PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 32: Chap. 9 of F&W

Linear and non-linear sound waves

1. Introduction to non-linear effects
2. Analysis of instability – shock phenomena

24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#21
28	Wed, 11/01/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#22
29	Fri, 11/03/2023	Chap. 9	Ideal gas fluids	#23
31	Mon, 11/06/2023	Chap. 9	Traveling and standing waves in the linear approximation	#24
32	Wed, 11/08/2023	Chap. 9	Non-linear and other wave properties	Topic
33	Fri, 11/10/2023	Chap. 9	Analysis of non-linear waves and shock waves	#25
34	Mon, 11/13/2023	Chap. 10	Surface waves in fluids	
35	Wed, 11/15/2023	Chap. 10	Surface waves in fluids; soliton solutions	
36	Fri, 11/17/2023	Chap. 11	Heat conduction	
37	Mon, 11/20/2023	Chap. 12	Viscous effects in hydrodynamics	
	Wed, 11/22/2023	Thanksgiving		
	Fri, 11/24/2023	Thanksgiving		
	Mon, 11/27/2023		Presentations I	
	Wed, 11/29/2023		Presentations 2	
	Fri, 12/01/2023		Presentations 3	
38	Mon, 12/04/2023	Chap. 12	Viscous effects in hydrodynamics	
39	Wed, 12/06/2023		Review	
40	Fri, 12/08/2023		Review	

Presentation signup

PHY 711 -- Assignment #25

Assigned: 11/10/2023 Due: 11/13/2023

Finish reading Chapter 9 in **Fetter & Walecka**.

1. In class, we discussed how to visualize the non-linear behavior of an adiabatic ideal gas with parameter γ. Using Maple or Mathematica or other software and using a parametric plot formalism, create an animated gif file to show the traveling waveform $s(w)$, where s is a shape of your choice and $w=x-u(s(w))t$. You will also need to choose the value of γ as well.

Now consider some non-linear effects in sound

Examples?

We will consider the simple case –

1. One dimension for motion
2. Fluid is assumed to be an ideal gas
3. Adiabatic conditions
4. All variables will be expressed in terms of the density $\rho(x,t)$

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume spatial variation confined to x direction ;

assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0.$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Expressing p in terms of ρ: $\quad p = p(\rho)$

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}\equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where} \quad \frac{\partial p}{\partial \rho}\equiv c^2(\rho)$$

For adiabatic ideal gas: $\quad \frac{dp}{p}=\gamma\frac{d\rho}{\rho} \qquad p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho)=\frac{\gamma p}{\rho}=c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where} \quad c_0^2\equiv\frac{\gamma p_0}{\rho_0}$$

Digression – What is gamma?

Internal energy for ideal gas: $pV = Nk_BT$

$E_{\text{int}} = \frac{f}{2}Nk_BT$ $\quad f \equiv$ degrees of freedom; 3 for atom, 5 for diatomic molecule

In terms of specific heat ratio: $\gamma \equiv \frac{C_p}{C_V}$

$$dE_{\text{int}} = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}Nk_B$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}Nk_B + Nk_B$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} = 1 + \frac{2}{f} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1} \qquad E_{\text{int}} = \frac{1}{\gamma - 1}Nk_BT$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation of v in terms of $v\left(\rho\right)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

Some more algebra :

From Euler equation : $$\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

From continuity equation : $$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$$

Combined equation : $$\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\Rightarrow\left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow\frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process: $c^2 = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma - 1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\Rightarrow \; c = c_0 \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2}$$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm \frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0} \right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma - 1} \left(\left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - 1 \right)$$

Summary :

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm \frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Traveling wave solution:

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assume: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Solution in linear approxiation:

$$u = v + c \approx v_0 + c_0 = c_0\left(\frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1}\right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f(x - c_0 t)$$

Some details

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Apparently: $u(\rho) \Leftrightarrow \quad v \pm c$

Note that for $u = v + c$ (choice of + solution)

