PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 34: Chapter 10 in F & W

Surface waves

- Summary of linear surface wave solutions
- Non-linear contributions and soliton solutions

Physics Colloquium

Thursday

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November 16th, 2023

Improving your research workflow with AI tools

Have you ever sat down and mulled to yourself "What in the word is _____?" Have you spent hours pulling together resources for a literature review? Have you developed a synthesis matrix to compare findings between multiple articles over the course of days and/or weeks? Do you use forward and backwards citation analysis and want to move from quantitative to qualitative metrics? AI tools like scite.ai, elicit.com, Research Rabbit, and others have become game changers in saving researchers time and energy. Although generative AI tools such as ChatGPT are in the proverbial limelight, other AI tools provide STEM researchers with precise results as well as time savings.

J. Denice Lewis
Research and Instruction Librarian for Engineering and Science
Wake Forest University

4 pm - Olin 101
Refreshments will be served in Olin Lobby beginning at 3:30pm.

This material is covered in Chapter 10 of your textbook using similar notation.

25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#21
28	Wed, 11/01/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#22
29	Fri, 11/03/2023	Chap. 9	Ideal gas fluids	#23
31	Mon, 11/06/2023	Chap. 9	Traveling and standing waves in the linear approximation	#24
32	Wed, 11/08/2023	Chap. 9	Non-linear and other wave properties	Topic
33	Fri, 11/10/2023	Chap. 9	Analysis of non-linear waves and shock waves	#25
34	Mon, 11/13/2023	Chap. 10	Surface waves in fluids	#26
35	Wed, 11/15/2023	Chap. 10	Surface waves in fluids; soliton solutions	#27
36	Fri, 11/17/2023	Chap. 11	Heat conduction	
37	Mon, 11/20/2023	Chap. 12	Viscous effects in hydrodynamics	
	Wed, 11/22/2023	Thanksgiving		
	Fri, 11/24/2023	Thanksgiving		
	Mon, 11/27/2023		Presentations I	
	Wed, 11/29/2023		Presentations 2	
	Fri, 12/01/2023		Presentations 3	
38	Mon, 12/04/2023	Chap. 12	Viscous effects in hydrodynamics	
39	Wed, 12/06/2023		Review	
40	Fri, 12/08/2023		Review	

PHY 711 – Homework # 27

Assigned: 11/15/2023 Due: 11/20/2023

Read Chapter 10 of **Fetter and Walecka**.

1. In your textbook and In class, we discussed the traveling wave soliton solutions to the surface height function $\eta(u)$ as a function of position x and time t, $u = x - ct$ takes the form

$$\eta(u) = \frac{\eta_0}{\cosh^2(\sqrt{\frac{3\eta_0}{4h^3}}u)}.$$

Here η_0 is a scale factor for the height which is related to the acceleration of gravity g, the average water height h, and the speed parameter of the soliton c according to

$$\eta_0 = h\left(1 - \frac{gh}{c^2}\right).$$

With this information, show that $\eta(u)$ is a solution to the non-linear equation

$$\frac{\eta_0}{h}\eta(u) - \frac{3}{2h}\eta^2(u) - \frac{h^2}{3}\frac{d^2\eta(u)}{du^2} = 0.$$

PHY 711 Presentation Schedule

Monday 11/27/2023

	Presenter Name	Topic
10:00-10:16	Gabby Tamayo	Three body problem / Ejection
10:17-10:33	Thilini Karunarathna	Lagrangian and Hamiltonian Equations
10:34-10:50	Joe Granlie	Canonical transformations (+Ham. Jacobi maybe)

Wednesday 11/29/2023

	Presenter Name	Topic
10:00-10:24	Athul Prem	Numerical simulation of particle dynamics
10:25-10:50	David Carchipulla-Morales	Eddy Covariance Momentum Conservation

Friday 12/01/2023

	Presenter Name	Topic
10:00-10:24	Caela Flake	Analyze the equations of motion for the Foucoult Pendulum
10:25-10:50	Mitchell Turk	Comparison of Results Between Different Methodologies for N-Body Orbital Mechanics Simulations

