PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Notes on Lecture 35: Chap. 11 in F&W

Heat conduction

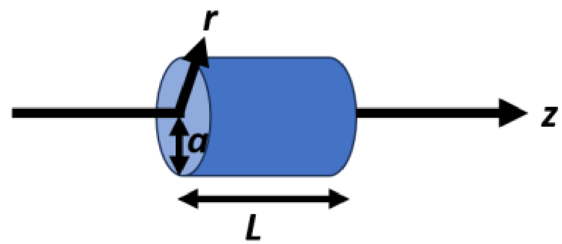
1. Basic equations
2. Boundary value problems

24	Mon, 10/23/2023	Chap. 7	Laplace transforms and complex functions	#18
25	Wed, 10/25/2023	Chap. 7	Complex integration	#19
26	Fri, 10/27/2023	Chap. 8	Wave motion in 2 dimensional membranes	#20
27	Mon, 10/30/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#21
28	Wed, 11/01/2023	Chap. 9	Motion in 3 dimensional ideal fluids	#22
29	Fri, 11/03/2023	Chap. 9	Ideal gas fluids	#23
31	Mon, 11/06/2023	Chap. 9	Traveling and standing waves in the linear approximation	#24
32	Wed, 11/08/2023	Chap. 9	Non-linear and other wave properties	Topic
33	Fri, 11/10/2023	Chap. 9	Analysis of non-linear waves and shock waves	#25
34	Mon, 11/13/2023	Chap. 10	Surface waves in fluids	#26
35	Wed, 11/15/2023	Chap. 10	Surface waves in fluids; soliton solutions	#27
36	Fri, 11/17/2023	Chap. 11	Heat conduction	#28
37	Mon, 11/20/2023	Chap. 12	Viscous effects in hydrodynamics	
	Wed, 11/22/2023	Thanksgiving		
	Fri, 11/24/2023	Thanksgiving		
	Mon, 11/27/2023		Presentations I	
	Wed, 11/29/2023		Presentations 2	
	Fri, 12/01/2023		Presentations 3	
38	Mon, 12/04/2023	Chap. 12	Viscous effects in hydrodynamics	
39	Wed, 12/06/2023		Review	
40	Fri, 12/08/2023		Review	

PHY 711 – Homework # 28

Assigned: 11/17/2023 Due: 11/20/2023

Read Chapter 11 of **Fetter and Walecka**.



1. A cylindrical solid material with cylindrical radius a and length L and thermal diffusivity κ has a time-dependent cylindrically symmetric temperature profile $T(r, z, t)$. In these cylindrical coordinates, the material is contained within $a \geq r \geq 0$ and $L \geq z \geq 0$. In the absense of external heating, the temperature profile is is well-described by the equation of heat conduction

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

Somehow for $t \leq 0$, the material is prepared so that its temperature profile is given by

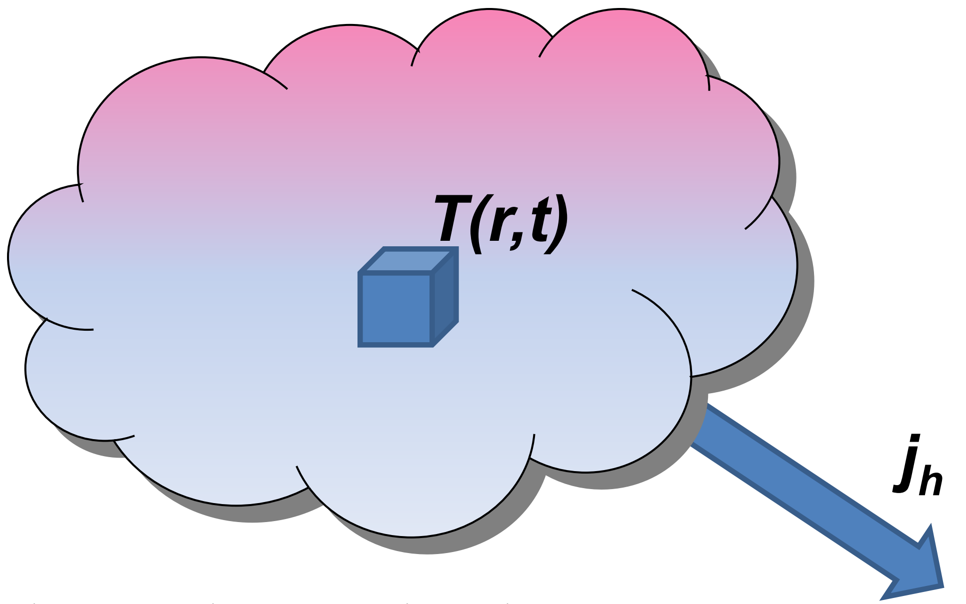
$$T(r, z, t \leq 0) = \begin{cases} 0 & \text{for} \quad r > a \quad \text{and/or} \quad z > L \\ A\cos(\pi z/L) & \text{for} \quad r \leq a \quad \text{and/or} \quad z \leq L. \end{cases}$$