$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$ is satisfied by a function of the form

$$\rho(x,t) = \rho_0 + f(x - u(\rho(x,t))t)$$

Let $w \equiv x - u(\rho(x,t))t$

$$\frac{df}{dw}\frac{\partial w}{\partial t} + u\frac{df}{dw}\frac{\partial w}{\partial x} = \frac{df}{dw}(-u + u) = 0$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_{w})$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w,t)$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations:

plot $s(w)$ vs $x(w,t)$ for range of w at each t

Summary

$$\frac{\partial\rho}{\partial t} + u(\rho)\frac{\partial\rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0\left(1 + s(x - u(\rho)t)\right)$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(x - ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$

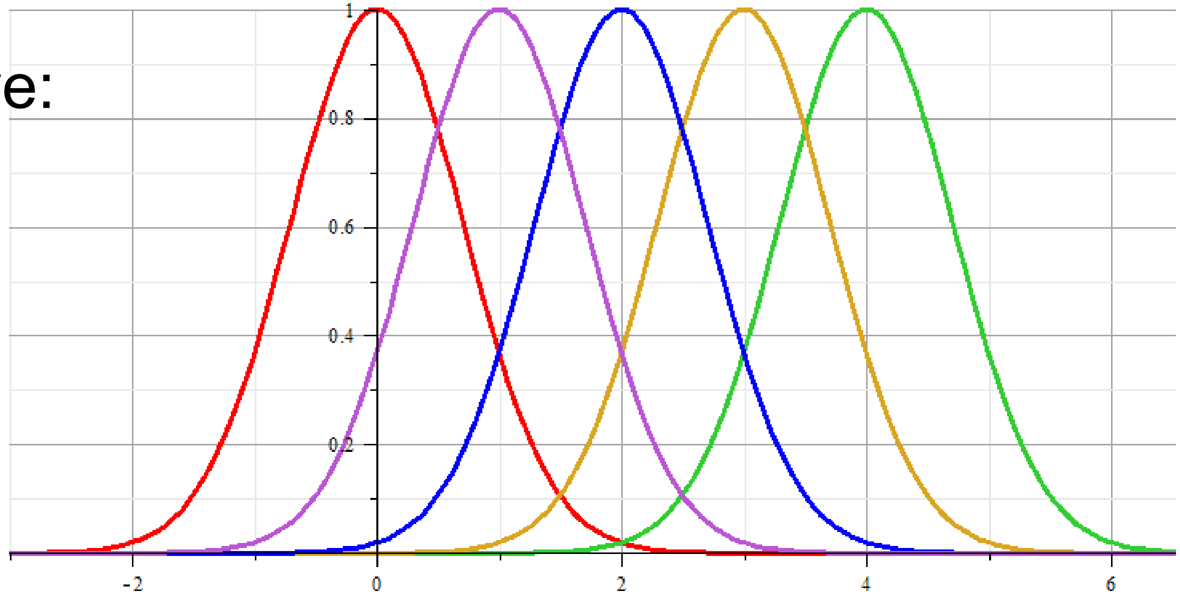
Plot $s(x - ut)$ for fixed t, as a function of x:

Let $w = x - ut \quad \Rightarrow \quad x = w + ut = w + u(w)t \equiv x(w,t)$

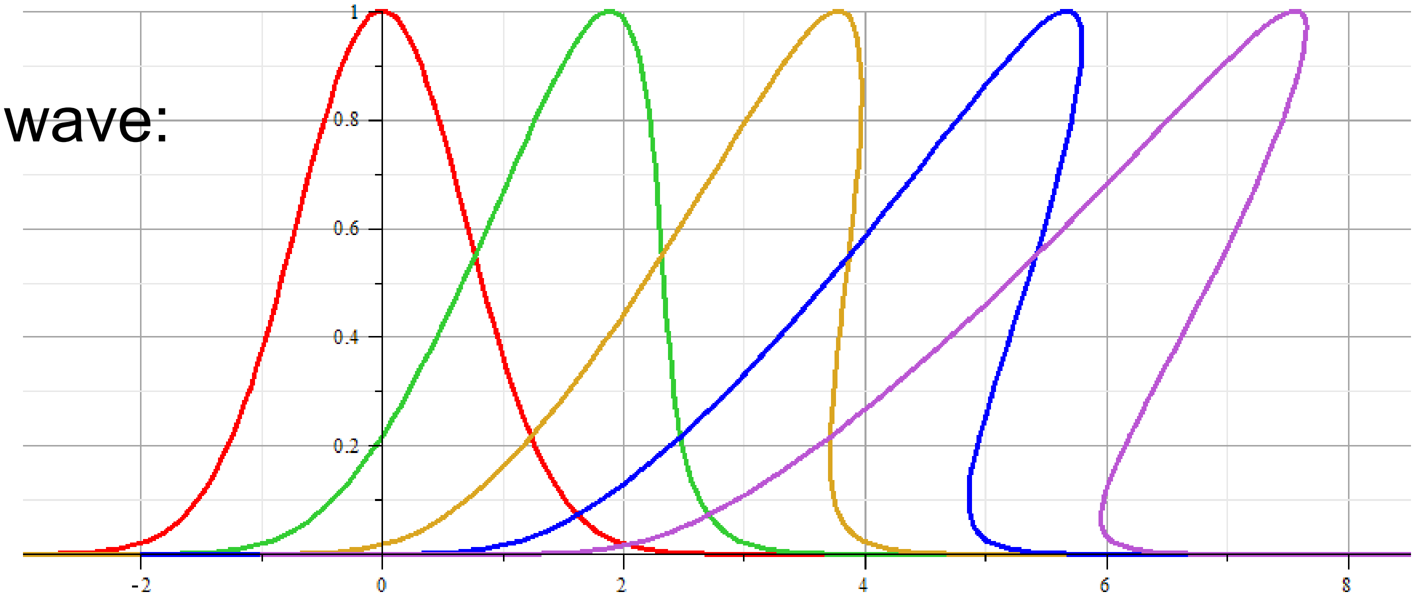
$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

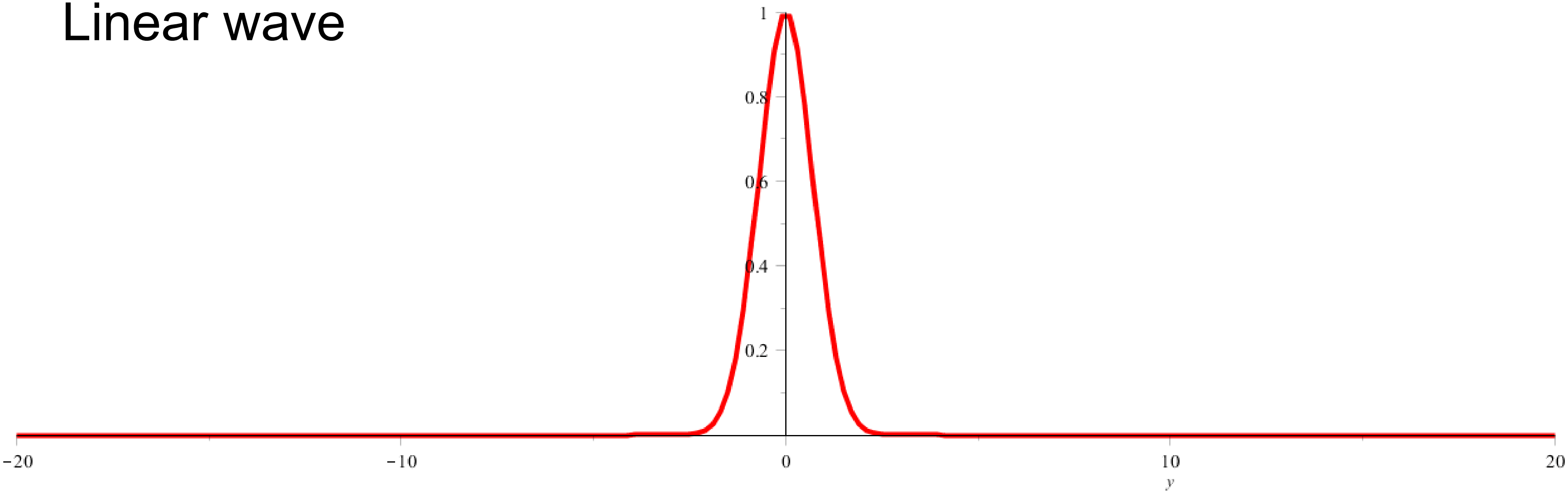
Linear wave:



Non-linear wave:



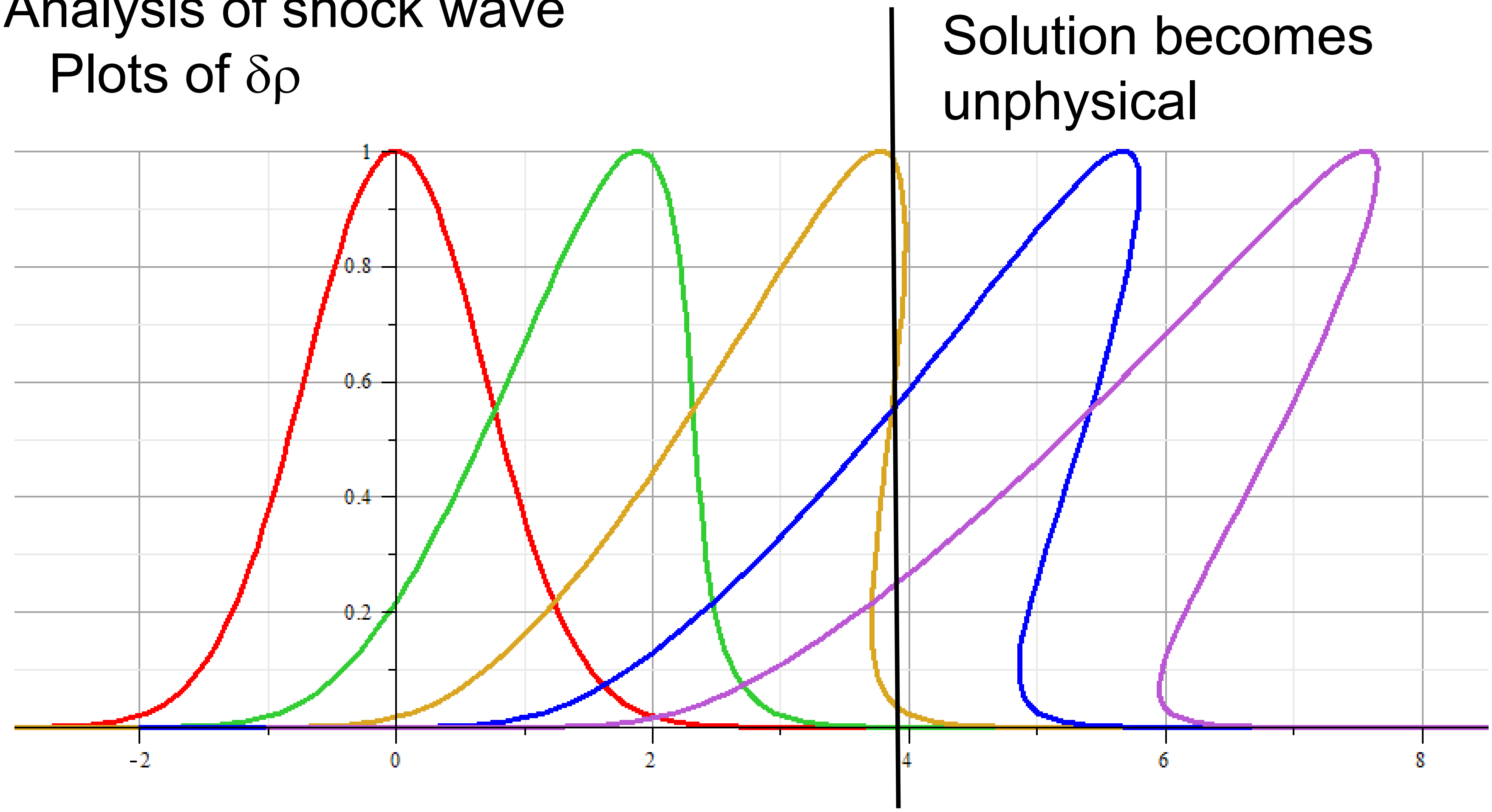
Linear wave



Non-linear wave