Note about presentations

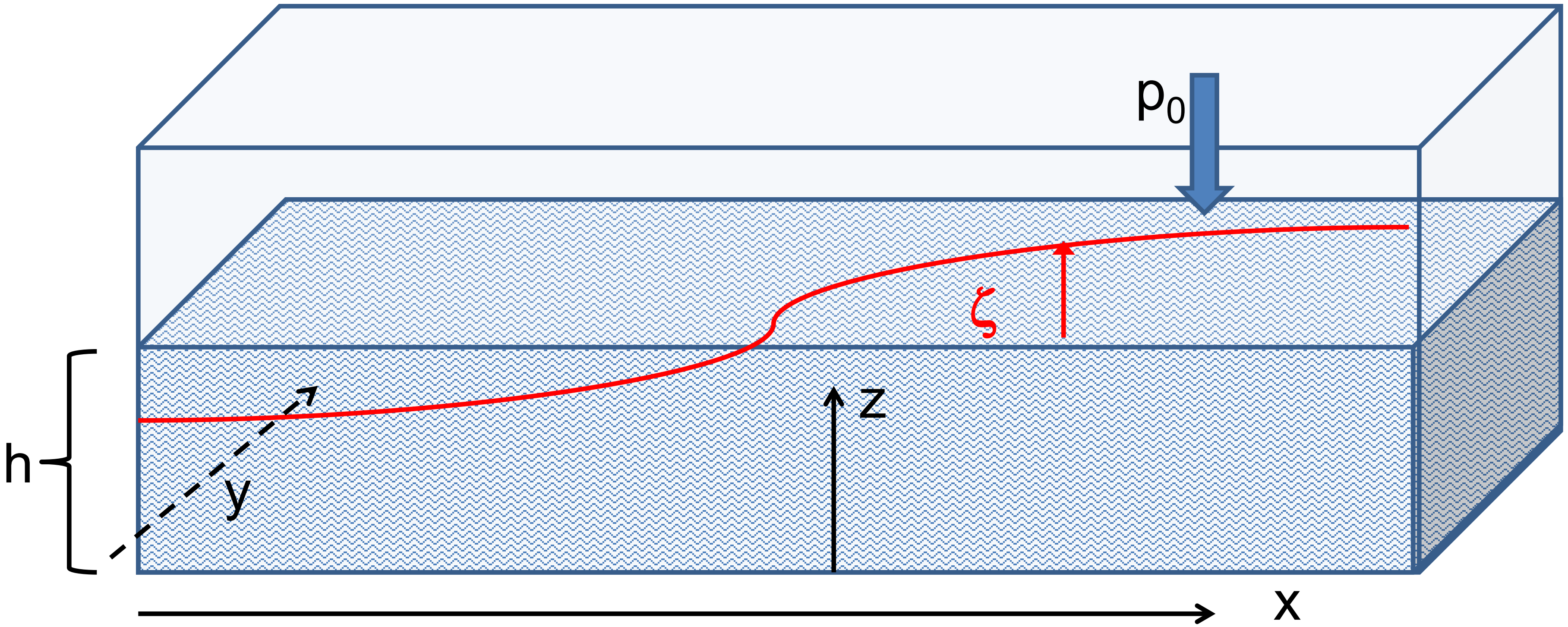
- Please consult with me if you have any questions about the content, format, expectations, etc..