Then, at $t = 0$ the cylindrical solid is placed in a thermally insulated container so that its temperature is well-described by the boundary conditions

$$\hat{\mathbf{n}} \cdot \nabla T(r, z, t) = 0$$

at all of its surfaces. Find an expression for the temperature profile of this system $T(r, z, t)$ for $t > 0$.

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Note that in this treatment we are considering a system at constant pressure *p*

Notation: Heat added to system -- $dQ = TdS$

External work done on system -- $dW = -pdV$

Internal energy -- $dE = dQ + dW = TdS - pdV$

Entropy -- dS

Enthalpy -- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \int \rho c_p d^3 r$$

More generally, note that c_p can depend on *T*; we are assuming that dependence to be trivial.

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Empirically: $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

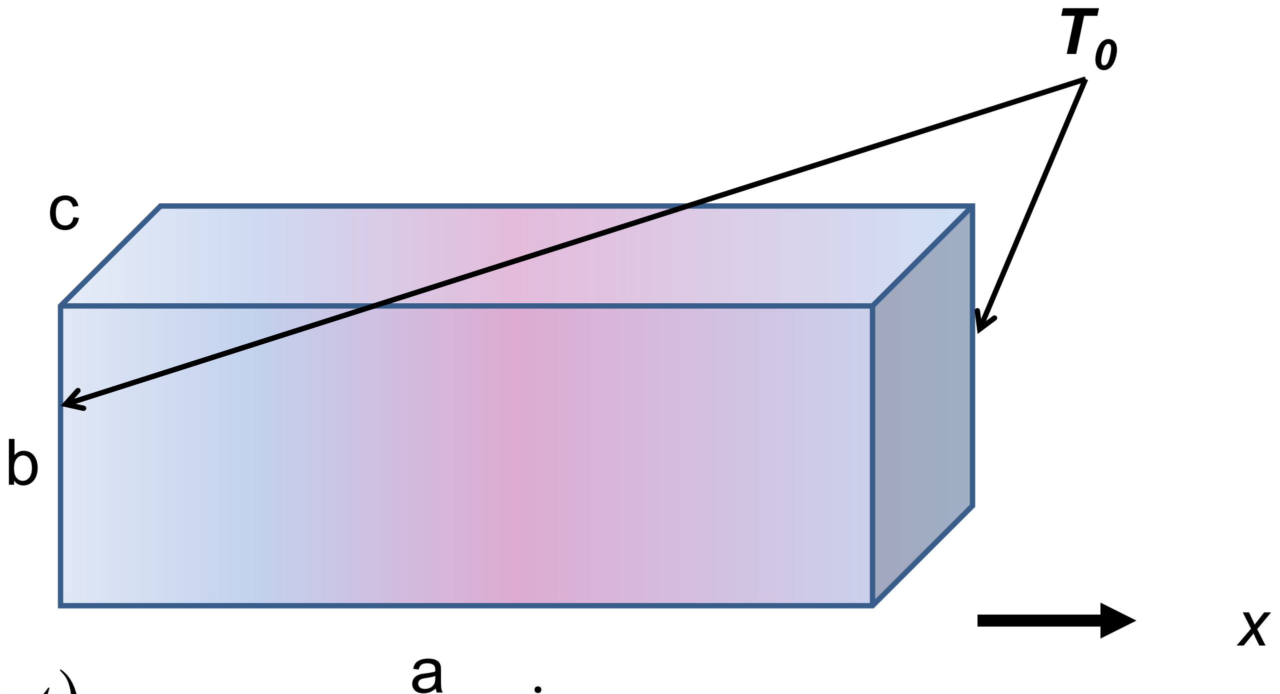
$$\kappa \equiv \frac{k_{th}}{\rho c_p}$$ thermal diffusivity

https://www.engineersedge.com/heat_transfer/thermal_diffusivity_table_13953.htm

Typical values (m^2/s)

