PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion of Lecture 4 – Chap. 3 F & W

Calculus of variation applied to classical mechanics

1. Hamilton's principle
2. D'Alembert's principle
3. Lagrange's equation in generalized coordinates

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023			
6	Fri, 9/08/2023			

PHY 711 -- Assignment #4

Assigned: 9/04/2023 Due: 9/11/2023

Continue reading Chapter 3, in **Fetter & Walecka**.

Consider a point particle of mass *m* moving (only) along the x axis according to a force $F_x = -K\,x$, where *K* is a positive constant. The particle trajectory as a function of time, $x(t)$, has the initial value $x(t=0)=C$, where *C* denotes a given length, and its initial velocity is 0.

a. Write down and solve Newton's second law for this system, finding the form of the trajectory $x(t)$.
b. Now write down the Lagrangian for this system and solve the Euler-Lagrange equations for the trajectory $x(t)$. How does your answer compare with (a)?

Comment – on this week's schedule

Normal for PHY 711

For the Colloquium, the originally scheduled colloquium for this week has been postponed until Dec. 7. We are thinking of just having the usual reception in the lobby instead…

Your questions –

From **Athul**:

1.I think I might be missing something but in slide 19, on the expansion of ma.ds how the second derivative of x becomes a first derivative? Also,
in generalized coordinates, the claim you made on the same slide, can you please explain the physical significance of it?
2.What is the relationship between generalized coordinates and degrees of freedom in a mechanical system?
3. Can we use generalized coordinates to represent the system near the boundaries of a system with many constraints?
4. I don't know if it make sense but when it comes to random walks can we define the generalized coordinates for a particle's position after a certain number of steps?

Summary of equations from calculus of variation --

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx \qquad\qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\right] = 0$$

or equivalently: $\frac{d}{dx}\left(f - \frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$

Application to particle dynamics

$x \to t$ (time)

$y \to q$ (generalized coordinate)

$f \to L$ (Lagrangian)

$I \to A$ or S (action)

Denote: $\dot{q} \equiv \frac{dq}{dt}$

$$S = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt$$

Application to particle dynamics

Hamilton's principle states that the dynamical trajectory of a system is given by the path that extremizes the action integral

$$S = \int_{t_1}^{t_2} L\left(\{q,\dot{q}\};t\right)dt \equiv \int_{t_1}^{t_2} L\left(\left\{y,\frac{dy}{dt}\right\};t\right)dt$$

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a*=-*g*.

Newton's formulation: $m\frac{d^2y}{dt^2} = -mg$

Resultant trajectory: $y(t) = y_i + v_i t - \frac{1}{2}gt^2$

Lagrangian for this case:

$$L = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Note that we have not yet justified this, but let us try it out.

http://www-history.mcs.st-and.ac.uk/Biographies/Hamilton.html

Sir William Rowan Hamilton

Wednesday, September 11th, 2013

Tribute to Sir William Hamilton (1805–1865)

Hello and welcome! This page is dedicated to the life and work of Sir William Rowan Hamilton.

William Rowan Hamilton was Ireland's greatest scientist. He was an mathematician, physicist, and astronomer and made important works in optics, dynamics, and algebra.

His contribution in dynamics plays a important role in the later developed quantum mechanics. His name was perpetuated in one of the fundamental concepts in quantum mechanics, called "Hamiltonian".

The Discovery of Quaternions is probably is his most familiar invention today.

2005 was the Hamilton Year, celebrating his 200th birthday. The year was dedicated to celebrate Irish Science. 2005 was called the Einstein year also, reminding of three great papers of the year 1905. So UNESCO designated 2005 to the World Year of Physics

Thanks for visiting this site! Please enjoy your stay while browsing through the pages.