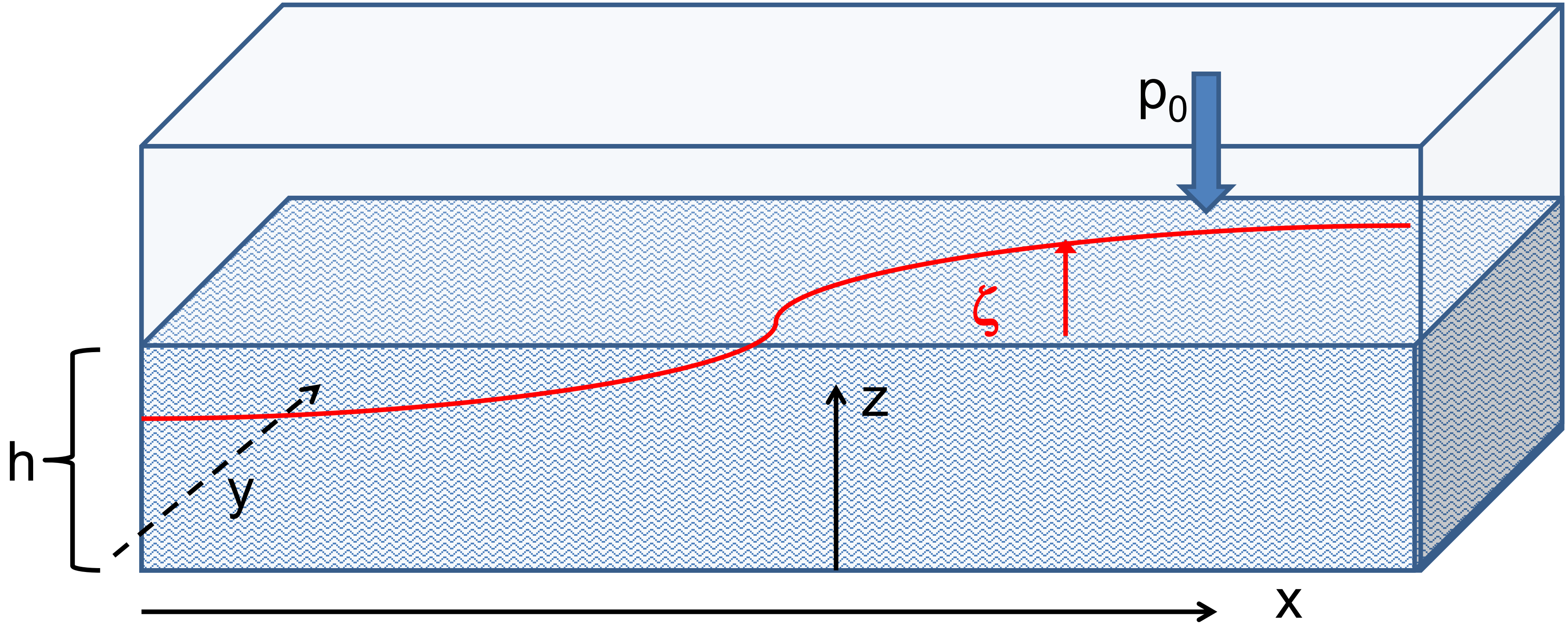
- Please prepare a ~ 10 (Mon) or 15 (Wed/Fri) minute presentation using powerpoint or equivalent software, leaving ~ 5 minutes for questions

- Please turn in your presentation, your preparation notes, mathematica, maple, matlab,… work if appropriate. If your topic follows a paper or write-up from the literature, please also include a copy of that.

Consider a container of water with average height h and surface h+$\zeta(x,y,t)$

Atmospheric pressure p_0 is in equilibrium at the surface





Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -\nabla U - \frac{\nabla p}{\rho}$$

Continuity equation within the fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0$$

For irrotational flow -- $\mathbf{v} = -\nabla \Phi$

Linearized equation: $\nabla \left(-\frac{\partial \Phi}{\partial t} + g(z-h) + \frac{p}{\rho} \right) = 0$

At surface: $z = h + \zeta \qquad -\frac{\partial \Phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

Keep only linear terms and assume that horizontal variation is only along *x*:

For $0 \le z \le h + \zeta$: $\nabla^2 \Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x,z,t) = 0$

Consider and periodic waveform: $\Phi(x,z,t) = Z(z) \cos\left(k(x - ct)\right)$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x,0,t) = 0$

$$\Rightarrow \frac{dZ}{dz}\left(0\right) = 0 \qquad Z(z) = A\cosh(kz)$$

At surface: $z = h + \zeta$ $\quad \frac{\partial \zeta}{\partial t} = v_z\left(x, h+\zeta, t\right) = -\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z}$

Also: $\quad -\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

$$\Rightarrow -\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x, h+\zeta, t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x, h+\zeta, t\right)}{\partial z} = 0$$

Velocity potential: $\Phi(x,z,t) = A\cosh(kz)\cos\left(k(x-ct)\right)$

At surface: $\Phi(x,\left(h+\zeta\right),t) = A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)$

$$\frac{\partial \zeta}{\partial t} = v_z\left(x,h+\zeta,t\right) = -\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z} \qquad -\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$$

