PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 37
Continued discussion of viscous fluids: Chap. 12 in F & W

1. Some general comments
2. Navier-Stokes equation
3. Review of results from last time – Stokes "law"
4. Effects on linearize sound waves

35	Fri, 11/17/2023	Chap. 11	Heat conduction	#28
36	Mon, 11/20/2023	Chap. 12	Viscous effects in hydrodynamics	
	Wed, 11/22/2023	Thanksgiving		
	Fri, 11/24/2023	Thanksgiving		
	Mon, 11/27/2023		Presentations I	
	Wed, 11/29/2023		Presentations 2	
	Fri, 12/01/2023		Presentations 3	
37	Mon, 12/04/2023	Chap. 12	Viscous effects in hydrodynamics	
38	Wed, 12/06/2023		Review	
39	Fri, 12/08/2023		Review	

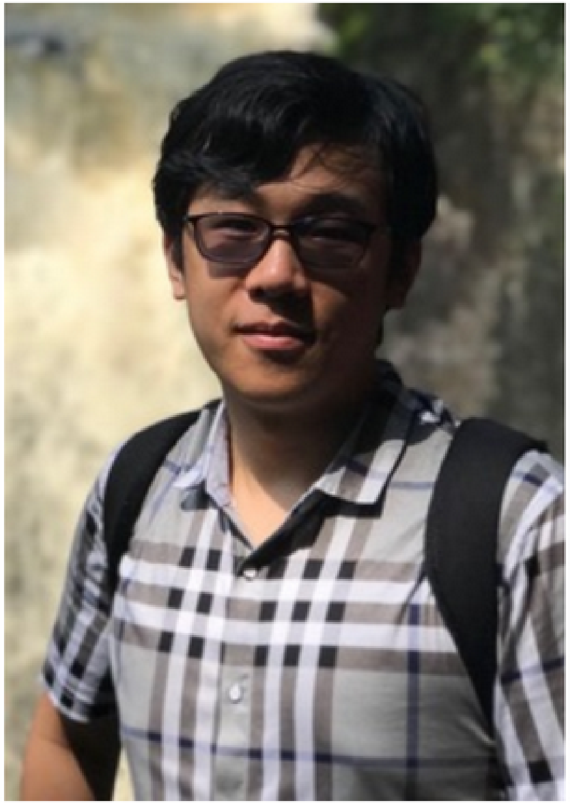
Please fill out the course evaluation form for PHY 711 in class on Friday or on your own at your leisure.

Final exam during finals week
12/11/2023-12/16/2023
(final grades due 12/20/2023 at noon)

Physics Colloquium – December 7, 2023
4-5 PM in Olin 101

Nuclear Quantum Effects: Insights from First-Principles Theory

In electronic structure theory, atomic nuclei are generally treated as classical point charges. However, there is a growing realization that the quantum-mechanical nature of light atomic nuclei like protons is essential for predicting certain properties. In recent years, this so-called nuclear quantum effect (NQE) has become an important topic in condensed matter physics and chemistry. In this talk, I will discuss how we examine different aspects of NQE by advancing first-principles electronic structure theory. I will first focus on the use of the path integral approach with first-principles molecular dynamics simulation based on density functional theory (DFT) for examining the NQE in liquid water and ionic solution. I will then discuss how multi-component DFT can be used with the nuclear electronic orbital (NEO) method for studying the coupled quantum dynamics of electrons and protons in heterogeneous matter in the context of real-time time-dependent DFT.



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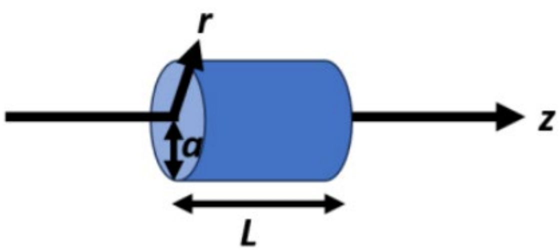
Comment on HW #28

Assigned: 11/17/2023 Due: 11/20/2023

Read Chapter 11 of **Fetter and Walecka**.



1.