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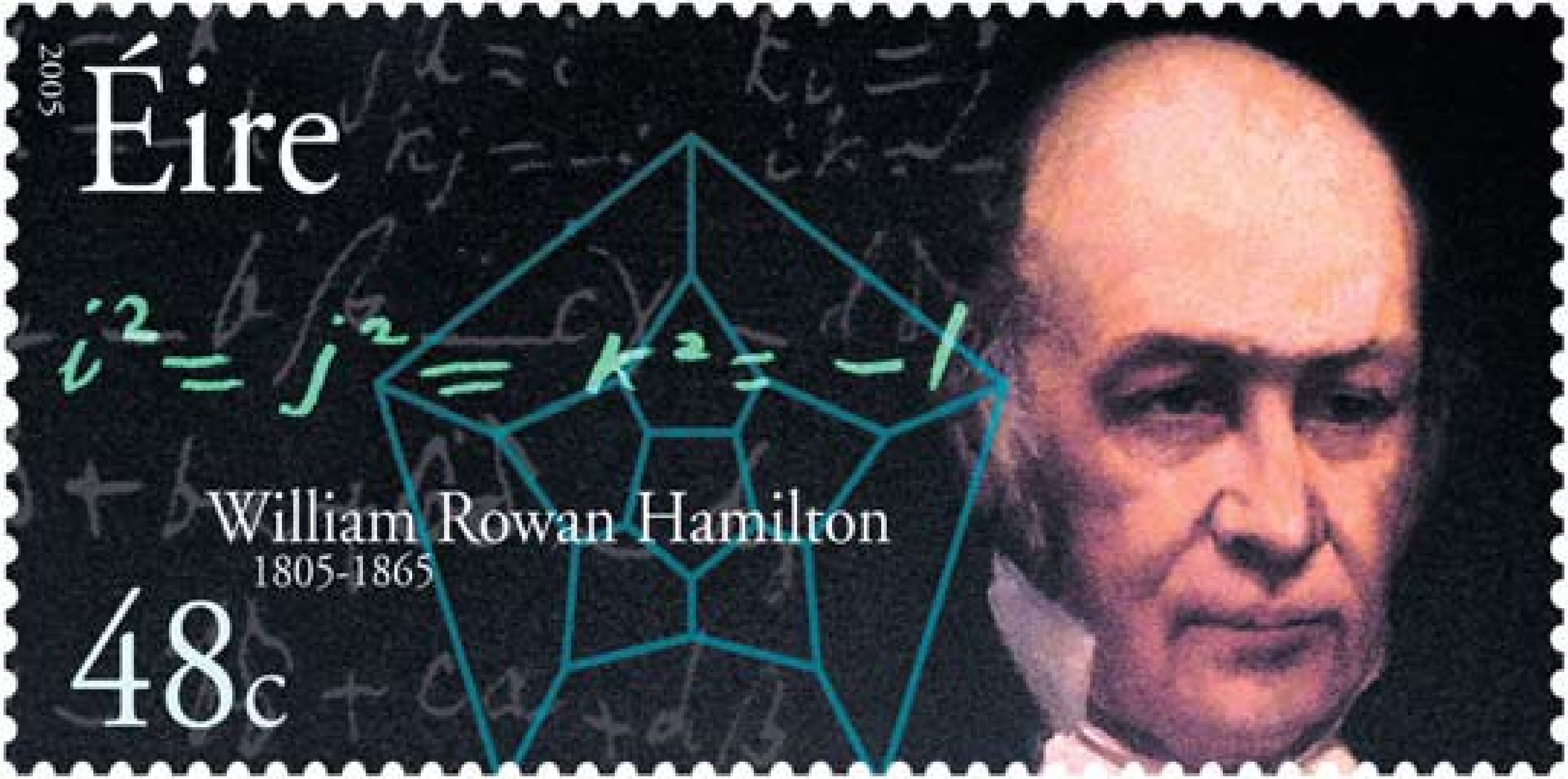


https://irishpostalheritagegpo.wordpress.com/2017/06/08/william-rowan-hamilton-irish-mathematician-and-scientist/

Now consider the Lagrangian defined to be :

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy

Potential energy

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) dt \quad \text{is minimized for physical } y(t):$$

Condition for minimizing the action in example:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler-Lagrange relations:

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{dy}{dt} = -g \qquad y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Perhaps looks familiar?