$$\Rightarrow -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z} = 0$$

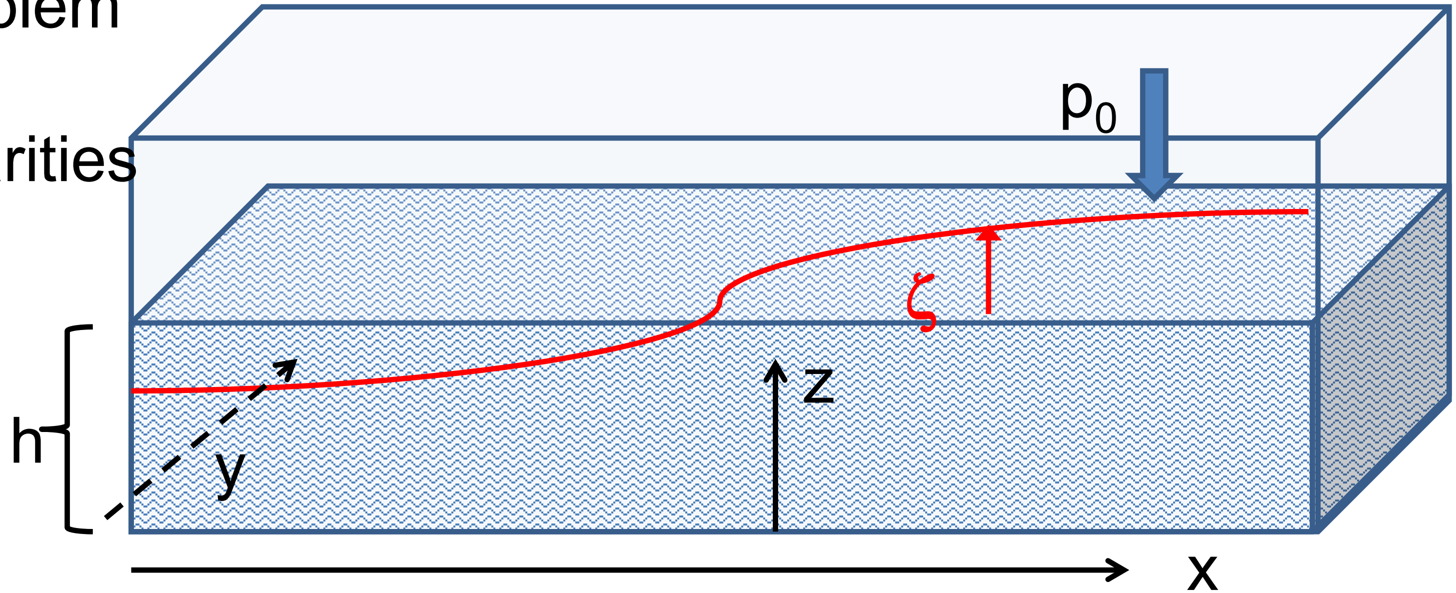
$$A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)\left(k^2c^2 - gk\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))}\right) = 0$$

$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))} \approx \frac{g}{k}\tanh(kh)$$

Note that this solution represents a pure plane wave. More likely, there would be a linear combination of wavevectors *k*. Additionally, your text considers the effects of surface tension. In this lecture, we will focus on the effects of the non-linear effects of Euler and continuity equations.

Surface waves in an incompressible fluid

General problem
including
non-linearities



Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g\left(z - h\right) = \text{constant} \qquad \Phi = \Phi(x, y, z, t)$$

$$-\nabla^2 \Phi = 0 \qquad \mathbf{v} = \mathbf{v}(x, y, z, t) = -\nabla \Phi(x, y, z, t)$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

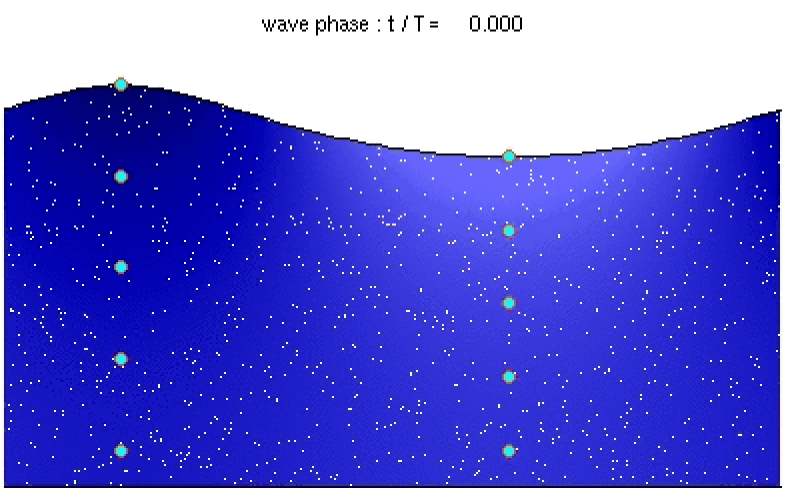
$$v_z(h+\zeta) = \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x, y, z, t)}{\partial z}\right|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}\left(x, y, h+\zeta, t\right)$$