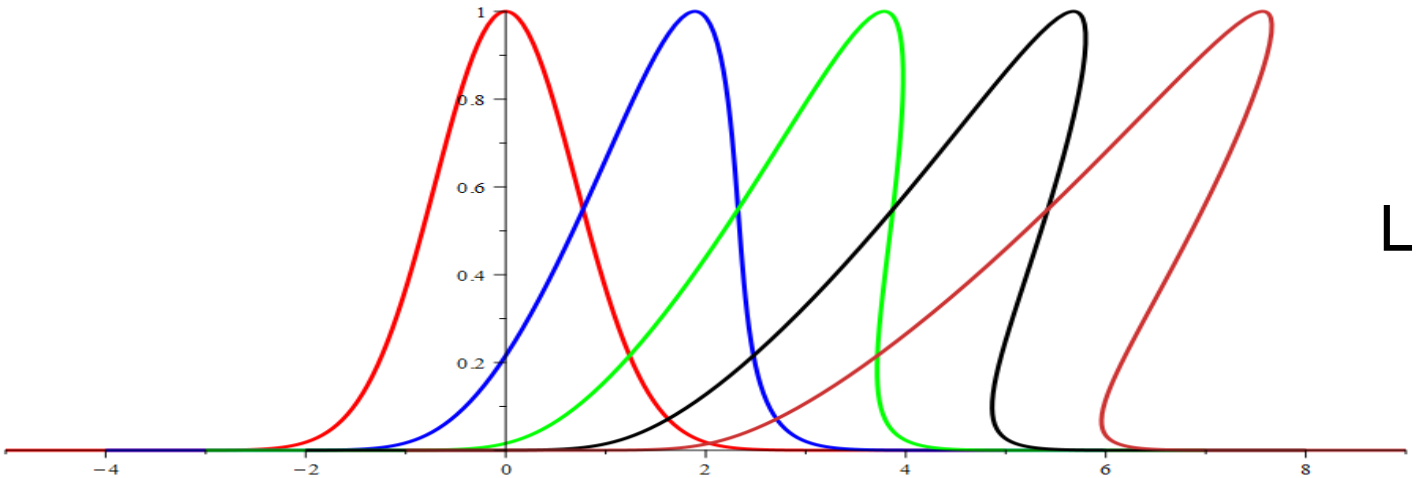
Analysis of shock wave
Plots of $\delta\rho$