Check:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0, \;\; y_i = h \equiv \frac{1}{2} gT^2; \quad t_f = T, \;\; y_f = 0$

Trial trajectories: $y_1(t) = \frac{1}{2} gT^2 \left(1 - t/T\right) = h - \frac{1}{2} gTt$

$$y_2(t) = \frac{1}{2} gT^2 \left(1 - t^2/T^2\right) = h - \frac{1}{2} gt^2$$

$$y_3(t) = \frac{1}{2} gT^2 \left(1 - t^3/T^3\right) = h - \frac{1}{2} gt^3/T$$

Maple says:

$$S_1 = -0.125 mg^2 T^3$$

$$S_2 = -0.167 mg^2 T^3$$

$$S_3 = -0.150 mg^2 T^3$$

Note that, showing that our construction is consistent with Newton's laws **is not a proof**.   You will get the chance to consider another example to check if that works (or not) as well.

Jean d'Alembert 1717-1783
French mathematician and philosopher

"Deriving" Lagrangian mechanics from Newton's laws.

The Lagrangian function is:

$$L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right) \equiv T - U \quad q_i(t) \text{ are generalized coordinates}$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} L\left(\left\{\{q_i(t)\},\left\{\frac{dq_i}{dt}\right\}\right\},t\right) dt \quad \text{is minimized for physical } q_i(t):$$

Digression -- notion of generalized coordinates

Referenced to cartesian coordinates: $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$

Cylindrical coordinates

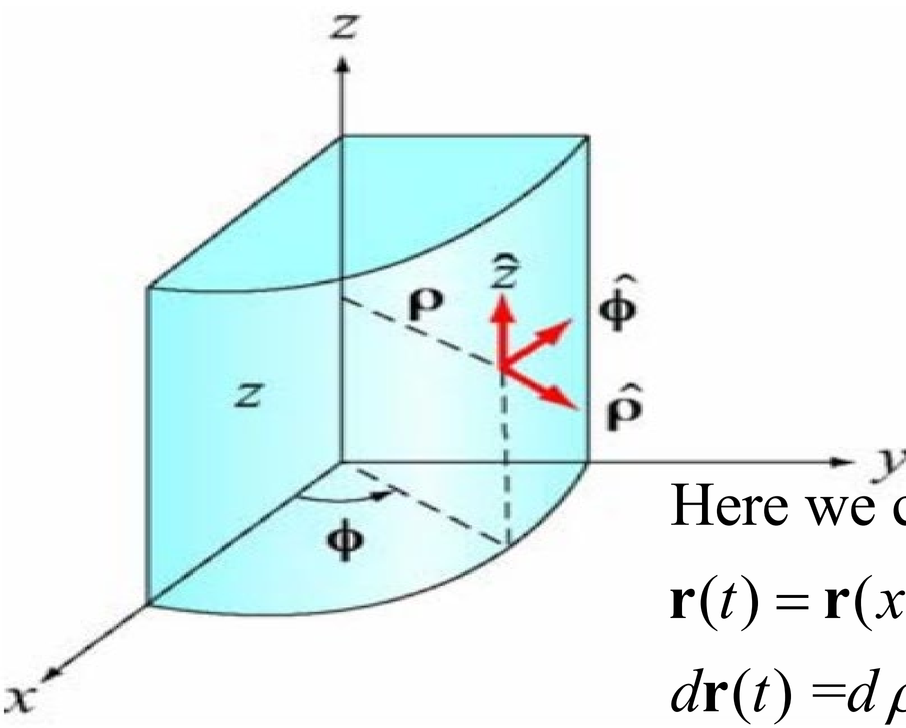


Figure B.2.4 Cylindrical coordinates

$$x = \rho\cos\phi \equiv x(\rho,\phi)$$

$$y = \rho\sin\phi \equiv y(\rho,\phi)$$

$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \arctan(y/x)$$

$$z = z$$

Here we can write

$$\mathbf{r}(t) = \mathbf{r}(x(t), y(t), z(t)) = \mathbf{r}(\rho(t), \phi(t), z(t))$$

$$d\mathbf{r}(t) = d\rho(t)\hat{\boldsymbol{\rho}}(t) + \rho(t)d\phi(t)\hat{\boldsymbol{\varphi}}(t) + z(t)\hat{\mathbf{z}}$$

(Figure taken from 8.02 handout from MIT.)