Some relationships at surface --

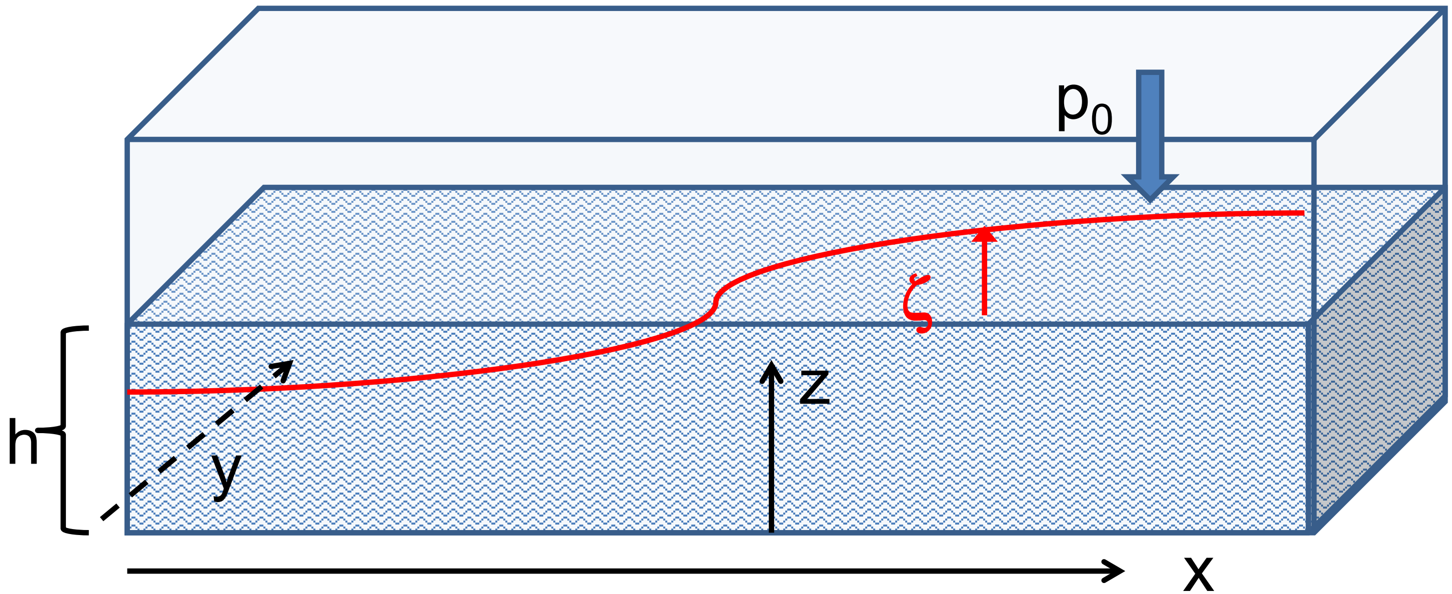
At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x, y, z, t)}{\partial z}\right|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Note that $v_z(x, y, h+\zeta, t) = \frac{d\zeta}{dt}$