A cylindrical solid material with cylindrical radius a and length L and thermal diffusivity κ has a time-dependent cylindrically symmetric temperature profile $T(r, z, t)$. In these cylindrical coordinates, the material is contained within $a \geq r \geq 0$ and $L \geq z \geq 0$. In the absense of external heating, the temperature profile is is well-described by the equation of heat conduction

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

Somehow for $t \leq 0$, the material is prepared so that its temperature profile is given by

$$T(r, z, t \leq 0) = \begin{cases} 0 & \text{for} \quad r > a \quad \text{and/or} \quad z > L \\ A\cos(\pi z/L) & \text{for} \quad r \leq a \quad \text{and/or} \quad z \leq L. \end{cases}$$

Then, at $t = 0$ the cylindrical solid is placed in a thermally insulated container so that its temperature is well-described by the boundary conditions

$$\hat{\mathbf{n}} \cdot \nabla T(r, z, t) = 0$$

at all of its surfaces. Find an expression for the temperature profile of this system $T(r, z, t)$ for $t > 0$.

The diffusion (or heat conduction) equation for the temperature profile $T(\mathbf{r},t)$:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

For cylindrical coordinates -- $T(\mathbf{r},t) = T\left(r,\varphi,z,t\right)$

and the diffusion equation takes the form:

$$\frac{\partial T\left(r,\varphi,z,t\right)}{\partial t} = \kappa \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right) T\left(r,\varphi,z,t\right)$$

Partial differential equation:

$$\frac{\partial T(r,\varphi,z,t)}{\partial t} = \kappa\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) T(r,\varphi,z,t)$$

Assume separable form: $T(r,\varphi,z,t) = R(r)\Phi(\varphi)Z(z)f(t)$

In this particular case, the φ dependence is trivial, so that it is reasonable to assume that $T(r,\varphi,z,t) = T(r,z,t) = R(r)Z(z)f(t)$

Then $\dfrac{\partial T(r,z,t)}{\partial t} = R(r)Z(z)\dfrac{df(t)}{dt}$

$$\nabla^2 T(r,z,t) = Z(z)f(t)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)R(r) + R(r)f(t)\frac{d^2 Z(z)}{dz^2}$$

The separable form works best, when each factor solves a differential eigenvalue problem.

Suppose $\frac{df(t)}{dt} = -\lambda f(t)$ and $\frac{d^2 Z(z)}{dz^2} = -\alpha Z(z)$

Then $R(r)$ must solve the equation:

$$-\lambda R(r) = \kappa\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \alpha\right)R(r)$$

or $$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left(\frac{\lambda}{\kappa} - \alpha\right)\right)R(r) = 0$$

Solution of ordinary differential equation:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\left(\frac{\lambda}{\kappa}-\alpha\right)\right)R(r)=0$$

Recall that the regular solution of the Bessel equation of order 0 is a solution of the differential equation:

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1\right)J_0(x)=0$$

Therefore, $R(r)=CJ_0(\mu r)$ where C is a constant and

$$\mu^2=\frac{\lambda}{\kappa}-\alpha$$

More generally, multiple solutions μ_n may be viable, in which case the solution has the form $R(r)=\sum_n C_n J_0(\mu_n r)$.

Full general solution:

$$T(r,z,t) = \sum_n C_n J_0(\mu_n r) \cos\left(\frac{\pi z}{L}\right) e^{-\lambda_n t}$$

where $\lambda_n = \kappa\left(\mu_n^2 + \frac{\pi^2}{L^2}\right)$

Somehow for $t \leq 0$, the material is prepared so that its temperature profile is given by

$$T(r,z,t \leq 0) = \begin{cases} 0 & \text{for} \quad r > a \quad \text{and/or} \quad z > L \\ A\cos(\pi z/L) & \text{for} \quad r \leq a \quad \text{and [illegible]} \quad z \leq L. \end{cases}$$

Then, at $t = 0$ the cylindrical solid is placed in a thermally insulated container so that its temperature is well-described by the boundary conditions