Solution becomes unphysical

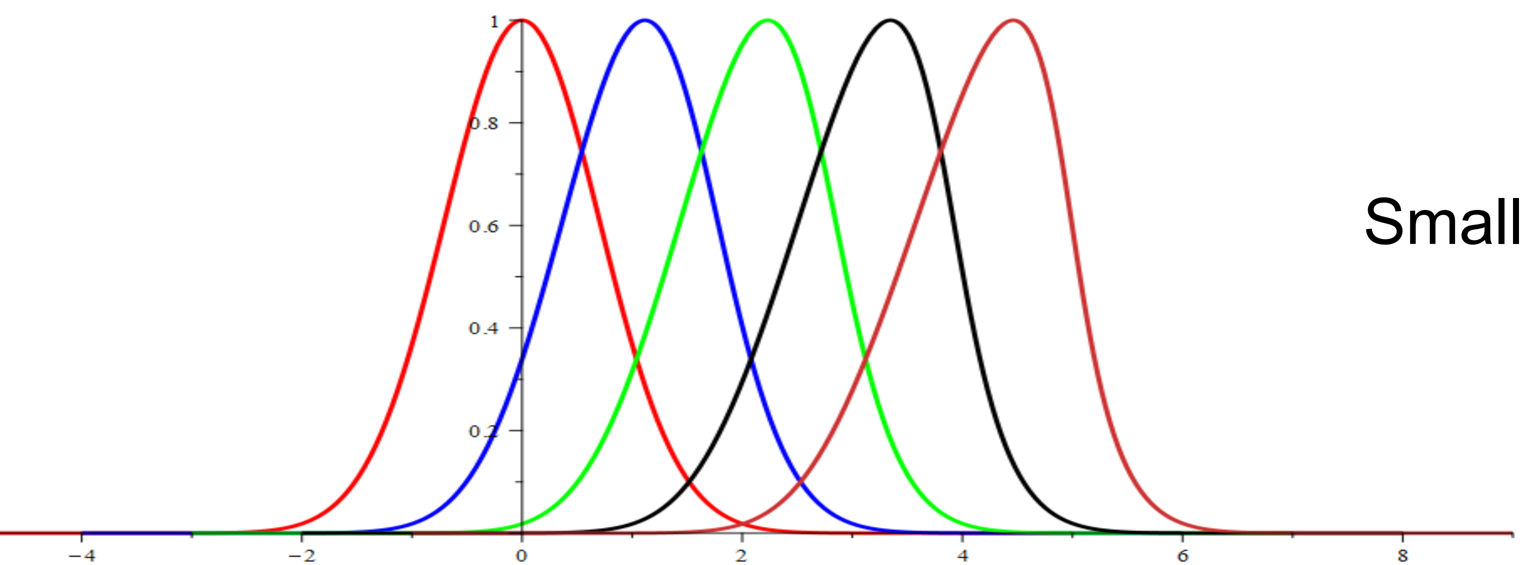


shock

Effects of amplitude of $\delta\rho$

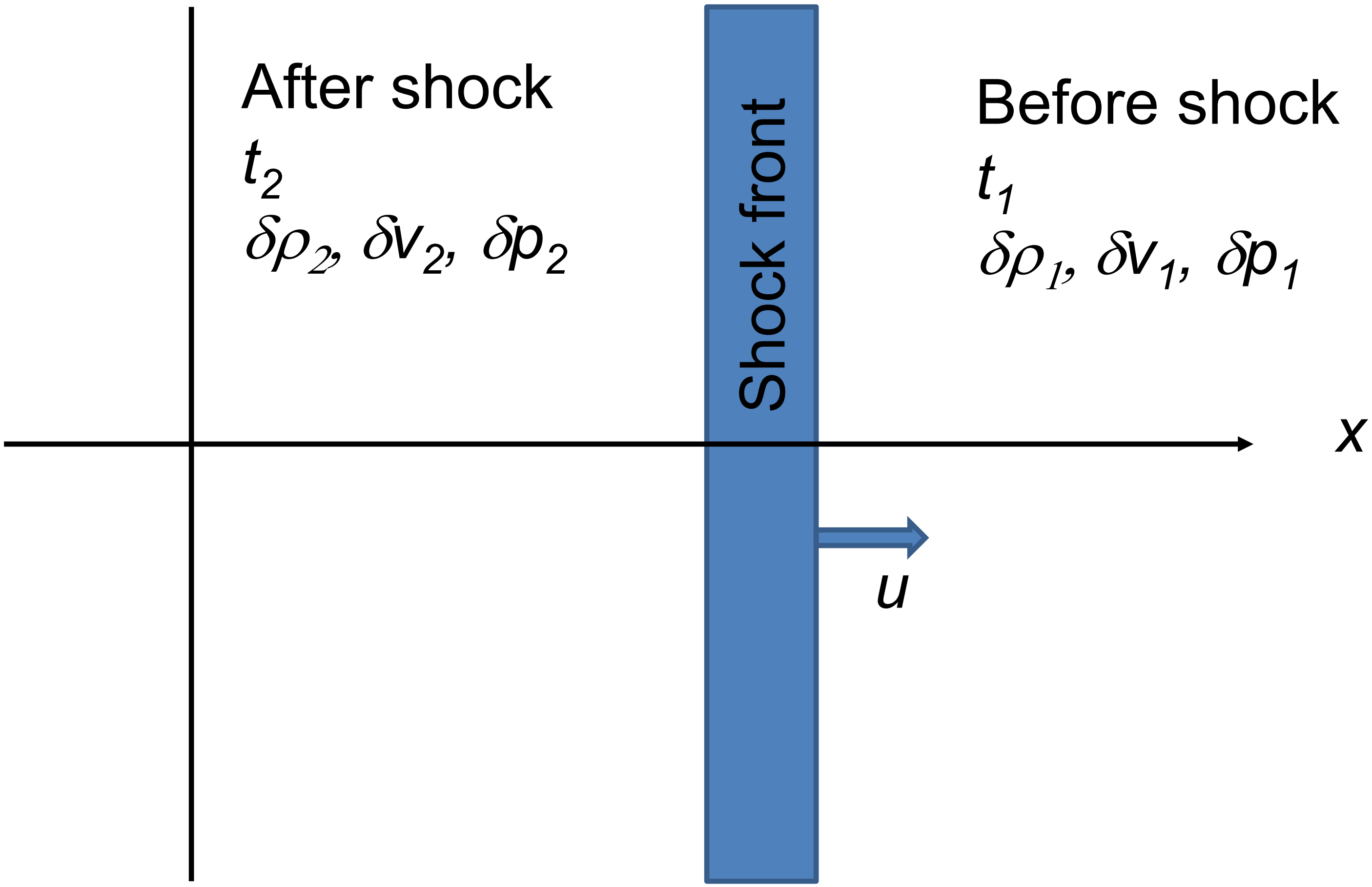


Large amplitude



Smaller amplitude

Analysis of shock wave – assumed to moving at velocity u



Note that in this case u is assumed to be a given parameter of the system.

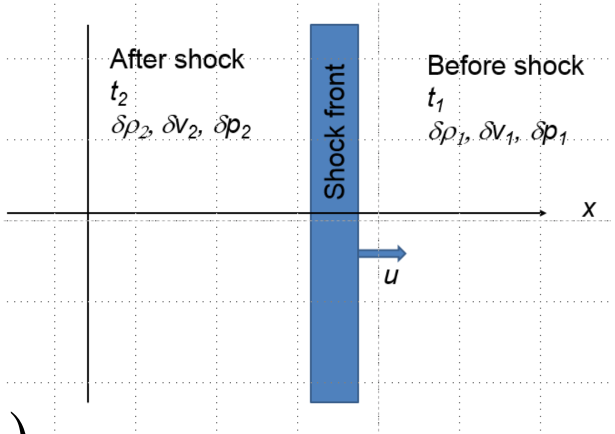
Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Assume $\rho(x,t) = \rho(x - ut)$

$$p(x,t) = p(x - ut)$$

$$v(x,t) = v(x - ut)$$



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 = \frac{\partial(\rho v - \rho u)}{\partial x} \qquad \Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

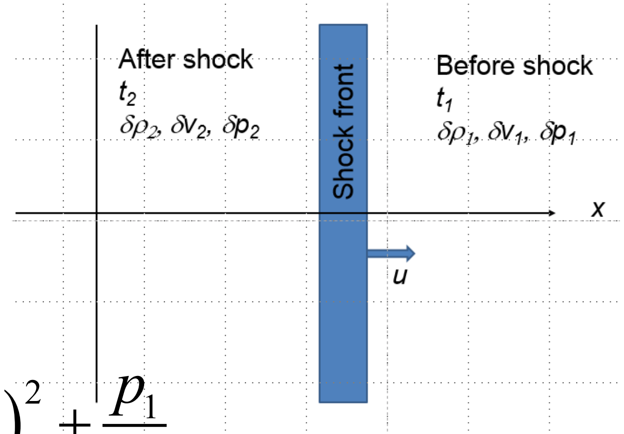
Conservation of energy and momentum:

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2



Summary of equations

$$\Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Assume that within each regions (1 & 2), the ideal gas equations apply