From wikipedia



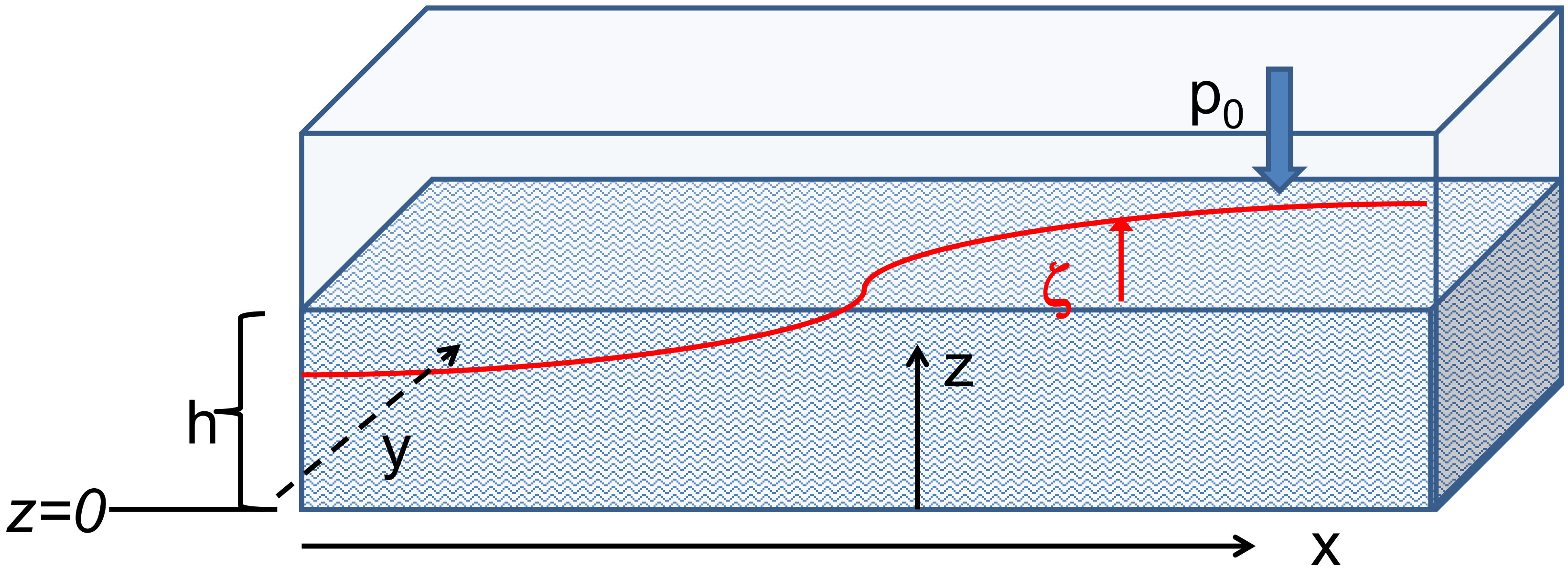
Further simplifications; assume trivial *y*-dependence

$$\Phi = \Phi(x,z,t) \qquad\qquad \zeta = \zeta(x,t)$$

Within fluid: $\quad 0 \le z \le h + \zeta$

At surface: $\quad v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$

Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$

Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x,z,t)$; $\mathbf{v}(x,z,t) = -\nabla\Phi(x,z,t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

Defining equations for $\Phi(x,z,t)$ and $\zeta(x,t)$

where $0 \leq z \leq h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2}_{v_x^2} + \underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2}_{v_z^2}\right] + g(z-h) = 0.$$

Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\Big|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v}\cdot\nabla\zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta} = 0$$

Analysis assuming water height *z* is small relative to variations in the direction of wave motion *(x)*

Taylor's expansion about $z = 0$:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Note that the zero vertical velocity at the bottom suggest that to a good approximation, that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the Taylor expansion. In addition, the Laplace equation allows us to convert all even derivatives with respect to *z* to derivatives with respect to *x*.

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

$$\Rightarrow \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Modified Taylor's expansion: $\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial x^4}(x,0,t)\cdots$

Some details --

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

At bottom: $z = 0$ and $v_z(x,0,t) = 0 \quad \Rightarrow \frac{\partial \Phi}{\partial z}(x,0,t) = 0$

Further, your textbook argues that using Fourier transforms,

$$\Phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \cosh(kz) e^{ikx} \tilde{f}(k,t) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(1 + \frac{(kz)^2}{2!} + \frac{(kz)^4}{4!} + \ldots\right) e^{ikx} \tilde{f}(k,t)$$

$$\Phi(x,z,t) \approx \Phi(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z=h+\zeta(x,t)$

$$-\frac{\partial\Phi(x,h,t)}{\partial t}+g\zeta(x,t)=0$$

Consistent vertical velocity at $z=h+\zeta(x,t)$

$$-\frac{\partial\Phi(x,z,t)}{\partial z}-\frac{\partial\zeta(x,t)}{\partial t}\bigg|_{z=h+\zeta}=0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial\Phi(x,h,t)}{\partial z}\approx h\frac{\partial^2\Phi(x,0,t)}{\partial x^2}=-\frac{\partial\zeta(x,t)}{\partial t} \qquad -\frac{\partial\Phi(x,h,t)}{\partial t}\approx-\frac{\partial\Phi(x,0,t)}{\partial t}=-g\zeta(x,t)$$

Decoupled equations: $$\frac{\partial^2\Phi(x,0,t)}{\partial t^2}=gh\frac{\partial^2\Phi(x,0,t)}{\partial x^2}.$$

➔linear wave equation with $c^2=gh$

Analysis of non-linear equations --

Bernoulli equation evaluated at surface:

$$-\frac{\partial\Phi(x,z,t)}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi(x,z,t)}{\partial x}\right)^2+\left(\frac{\partial\Phi(x,z,t)}{\partial z}\right)^2\right]\Bigg|_{z=h+\zeta}+g\zeta(x,t)=0.$$