Air	2×10^{-5}
Water	1×10^{-7}
Copper	1×10^{-4}

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

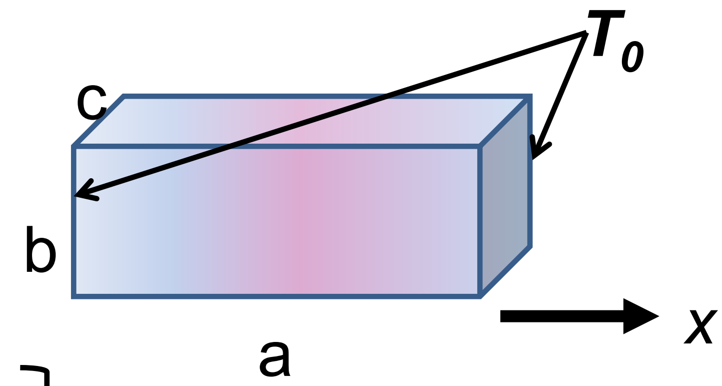
Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Have you ever encountered the following equation in other contexts and if so where?

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

Boundary value problems for heat conduction

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$



$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

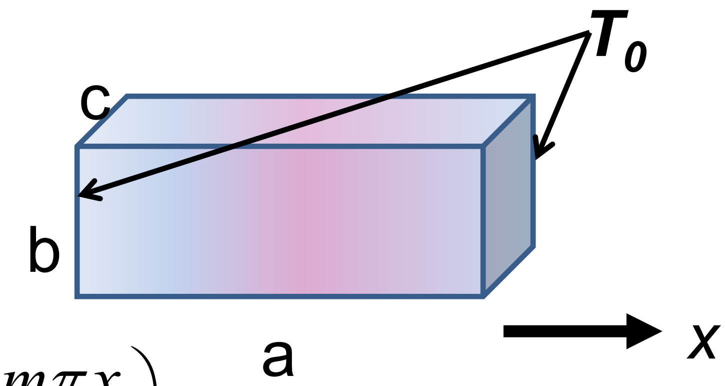
Assuming thermally insulated boundaries

Separation of varables: $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa\left(\alpha^2 + \beta^2 + \gamma^2\right) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

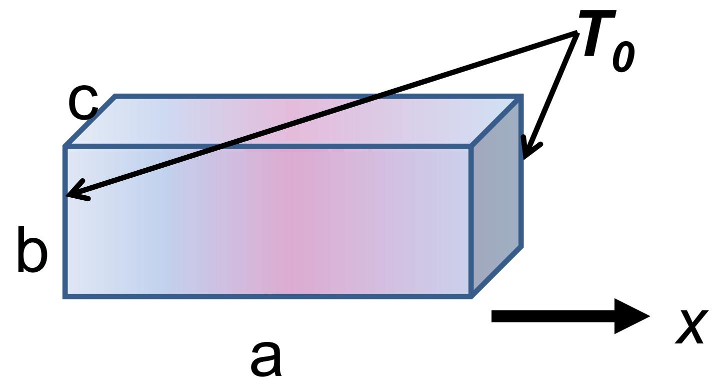
$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



Full solution:

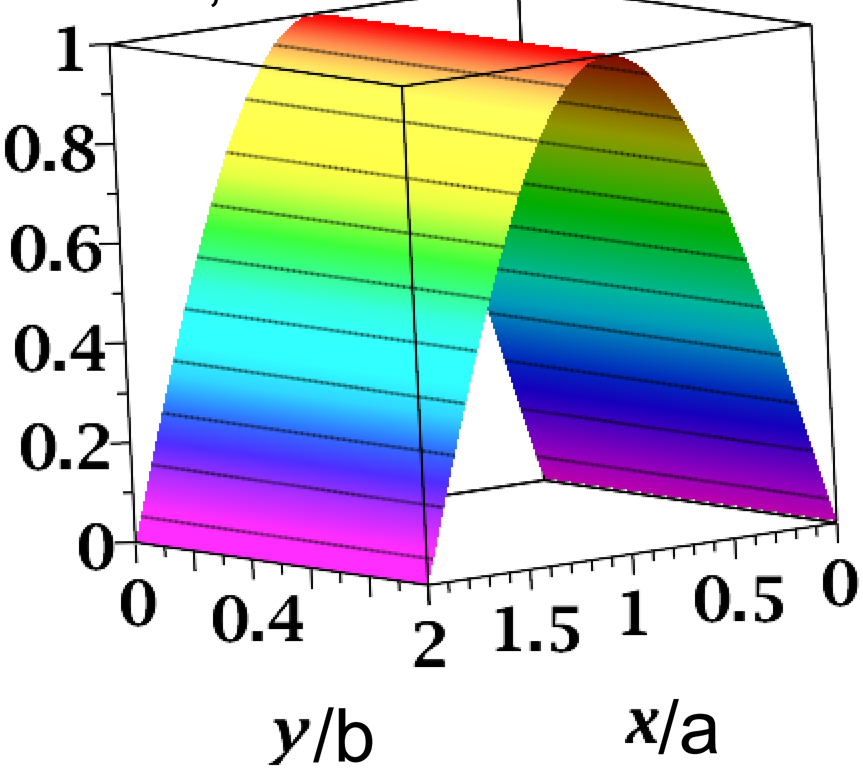
$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