Spherical coordinates

$$x = r\sin\theta\cos\phi \equiv x(r,\theta,\phi)$$
$$y = r\sin\theta\sin\phi \equiv y(r,\theta,\phi)$$
$$z = r\cos\theta \equiv z(r,\theta,\phi)$$
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$
$$\phi = \arctan(y / x)$$

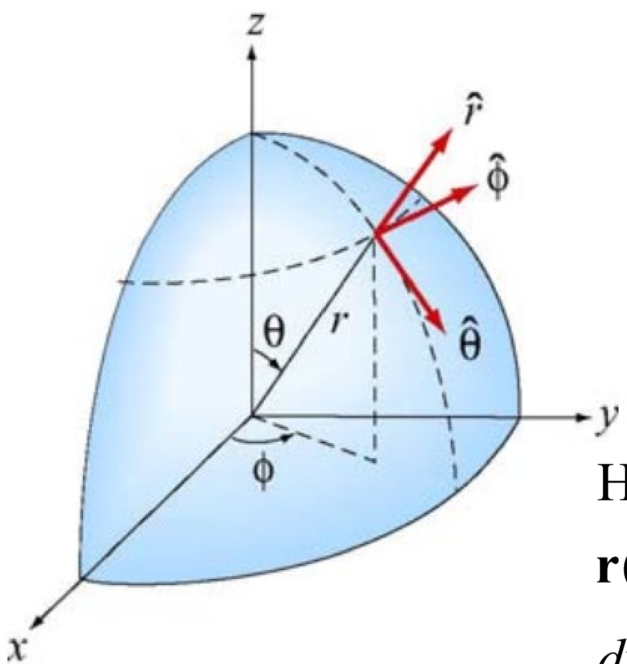


Figure B.3.1 Spherical coordinates

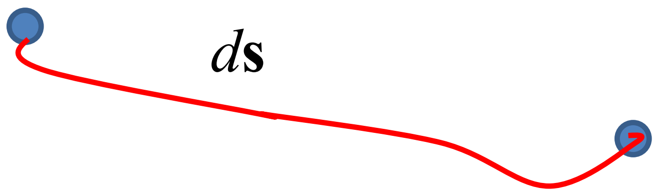
Here we can write

$$\mathbf{r}(t) = \mathbf{r}(x(t), y(t), z(t)) = \mathbf{r}(r(t), \theta(t), \phi(t))$$
$$d\mathbf{r}(t) = dr(t)\hat{\mathbf{r}}(t) + r(t)d\theta(t)\hat{\boldsymbol{\theta}}(t) + r(t)\sin\theta(t)d\phi(t)\hat{\boldsymbol{\varphi}}(t)$$

(Figure taken from 8.02 handout from MIT.)

Note that the notion of "generalized coordinates" could be a single coordinate for a single particle in one dimension, d coordinates for a single particle in d dimensions, or dN coordinates for N particles in d dimensions. Cartesian coordinates are also "generalized coordinates".

D'Alembert's principle:



Note that: $d\mathbf{s} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

Generalized coordinates:

$q_\sigma(\{x_i\}) \leftrightarrow x_i(\{q_\sigma\})$

Note that

$q_\sigma(t)$ can be $x(t), \theta(t)$....

$$dx \equiv dx_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

Newton's laws:

$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a}) \cdot d\mathbf{s} = 0$

This is D'Alembert's principle.

$$\mathbf{F} \cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

For a conservative force: $F_i = -\dfrac{\partial U}{\partial x_i}$

$$\mathbf{F} \cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma} \delta q_\sigma$$

You might ask why we need "generalized" coordinates. In fact, Cartesian coordinates are often just fine, but using the more flexible possibilities reveals important aspects of the formalism. Cartesian coordinates are a special case of generalized coordinates.