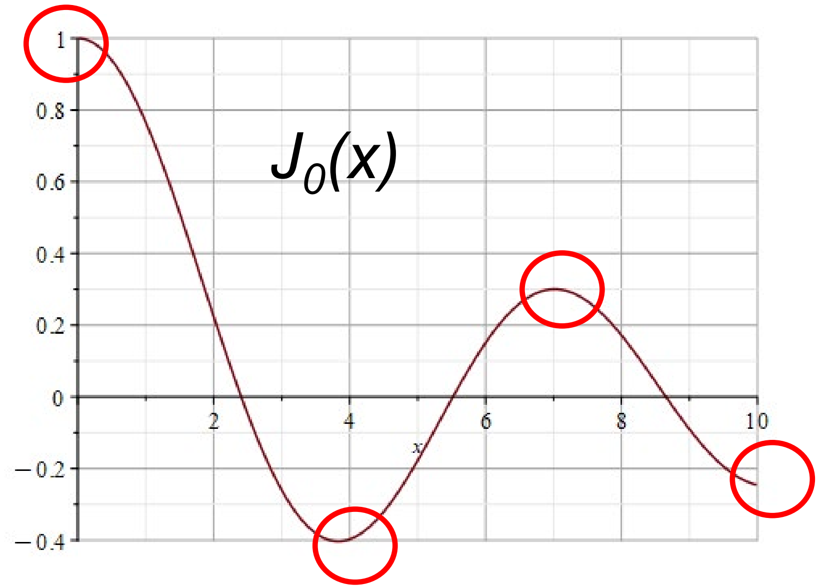
$$\hat{\mathbf{n}} \cdot \nabla T(r,z,t) = 0$$

at all of its surfaces. Find an expression for the temperature profile of this system $T(r,z,t)$ for $t > 0$.

Finishing up --

$$T(r,z,t) = \sum_n C_n J_0(\mu_n r)\cos\left(\frac{\pi z}{L}\right) e^{-\lambda_n t} \text{ with } \lambda_n = \kappa\left(\mu_n^2 + \frac{\pi^2}{L^2}\right)$$

Satisfies the differential equation, but does not satisfy boundary and initial conditions



Need to find μ_n and C_n.

For boundary value at $r = a$

$$\left.\frac{dJ_0(\mu_n r)}{dr}\right|_{r=a} = 0$$

Define $\frac{dJ_0(x_n')}{dx} = 0$

$$\mu_n = \frac{x_n'}{a}$$

Note that the functions $J_0(\mu_n r)$ form a set of orthogonal functions over the range $0 \le r \le a$.

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\mu_n^2\right)J_0(\mu_n r)=0$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\mu_m^2\right)J_0(\mu_m r)=0$$

$$J_0(\mu_m r)\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)J_0(\mu_n r)-J_0(\mu_n r)\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)J_0(\mu_m r)=$$

$$\left(\mu_m^2-\mu_n^2\right)J_0(\mu_n r)J_0(\mu_m r)$$

If $\mu_n=\mu_m$, then the equality is trivial. If $\mu_n \ne \mu_m$, the integrating both sides of the equation $0 \le r \le a$ implies that

$$\int_0^a dr\ rJ_0(\mu_n r)J_0(\mu_m r)=0$$

Full general solution:

$$T(r,z,t)=\sum_n C_n J_0(\mu_n r)\cos\left(\frac{\pi z}{L}\right)e^{-\lambda_n t}$$

where $\lambda_n = \kappa\left(\mu_n^2+\frac{\pi^2}{L^2}\right)$

and where $C_n = A\frac{\int_0^a dr\ rJ_0(\mu_n r)}{\int_0^a dr\ rJ_0^2(\mu_n r)}$

Back to discussion of fluids with the inclusion of viscosity --

Equations for motion of non-viscous fluid --

Modified Newton-Euler equation in terms of fluid momentum:

$$\frac{\partial(\rho \mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j \mathbf{v})}{\partial x_j}=\rho \mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j \mathbf{v})}{\partial x_j}+\nabla p=\rho \mathbf{f}_{applied}$$

Fluid momentum: $\rho \mathbf{v}$

Stress tensor: $T_{ij} \equiv \rho v_i v_j + p\delta_{ij}$

i^{th} component of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

Now consider the effects of viscosity

In terms of stress tensor:

$$T_{ij} = T_{ij}^{\text{ideal}} + T_{ij}^{\text{viscous}}$$

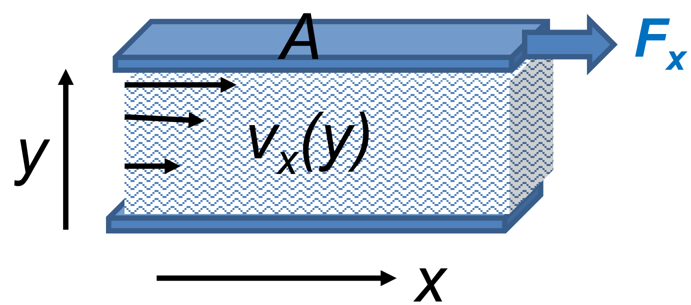
$$T_{ij}^{\text{ideal}} = \rho v_i v_j + p\delta_{ij} = T_{ji}^{\text{ideal}}$$