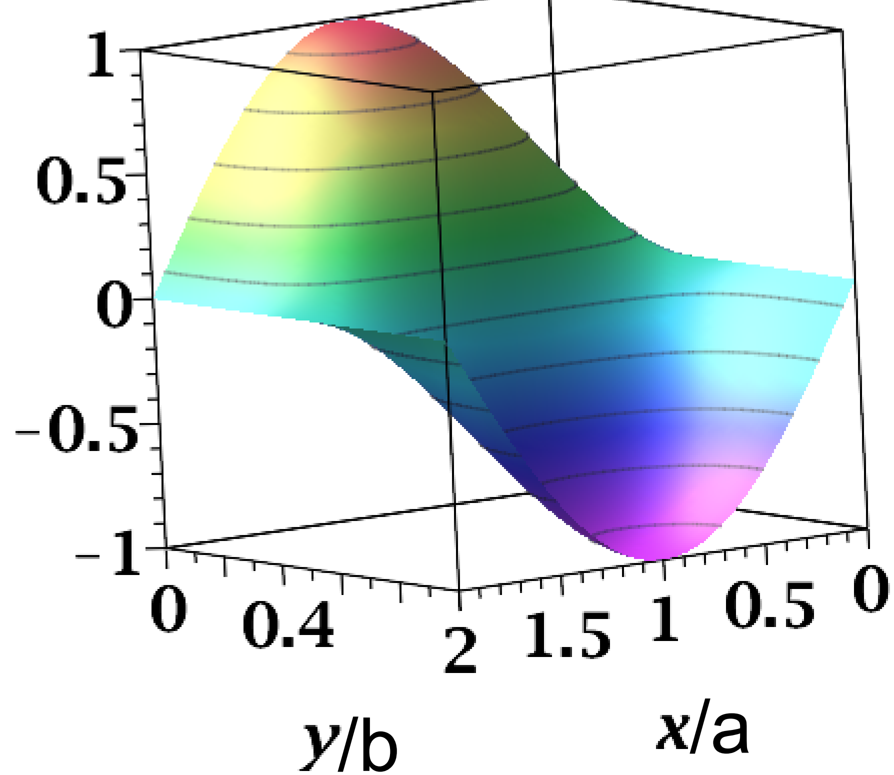
Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