Comment on notation -- $d\mathbf{s} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$

For convenience let $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1, \;\; \hat{\mathbf{y}} = \hat{\mathbf{x}}_2, \;\; \hat{\mathbf{z}} = \hat{\mathbf{x}}_3$

Then $\mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{3} F_i dx_i$

But now we want to change coordinates $q_\sigma\left(\{x_i\}\right) \leftrightarrow x_i(\{q_\sigma\})$

$$dx_i = \sum_\sigma \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma \qquad\qquad \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{3} F_i dx_i = \sum_\sigma \sum_{i=1}^{3} F_i \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

Summary up to now --

$$\mathbf{F}\cdot d\mathbf{s} = \sum_{\sigma}\sum_{i} F_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

For a conservative force: $F_i = -\frac{\partial U}{\partial x_i}$

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_{\sigma}\sum_{i} \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_{\sigma}\frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

Generalized coordinates:

$$q_\sigma\left(\{x_i\}\right)$$

$$x \Leftrightarrow x_1$$
$$y \Leftrightarrow x_2$$
$$z \Leftrightarrow x_3$$

Newton's laws:

$$\mathbf{F} - m\mathbf{a} = 0 \qquad \Rightarrow \left(\mathbf{F} - m\mathbf{a}\right)\cdot d\mathbf{s} = 0$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt}\left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma}\right) - m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}\right)\delta q_\sigma$$

Claim: $\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\dfrac{d}{dt}\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial}{\partial q_\sigma}\dfrac{dx_i}{dt} \equiv \dfrac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma}\right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial q_\sigma}\right)\delta q_\sigma$$

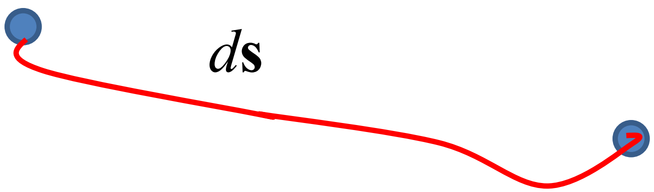
Some details

$$\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} = \frac{d\dot{x}_i}{dt}\frac{\partial x_i}{\partial q_\sigma} = \frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - \dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}$$

You may be still wondering why we need to introduce "generalized" coordinates when cartesian coordinates are an example. What the generalized coordinates allow us to show is that

$$m\mathbf{a} \cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma$$

where $T \equiv \sum_i \frac{1}{2} m \dot{x}_i^{\,2}$ (kinetic energy)



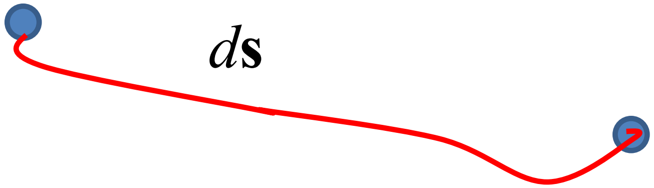
$$x_i = x_i\left(\{q_\sigma(t)\}, t\right)$$

Claim: $\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$

Details: $\dot{x}_i = \sum_\sigma \dfrac{\partial x_i}{\partial q_\sigma}\dot{q}_\sigma + \dfrac{\partial x_i}{\partial t}$ Therefore: $\dfrac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \dfrac{\partial x_i}{\partial q_\sigma}$

Claim: $\dfrac{d}{dt}\dfrac{\partial x_i}{\partial q_\sigma} = \dfrac{\partial}{\partial q_\sigma}\dfrac{dx_i}{dt} \equiv \dfrac{\partial \dot{x}_i}{\partial q_\sigma}$

$$\sum_{\sigma'}\frac{\partial^2 x_i}{\partial q_{\sigma'}\partial q_\sigma}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial t \partial q_\sigma} \qquad \sum_{\sigma'}\frac{\partial^2 x_i}{\partial q_\sigma \partial q_{\sigma'}}\dot{q}_{\sigma'} + \frac{\partial^2 x_i}{\partial q_\sigma \partial t}$$



Generalized coordinates:

$$q_\sigma\left(\{x_i\}\right)$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2} m\dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial\left(\frac{1}{2} m\dot{x}_i^{\,2}\right)}{\partial q_\sigma} \right)\delta q_\sigma$$

$$\text{Define -- kinetic energy:} \quad T \equiv \sum_i \tfrac{1}{2} m\dot{x}_i^{\,2}$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma$$

