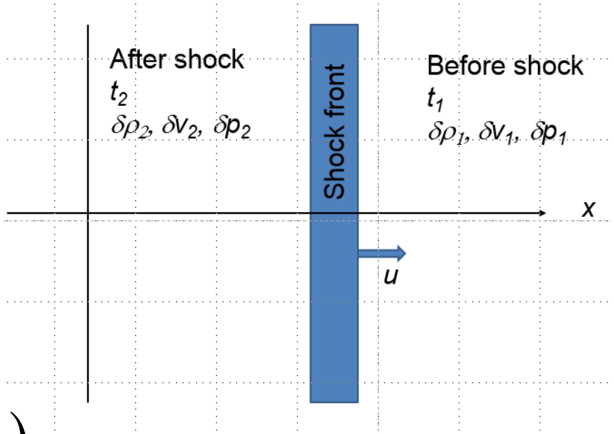
Analysis of shock wave – assumed to moving at velocity u



Note that in this case u is assumed to be a given parameter of the system.

Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2



Assume $\rho(x,t) = \rho(x-ut)$

$$p(x,t) = p(x-ut)$$

$$v(x,t) = v(x-ut)$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 = \frac{\partial(\rho v - \rho u)}{\partial x} \quad \Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

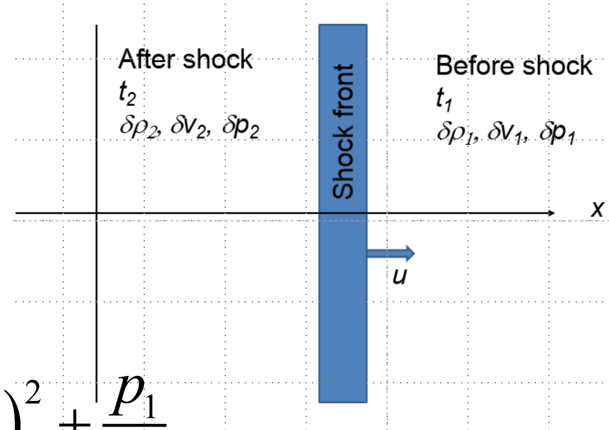
Conservation of energy and momentum:

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2



Summary of equations

$$\Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Assume that within each regions (1 & 2), the ideal gas equations apply

$$\epsilon_1 + \frac{p_1}{\rho_1} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} \qquad \epsilon_2 + \frac{p_2}{\rho_2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}$$

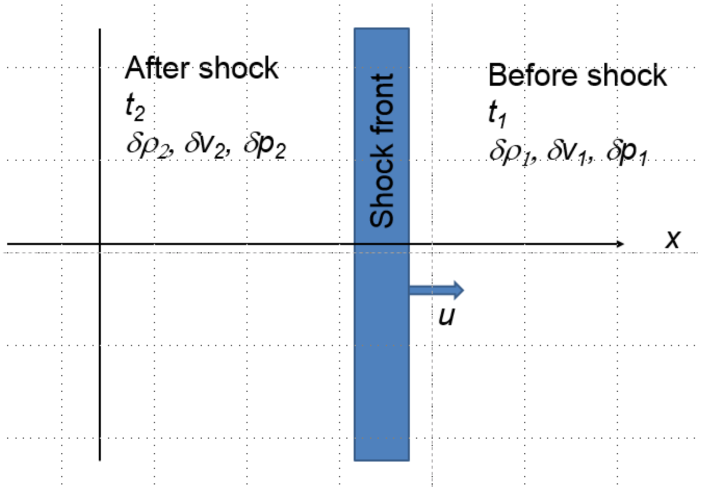
It follows that

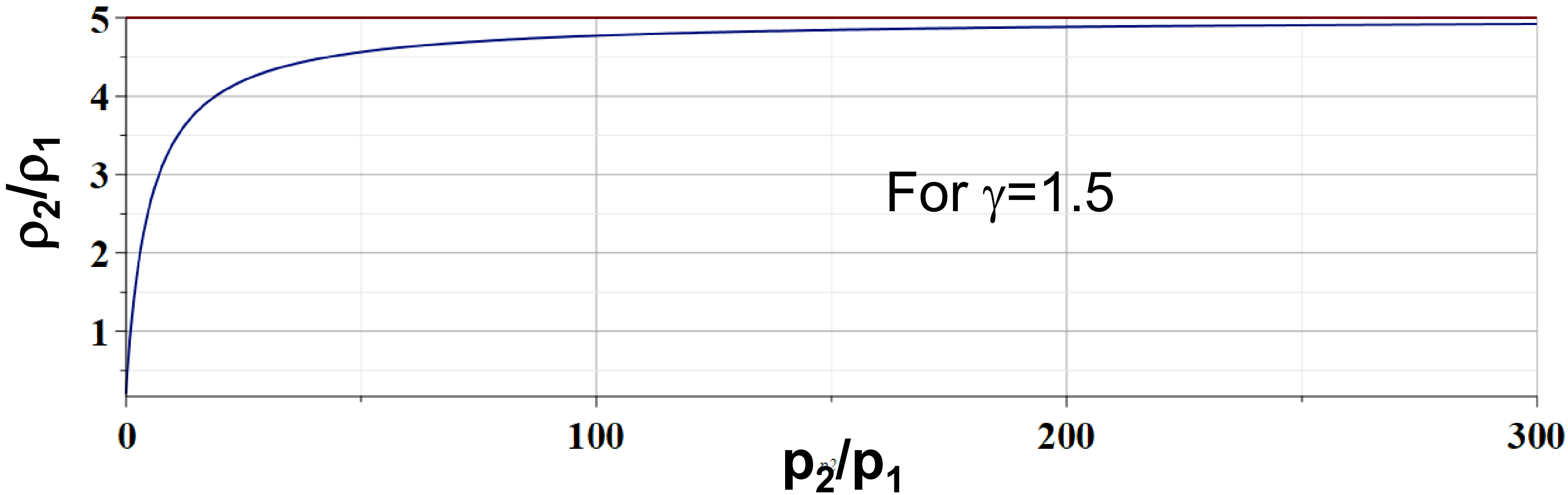
$$\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}(v_2 - u)^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}(v_1 - u)^2$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$





Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Ideal gas law: $\frac{p}{\rho} = \frac{k_B T}{M_0}$ Adiabatic ideal gas: $\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}$

Internal energy density: $\varepsilon \equiv \frac{E_{int}}{M} = \frac{p}{(\gamma - 1)\rho} = \frac{k_B T}{(\gamma - 1) M_0} \equiv c_V T$

First law of thermo: $d\varepsilon = T ds - p d\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) + p d\left(\frac{1}{\rho}\right)\right) = \frac{p}{(\gamma-1)\rho T}\left(\frac{dp}{p} - \gamma \frac{d\rho}{\rho}\right) = c_V d \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = c_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = c_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < c_V \left(\ln\left(\frac{p_2}{p_1}\right) - \gamma \ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$