At t=0,z=0



$m=1, n=0, p=0$



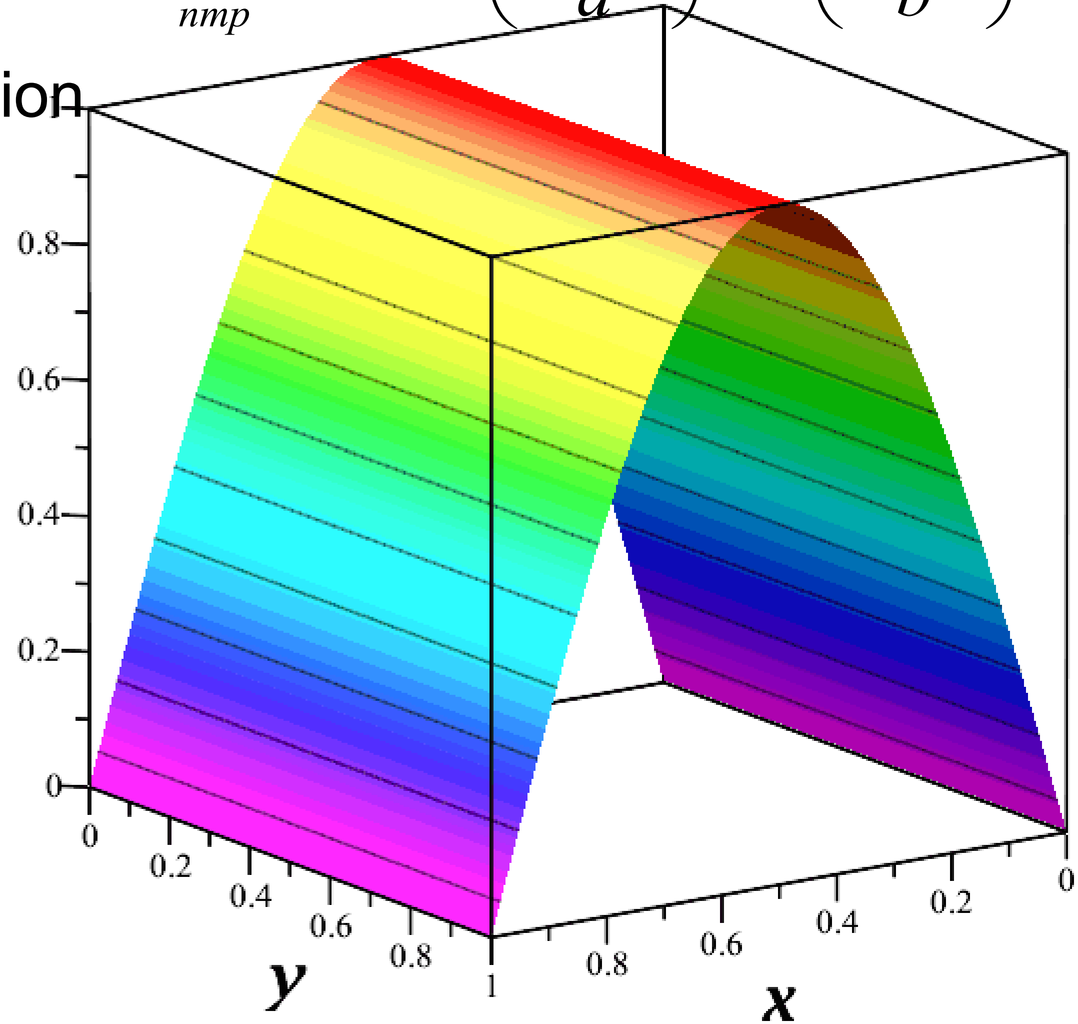
$m=1, n=1, p=0$

Full solution:

$t = 0.$

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

Time evolution
nmp=100
at *z=0*



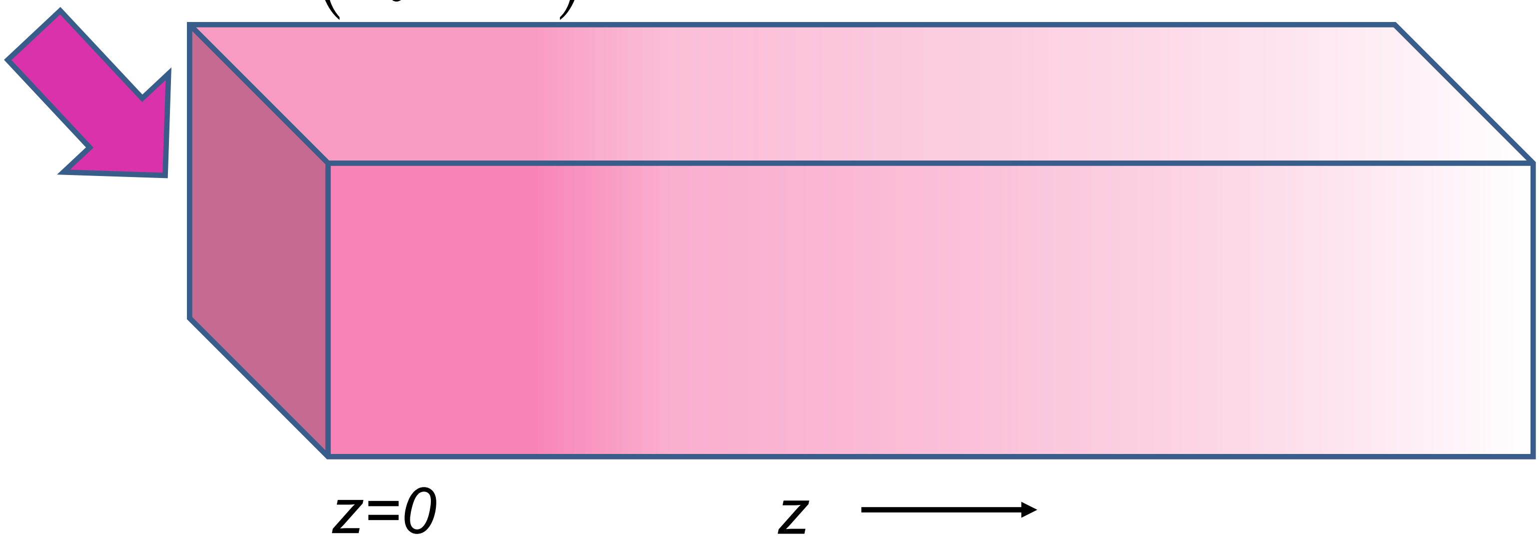
What real system could have such a temperature distribution?