As an example of a viscous effect, consider --

Newton's "law" of viscosity

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}$$

material dependent parameter



Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{drag}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

viscosity

bulk (or dilational) viscosity

Total stress tensor: $T_{kl} = T_{kl}^{ideal} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p\delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_i v_j)}{\partial x_j}=\rho f_i-\frac{\partial p}{\partial x_i}+\eta\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2}+\left(\zeta+\frac{1}{3}\eta\right)\sum_{j=1}^{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j}$$

Continuity equation

$$\frac{\partial\rho}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j)}{\partial x_j}=0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

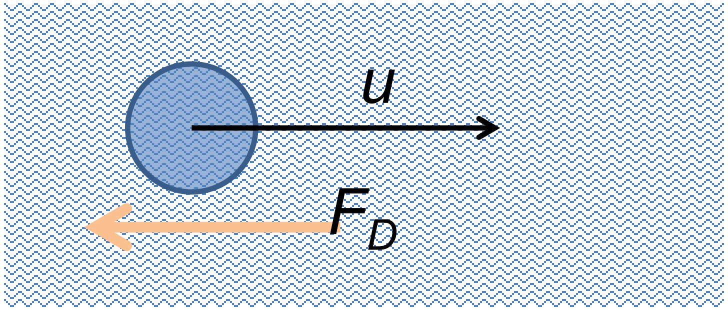
Typical viscosities at 20° C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00×10^{-6}	1×10^{-3}
Air	14.9×10^{-6}	0.018×10^{-3}
Ethyl alcohol	1.52×10^{-6}	1.2×10^{-3}
Glycerine	1183×10^{-6}	1490×10^{-3}

More discussion of viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta\left(6\pi R u\right)$$



"Derivation"

1. Consider the general effects of viscosity on fluid equations
2. Solve the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow
4. Note that solution is special to the sphere geometry.

Additional effects of viscosity – allowing for changes in entropy
-- particularly in the case of sound waves in air

$$p(\rho, s) = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \nabla\right)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \mathbf{v}\right)$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\right) = 0$$

Newton-Euler equations for viscous fluids – effects on sound

Without viscosity terms:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v} \qquad \mathbf{f} = 0 \qquad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv p_0 + c^2\delta\rho$$

Linearized equations: $\dfrac{\partial \delta\mathbf{v}}{\partial t} = -\dfrac{c^2}{\rho_0}\nabla\delta\rho \qquad \dfrac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Sound waves without viscosity -- continued

Linearized equations: $\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0} \nabla \delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta \mathbf{v}) = 0$

Let $\quad \delta\mathbf{v} \equiv \delta\mathbf{v}_0 \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0 \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0} \nabla \delta\rho \qquad \Rightarrow \omega \delta \mathbf{v}_0 = \frac{c^2 \delta\rho_0}{\rho_0} \mathbf{k}$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta \mathbf{v}) = 0 \qquad \Rightarrow -\omega \delta\rho_0 + \rho_0 \mathbf{k} \cdot \delta \mathbf{v}_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \qquad \frac{\delta\rho_0}{\rho_0} = \frac{\hat{\mathbf{k}} \cdot \delta \mathbf{v}_0}{c}$$

➔Pure longitudinal harmonic wave solutions

Newton-Euler equations for viscous fluids – effects on sound

Recall full equations:

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad$ $\mathbf{f}=0$ $\quad$ $\rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

where $c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$

viscosity causes heat transfer

Newton-Euler equations for viscous fluids – effects on sound

Note that pressure now depends both on density and entropy so that entropy must be coupled into the equations

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \qquad\qquad \rho T\frac{\partial s}{\partial t}=k_{th}\nabla^2 T$$