Consistency of surface velocity

$$-\frac{\partial\Phi(x,z,t)}{\partial z}+\frac{\partial\Phi(x,z,t)}{\partial x}\frac{\partial\zeta(x,t)}{\partial x}-\frac{\partial\zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta}=0$$

Representation of velocity potential from Taylor′s expansion:

$$\Phi(x,z,t)\approx\Phi(x,0,t)-\frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x,0,t)+\frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x,0,t)\cdots$$

Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\Rightarrow \frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

These equations represent non-linear coupling of $\phi(x,t)$ and $\zeta(x,t)$.

Coupled equations: $$-\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

Traveling wave solutions with new notation:

$u \equiv x - ct \qquad \phi(x,t) \equiv \chi(u)$ and $\zeta(x,t) \equiv \eta(u)$

Note that the wave "speed" c will be consistently determined

$$c\frac{d\chi(u)}{du} - \frac{ch^2}{2}\frac{d^3\chi(u)}{du^3} + \frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2 + g\eta(u) = 0.$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right) - \frac{h^3}{6}\frac{d^4\chi(u)}{du^4} + c\frac{d\eta(u)}{du} = 0.$$

Integrating and re-arranging coupled equations

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\chi'=-\frac{g}{c}\eta+\frac{h^2}{2}\chi'''-\frac{1}{2c}(\chi')^2\approx-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

$$\Rightarrow(h+\eta)\frac{d\chi(u)}{du}-\frac{h^3}{6}\frac{d^3\chi(u)}{du^3}+c\eta(u)=0$$

Now we can express $\frac{d\chi(u)}{du}=\chi'$ in terms of η:

$$\chi'\approx-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

Integrating and re-arranging coupled equations – continued --
Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h+\eta)\left(-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2\right)+\frac{h^3 g}{6c}\eta''+c\eta=0$$

$$\Rightarrow\left(1-\frac{gh}{c^2}\right)\eta-\frac{gh^3}{3c^2}\eta''-\frac{g}{c^2}\left(1+\frac{gh}{2c^2}\right)\eta^2=0$$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Note: $c^2=gh+\ldots$

Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of η

$$\Rightarrow \left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)\quad \text{where } \eta_0 \text{ is a constant}$$

Steps to solution

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Let $1-\frac{hg}{c^2}\equiv\frac{\eta_0}{h}$ $\quad\Rightarrow \frac{\eta_0}{h}\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$

Multiply equation by $\eta'(u)$ $\quad\Rightarrow \frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)\right)=0$

Integrate wrt u and assume solution vanishes for $u\rightarrow\infty$

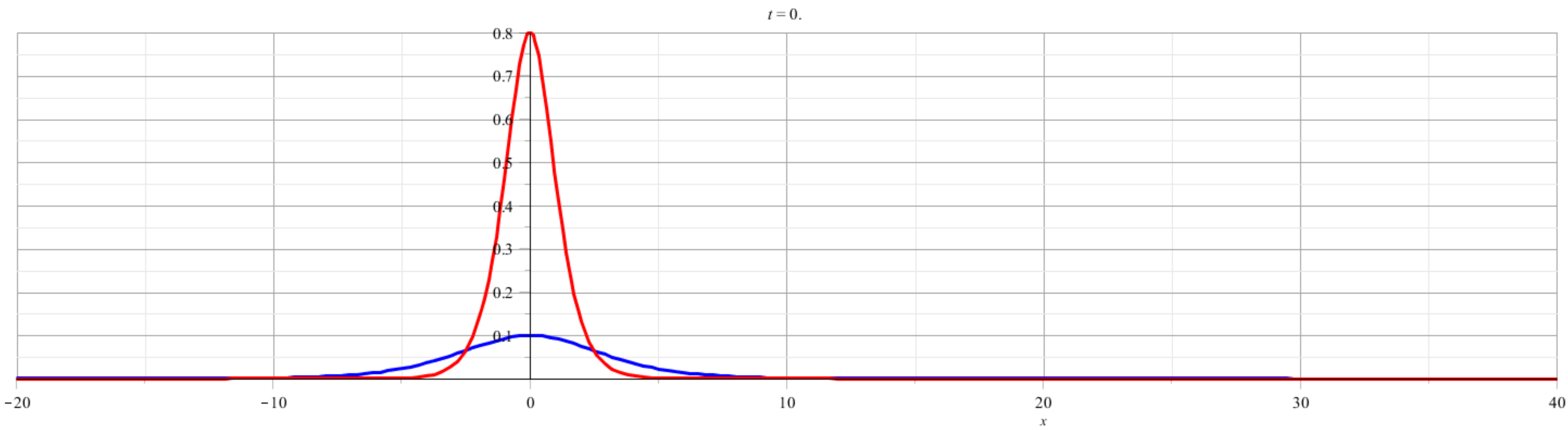
$$\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)=0$$