Recall:

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

$$\left(\mathbf{F} - m\mathbf{a}\right)\cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial \dot{q}_\sigma} \right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i^{\,2}\right)}{\partial q_\sigma} \right)\delta q_\sigma$$

Define -- kinetic energy: $\quad T \equiv \sum_i \frac{1}{2}m\dot{x}_i^{\,2}$

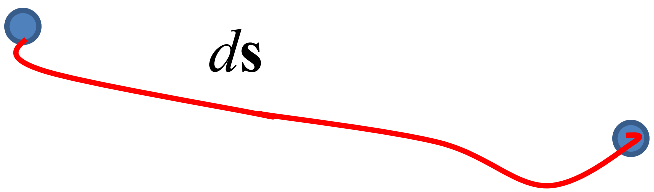
$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma$$

When do we need this term?

Single particle in 2 dimensions:

Cartesian coordinates: $\quad T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$

Polar coordinates: $\quad T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$



Generalized coordinates :

$q_\sigma(\{x_i\})$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

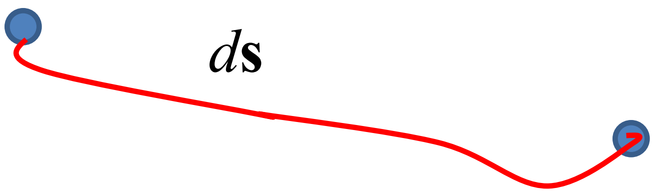
$$= -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$= -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note : This is only true if

$$\frac{\partial U}{\partial \dot{q}_\sigma} = 0$$



Generalized coordinates :

$$q_\sigma(\{x_i\})$$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t)$$

$$(\mathbf{F} - m\mathbf{a})\cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$\Rightarrow$ Minimization integral: $S = \int_{t_i}^{t_f} L(\{q_\sigma\},\{\dot{q}_\sigma\},t)\,dt$

➔Hamilton's principle from the "backwards" application of the Euler-Lagrange equations --

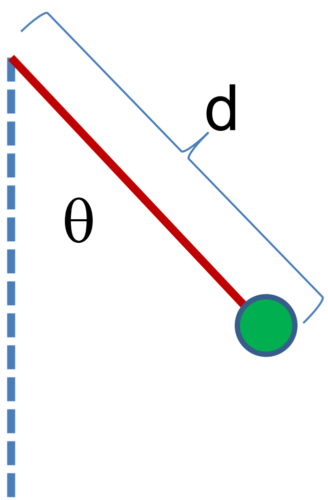
Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t)$$

$$\text{Euler–Lagrange equations:} \qquad L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) = T - U$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta,\dot{\theta}) = \tfrac{1}{2} m d^2 \dot{\theta}^2 - mg(d - d\cos\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \qquad \Rightarrow \frac{d}{dt} m d^2 \dot{\theta} + mgd\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{d}\sin\theta$$

Another example: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt}\left(I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt}\left(I_1\dot{\beta}\right) = \frac{\partial L}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}} = \frac{d}{dt}\left(I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\right) = 0$$