Comment – While one can imagine that the boundary conditions can be readily realized, the single normal mode patterns are much harder. On the other hand, we see that the smallest values of lambda have the longest time constants.

Oscillatory thermal behavior

Here we assume that the spatial variation is along z

$$T(z=0,t) = \Re\left(T_0 e^{-i\omega t}\right)$$



$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}$$

Assume: $T(z,t) = \Re\left(f(z) e^{-i\omega t}\right)$

$$(-i\omega) f = \kappa \frac{d^2 f}{dz^2}$$

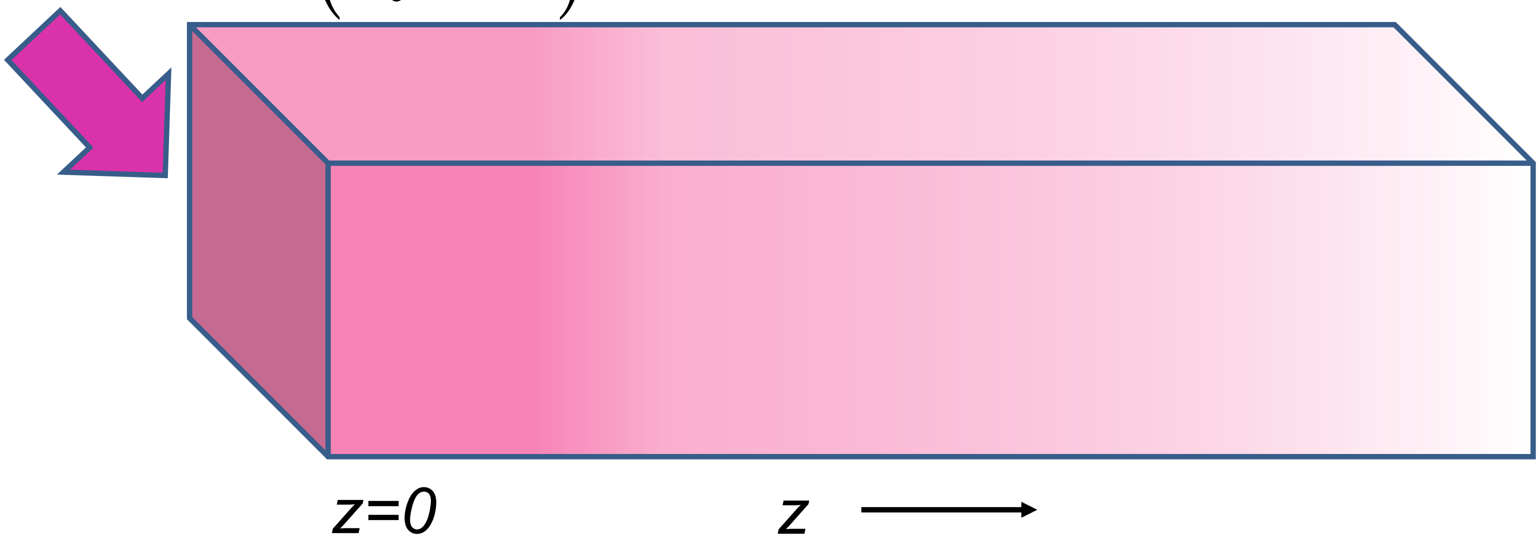
Let $f(z) = A e^{\alpha z}$

$$\alpha^2 = -\frac{i\omega}{\kappa} = e^{3i\pi/2} \frac{\omega}{\kappa}$$

$$\alpha = \pm(1-i)\sqrt{\frac{\omega}{2\kappa}}$$

Oscillatory thermal behavior -- continued

$$T(z=0,t)=\Re\left(T_0 e^{-i\omega t}\right)$$

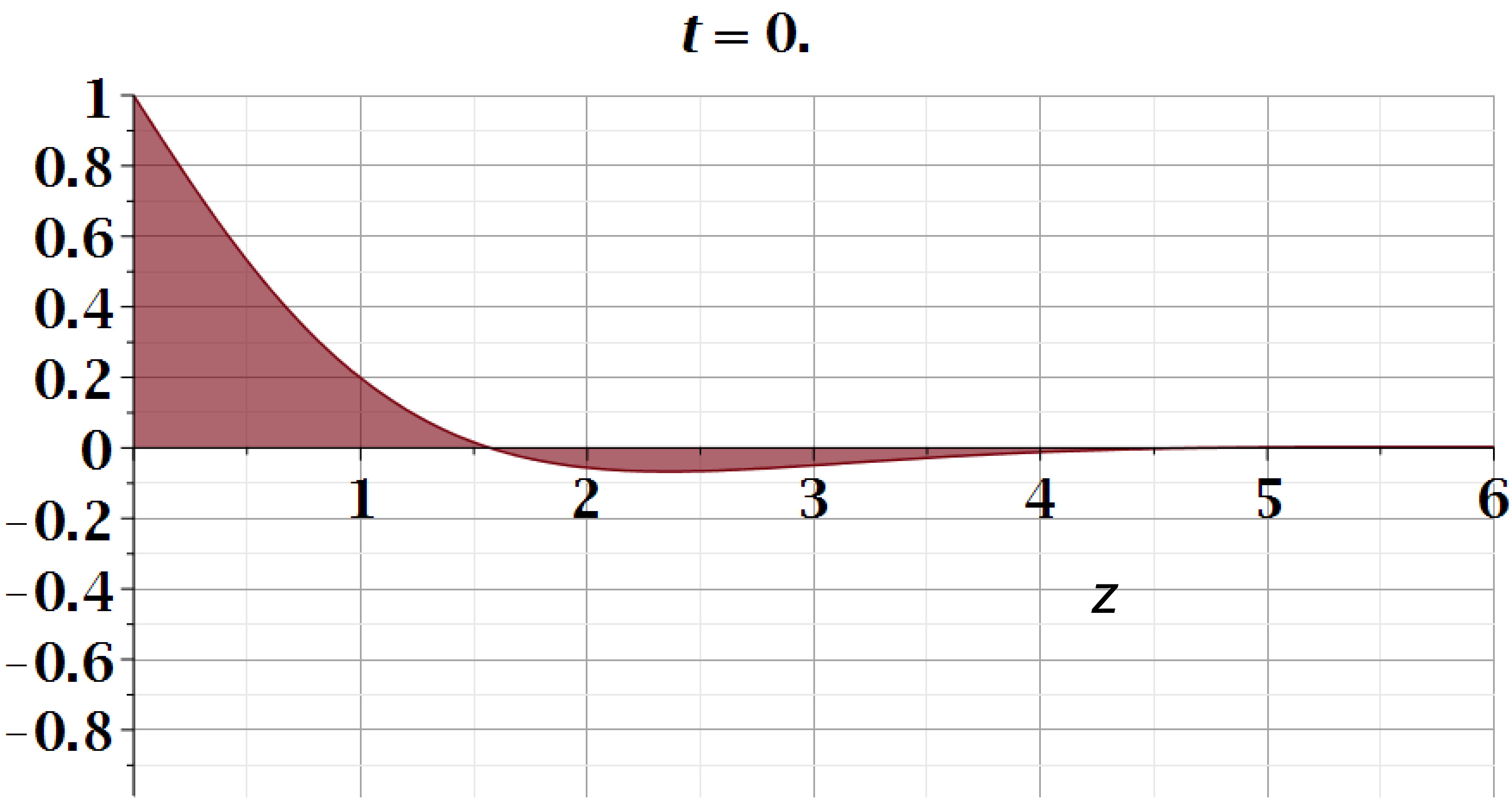


$$T(z,t)=\Re\left(A e^{\pm(1-i)z/\delta} e^{-i\omega t}\right)$$

$$\text{where} \quad \delta \equiv \sqrt{\frac{2\kappa}{\omega}}$$

$$\text{Physical solution:} \quad T(z,t)=T_0 e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t\right)$$

$$T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right)$$



Does this expression say the temperature transmits along the z axis?

Comment – In this case, our setup approximates trivial variation in the x-y plane so that all variation is along z. Given the oscillating boundary condition at z=0, the spatial form along z is a result of the form of the heat equation.

Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let}: \quad \widetilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\widetilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \widetilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$\tilde{T}(\mathbf{q},t)=\int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}}\tilde{T}(\mathbf{q},t)$$

$$\tilde{T}(\mathbf{q},t)=\tilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}}\tilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$\tilde{T}(\mathbf{q},0)=\tilde{f}(\mathbf{q})=\int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}}f(\mathbf{r})$$

$$T(\mathbf{r},t)=\int d^3r' G(\mathbf{r}-\mathbf{r}',t)T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t)\equiv\frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r' \, G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3} \int d^3q \, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},t) \Rightarrow T(z,t) \quad \text{with initial and boundary values:}$$

$$T(z,t) \equiv 0 \quad \text{for } z < 0$$

$$T(z,0) = 0 \quad \text{for } z > 0$$

$$T(0,t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution:} \quad T = T_0 \,\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

More details about the error function --
https://dlmf.nist.gov/7

§7.2(i) Error Functions

7.2.1
$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt,$$

7.2.2
$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt = 1 - \operatorname{erf} z,$$