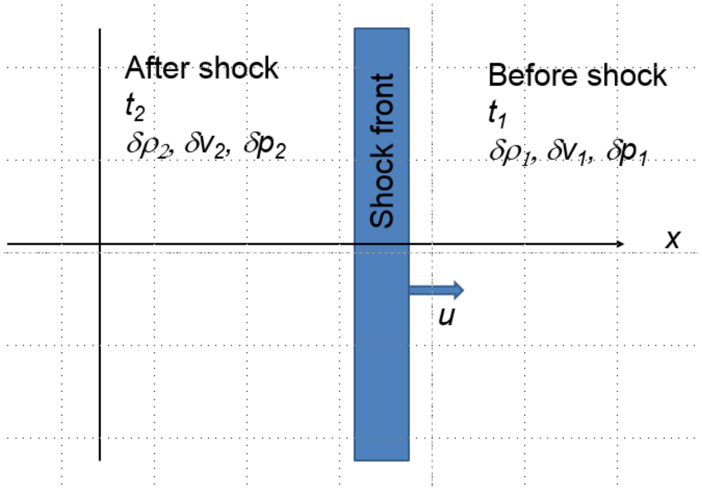
$$\epsilon_1 + \frac{p_1}{\rho_1} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} \qquad \epsilon_2 + \frac{p_2}{\rho_2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}$$

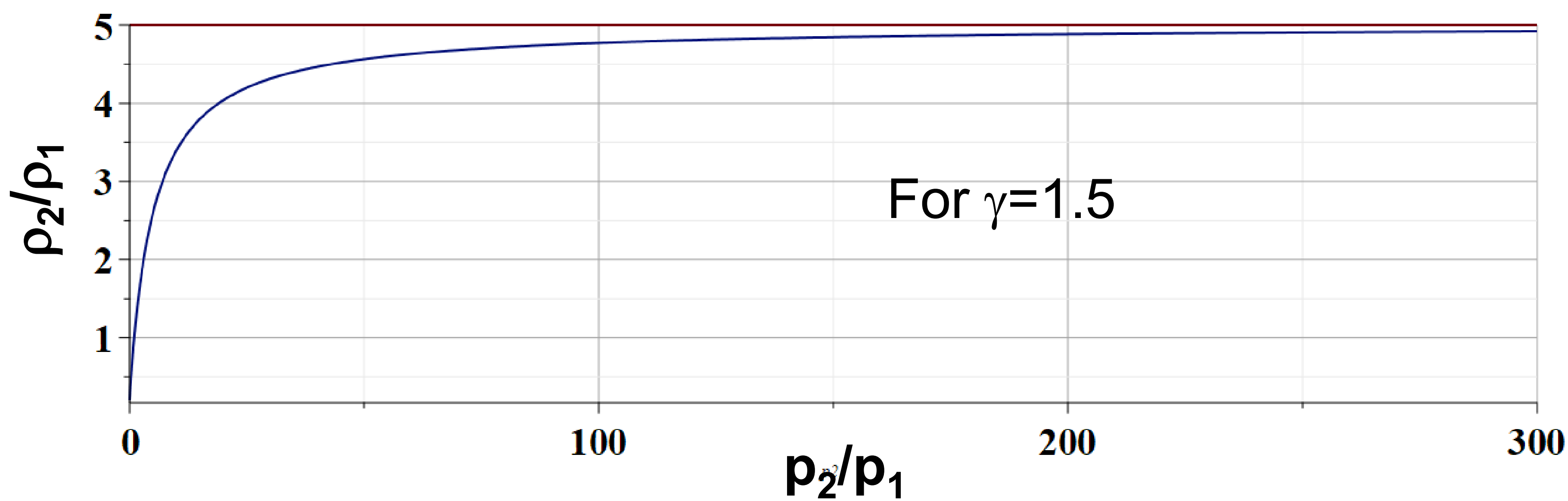
It follows that $$\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}(v_2 - u)^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}(v_1 - u)^2$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$





Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Ideal gas law: $\frac{p}{\rho} = \frac{k_B T}{M_0}$ Adiabatic ideal gas: $\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}$

Internal energy density: $\varepsilon \equiv \frac{E_{int}}{M} = \frac{p}{(\gamma-1)\rho} = \frac{k_B T}{(\gamma-1)M_0} \equiv c_V T$

First law of thermo: $d\varepsilon = Tds - pd\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) + pd\left(\frac{1}{\rho}\right)\right) = \frac{p}{(\gamma-1)\rho T}\left(\frac{dp}{p} - \gamma\frac{d\rho}{\rho}\right) = c_V d\ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = c_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = c_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < c_V\left(\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$