Assume: $\mathbf{v}=0+\delta\mathbf{v}$ $\qquad \mathbf{f}=0 \qquad \rho=\rho_0+\delta\rho$

$$p=p_0+\delta p=p_0+c^2\delta\rho+\left(\frac{\partial p}{\partial s}\right)_\rho\delta s \quad \text{where } c^2\equiv\left(\frac{\partial p}{\partial \rho}\right)_s$$

$$T=T_0+\delta T=T_0+\left(\frac{\partial T}{\partial \rho}\right)_s\delta\rho+\left(\frac{\partial T}{\partial s}\right)_\rho\delta s$$

$$s=s_0+\delta s$$

Newton-Euler equations for viscous fluids – linearized equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\Rightarrow \frac{\partial \delta\mathbf{v}}{\partial t} = \underbrace{-\frac{1}{\rho_0}\nabla\delta p}_{} + \frac{\eta}{\rho}\nabla^2\delta\mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\mathbf{v})$$

$$-\frac{1}{\rho_0}\left\{\left(\frac{\partial p}{\partial \rho}\right)_s \nabla\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \nabla\delta s\right\} = -\frac{c^2}{\rho_0}\nabla\delta\rho - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \nabla\delta s$$

Digression -- from the first law of thermodynamics:

$$d\epsilon = Tds + \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial}{\partial\rho}\left(\frac{\partial\epsilon}{\partial s}\right)_\rho\right)_s = \left(\frac{\partial T}{\partial\rho}\right)_s \quad\Leftrightarrow\quad \left(\frac{\partial}{\partial s}\left(\frac{\partial\epsilon}{\partial\rho}\right)_s\right)_\rho = \left(\frac{\partial p/\rho^2}{\partial s}\right)_\rho \approx \frac{1}{\rho_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho$$

Newton-Euler equations for viscous fluids – linearized equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0$$

$$\Rightarrow \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v} \right) = 0$$

$$\rho T \frac{\partial s}{\partial t} = k_{th} \nabla^2 T$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \frac{k_{th}}{\rho_0 T_0} \left(\left(\frac{\partial T}{\partial s} \right)_\rho \nabla^2 \delta s + \left(\frac{\partial T}{\partial \rho} \right)_s \nabla^2 \delta\rho \right)$$

Further relationships:

$$\left(\frac{\partial T}{\partial s} \right)_\rho \approx \frac{T_0}{c_v} \qquad \kappa = \frac{k_{th}}{\rho c_p}$$

heat capacity at constant volume

Newton-Euler equations for viscous fluids – linearized equations

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \left(\gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho \right) \quad \text{where } \gamma \equiv \frac{c_p}{c_v}$$

Newton-Euler equations for viscous fluids – effects on sound

Linearized equations (with the help of various thermodynamic relationships):

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

Here: $\gamma = \frac{c_p}{c_v}$ $\qquad \kappa = \frac{k_{th}}{c_p \rho_0}$

Linearized hydrodynamic equations

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

It can be shown that

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \frac{Tc^2\beta}{\rho c_p} \qquad \text{where} \qquad \beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \quad \text{(thermal expansion)}$$

Let $\delta \mathbf{v} \equiv \delta \mathbf{v}_0 \ e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ $\quad \delta\rho \equiv \delta\rho_0 \ e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ $\quad \delta s \equiv \delta s_0 \ e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

Linearized hydrodynamic equations; plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

In the absense of thermal expansion, $\beta = 0$

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0$$

➔Entropy and mechanical modes are independent

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega \delta \mathbf{v}_0 = \frac{c^2 \delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0 \beta c^2}{c_p}\delta s_0 \mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta \mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0 \; \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega \delta s_0 = -i\gamma\kappa k^2 \delta s_0 - \frac{i\kappa\beta c^2}{\rho_0} k^2 \delta\rho_0$$

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k} \neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0 \beta c^2 k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

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$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Approximate solution: $\quad k = \frac{\omega}{c} + i\alpha$

where $\quad \alpha \approx \frac{\omega^2}{2c^3\rho_0}\left(\frac{4}{3}\eta+\zeta\right) + \frac{\kappa T_0 \beta^2 \omega^2}{2c_p c}$

$$\delta\rho = \delta\rho_0 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}} e^{i\frac{\omega}{c}\left(\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}$$

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$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Transverse modes $(\delta\mathbf{v}\cdot\mathbf{k} = 0)$:

$$\delta\rho_0 = 0 \qquad \delta s_0 = 0$$

$$\left(\omega + \frac{i\eta k^2}{\rho_0}\right)\left(\delta\mathbf{v}\times\ \mathbf{k}\right) = 0 \qquad k = \pm\left(\frac{i\omega\rho_0}{\eta}\right)^{1/2}$$