Example – simple harmonic oscillator

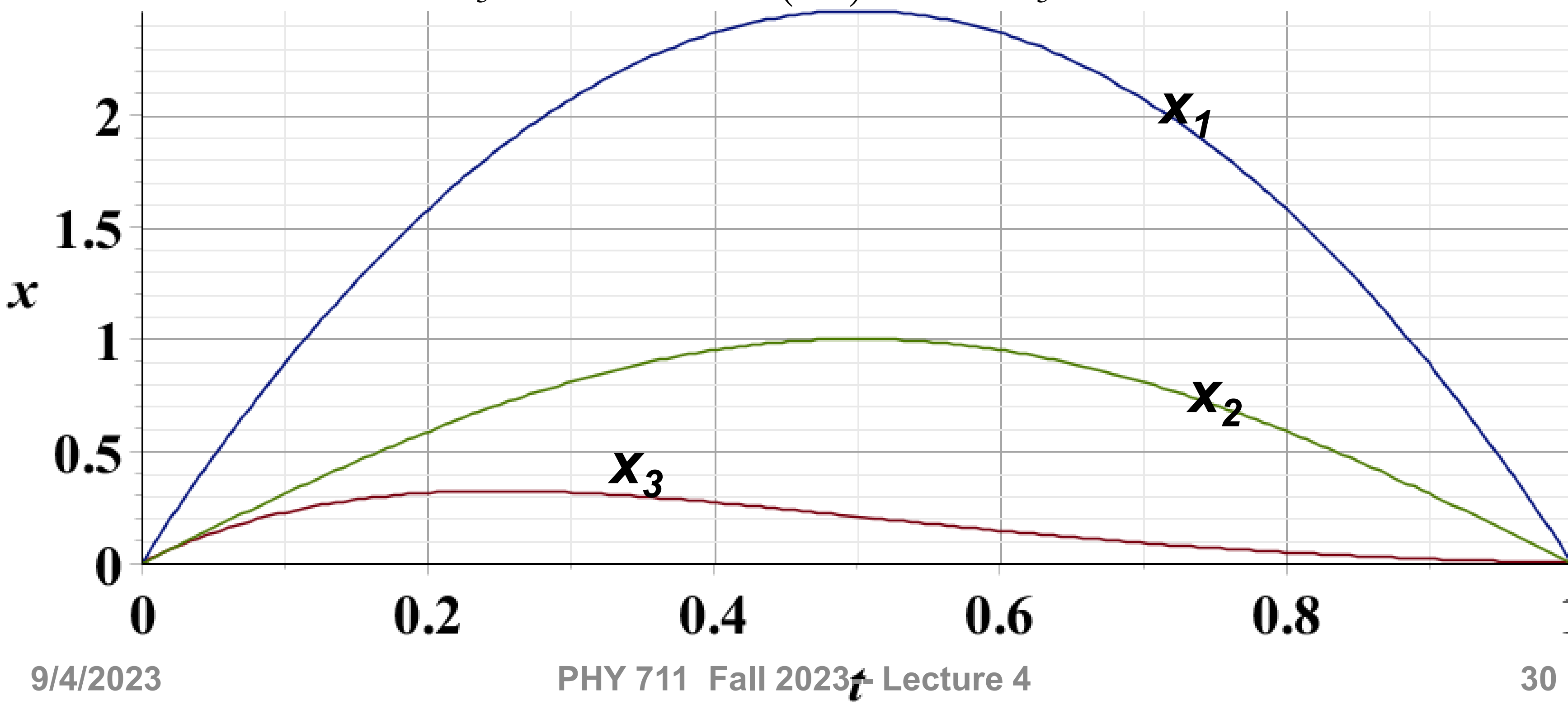
$T = \frac{1}{2} m\dot{x}^2 \qquad U = \frac{1}{2} m\omega^2 x^2$

Assume $x(0) = 0$ and $x(\frac{\pi}{\omega}) = 0 \qquad S = \frac{1}{2} m \int_0^{\pi/\omega} \left(\dot{x}^2 - \omega^2 x^2 \right) dt$

Trial functions $\quad x_1(t) = A\sin(\omega t) \qquad S_1 = 0$

$x_2(t) = A\omega t \cdot (\pi - \omega t) \qquad S_2 = 0.067 A^2 m\omega^2$

$x_3(t) = Ae^{-\omega t}\sin(\omega t) \qquad S_3 = 0.062 A^2 m\omega^2$



Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

are those which minimize the action: $S = \int L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \qquad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \quad \text{for} \quad L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

$\Rightarrow$ How can we represent velocity-dependent forces?

Why do we need velocity dependent forces?

a. Friction is sometimes represented as a velocity dependent force. (difficult to treat with Lagrangian mechanics.)
b. Lorentz force on a moving charged particle in the presence of a magnetic field.

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$

x − component: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Note: Here we are using cartesian coordinates for convenience.

x-component: $\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0$

Apparently: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

Answer: $U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$ $\quad$ $\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$