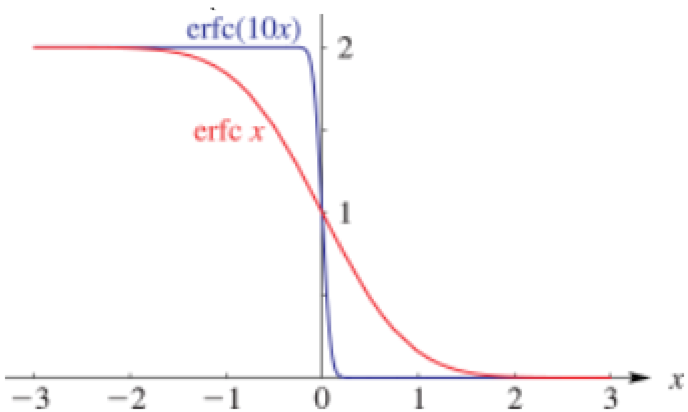


Figure 7.3.1: Complementary error functions $\operatorname{erfc} x$ and $\operatorname{erfc}(10x)$, $-3 \le x \le 3$.

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

$$\text{Solution:} \quad T = T_0 \,\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

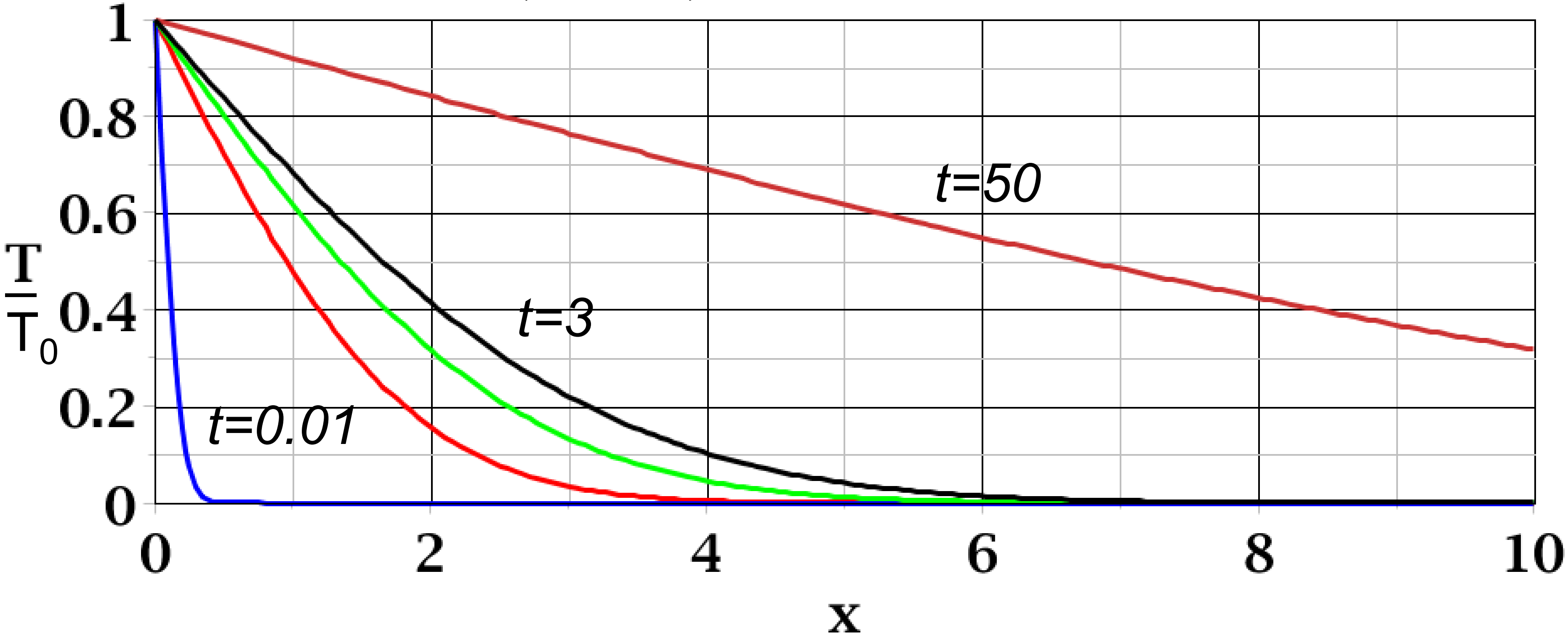
$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du$$

$$\text{Note that} \quad \frac{d\,\text{erfc}(x)}{dx} = \frac{d}{dx}\frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\frac{\partial}{\partial t}\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2}\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

$$T = T_0 \ \mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}} \right)$$



Temperature profile