$$\eta'^2(u)=\frac{3}{h^3}\eta^2(u)\left(\eta_0-\eta(u)\right)$$

$$\frac{d\eta}{\eta\left(\eta_0-\eta\right)^{1/2}}=\sqrt{\frac{3}{h^3}}du \qquad \Rightarrow \eta(u)=\frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}$$

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right)$$

Two soliton solutions with different amplitudes --



Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x},\bar{t}) = \frac{\beta}{2}\operatorname{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta\bar{t})\right].$$

More details

Modified surface amplitude equation in terms of η :

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Some identities: $\frac{\eta_0}{h}=1-\frac{gh}{c^2};\quad \frac{\partial\eta}{\partial t}=-c\frac{d\eta}{du};\quad \frac{\partial\eta}{\partial x}=\frac{d\eta}{du}.$

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta'-\frac{h^2}{3}\eta'''-\frac{3}{h}\eta\eta'=0.$$

Expression in terms of x and t:

$$-\frac{\eta_0}{ch}\frac{\partial\eta}{\partial t}-\frac{h^2}{3}\frac{\partial^3\eta}{\partial x^3}-\frac{3}{h}\eta\frac{\partial\eta}{\partial x}=0.$$

Expression in terms of $\overline{x}$ and $\overline{t}$:

$$\frac{\partial\eta}{\partial\overline{t}}+6\eta\frac{\partial\eta}{\partial\overline{x}}+\frac{\partial^3\eta}{\partial\overline{x}^3}=0.$$

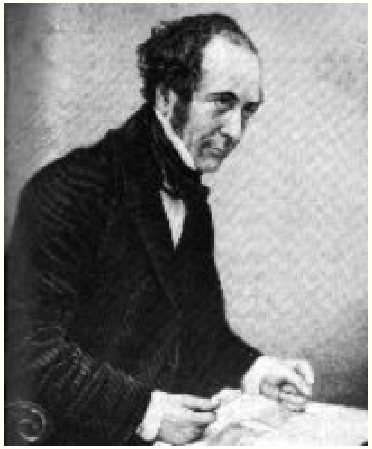
Summary

Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \; \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\, \frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

John Scott Russell and the solitary wave



Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

https://www.macs.hw.ac.uk/~chris/scott_russell.html

``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
(Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.

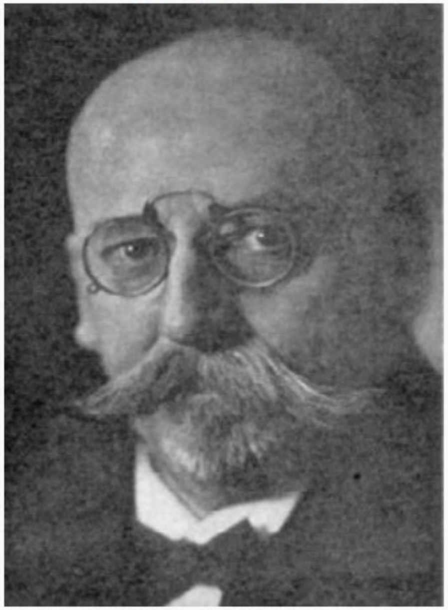
Photo of canal soliton http://www.ma.hw.ac.uk/solitons/
(link no longer active)

Diederik Korteweg

Diederik Johannes Korteweg

Born	31 March 1848 Den Bosch
Died	10 May 1941 (aged 93) Amsterdam
Nationality	Dutch
Alma mater	University of Amsterdam
Known for	Korteweg–de Vries equation, Moens–Korteweg equation[1]
	Scientific career
Fields	Mathematics
Institutions	University of Amsterdam

Gustav de Vries



Born	22 January 1866 Amsterdam
Died	16 December 1934 (aged 68)
Nationality	Dutch
Alma mater	University of Amsterdam
Known for	Korteweg–De Vries equation
	Scientific career
Fields	Mathematics