PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion of Lecture 5 – Chap. 3&6 in F&W

Lagrangian mechanics

1. Lagrange's equations in the presence of velocity dependent potentials – such as electromagnetic interactions.
2. Effects of constraints

Physics Colloquium Series

The originally scheduled colloquium for this week has been rescheduled for December 7, 2023

In order to keep up the departmental good spirits, please join **Physics Reception in the Olin Lobby at 3:30 PM**

WAKE FOREST UNIVERSITY

September 7, 2023

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W	Topic	HW
1	Mon, 8/28/2023		Introduction and overview	#1
2	Wed, 8/30/2023	Chap. 3(17)	Calculus of variation	#2
3	Fri, 9/01/2023	Chap. 3(17)	Calculus of variation	#3
4	Mon, 9/04/2023	Chap. 3	Lagrangian equations of motion	#4
5	Wed, 9/06/2023	Chap. 3 & 6	Lagrangian equations of motion	#5
6	Fri, 9/08/2023			
7	Mon, 9/11/2023			
8	Wed, 9/13/2023			

PHY 711 -- Assignment #5

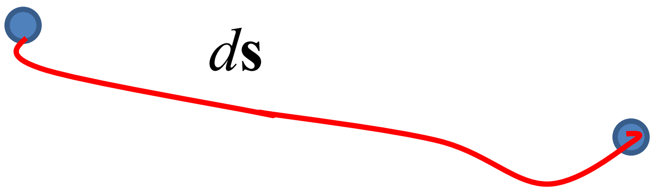
Assigned: 9/06/2023 Due: 9/11/2023

Continue reading Chapters 3 & 6 in **Fetter & Walecka**.

In class, we discussed two different examples of a time-independent vector potential **A**(**r**) that describes a constant magnetic field of magnitude B_0 directed along the **z** axis.

a. Find a third form for a vector potential **A**(**r**) for the same magnetic field.
b. Now write down the Lagrangian for this system and check whether you obtain a consistent trajectory for a point particle of mass *m* compared with the results discussed in class.

Previously derived form for the Lagrangian --



Generalized coordinates :

$q_\sigma(\{x_i\})$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

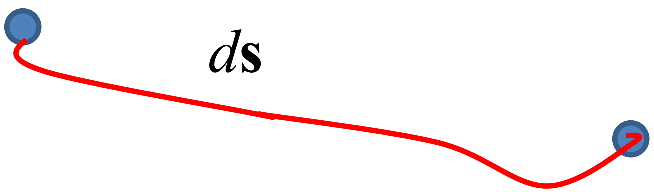
$$= \; -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$= \; -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note : This is only true if

$$\frac{\partial U}{\partial \dot{q}_\sigma} = 0$$



Generalized coordinates :

$$q_\sigma(\{x_i\})$$

Define -- Lagrangian : $\quad L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$(\mathbf{F}\text{-}m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$\Rightarrow$ Minimization integral: $\quad S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)\, dt$

➔Hamilton's principle from the "backwards" application of the Euler-Lagrange equations to

Define -- Lagrangian: $\quad L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

are those which minimize the action: $S = \int L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \qquad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \quad \text{for} \quad L = L\left(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\right) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

$\Rightarrow$ How can we represent velocity-dependent forces?

Why do we need velocity dependent forces?

a. Friction is sometimes represented as a velocity dependent force. (difficult to treat with Lagrangian mechanics.)
b. Lorentz force on a moving charged particle in the presence of a magnetic field.

Some details --

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right) = 0 \quad \text{for} \quad L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U$$

It was necessary to assume that:

$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

This comes from D'Alembert's analysis which gave us:

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = 0 = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma$$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = 0 = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma}\right)\delta q_\sigma$$

while we want to use: $0 = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma}\right)\delta q_\sigma$

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$

x − component: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Note: Here we are using cartesian coordinates for convenience.

x-component: $\left(\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x}\right) = 0$

Apparently: $F_x = -\dfrac{\partial U}{\partial x} + \dfrac{d}{dt}\dfrac{\partial U}{\partial \dot{x}}$

Answer: $U = q\Phi(\mathbf{r},t) - \dfrac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \dfrac{1}{c}\dfrac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$ $\qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$

More details --

Consider: $$0 = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial(T-U)}{\partial \dot{q}_\sigma} - \frac{\partial(T-U)}{\partial q_\sigma} \right)\delta q_\sigma$$

Suppose $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\Rightarrow 0 = \left(\frac{d}{dt}\frac{\partial(T-U)}{\partial \dot{x}} - \frac{\partial(T-U)}{\partial x} \right) = \frac{d}{dt}m\dot{x} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}} \right) + \frac{\partial U}{\partial x}$$

$$\Rightarrow m\ddot{x} = F_x = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}} \right) - \frac{\partial U}{\partial x}$$

Units for electromagnetic fields and forces

cgs Gaussian units -- (as used your textbook)

E and **B** fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

SI units --

E and **B** fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r},t) - q\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Lorentz forces, continued:

$$x-\text{component of Lorentz force: } F_x = q\left(E_x + \tfrac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$$

$$\text{Suppose:} \qquad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

$$\text{Consider:} \quad F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Some details on last step:

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

Note that: $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$ $\qquad$ $\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$

So that:

$$F_x(\mathbf{r},t) \quad = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$ $\qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t)$$

Example Lorentz force

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Suppose $\mathbf{E}(\mathbf{r},t) \equiv 0, \quad \mathbf{B}(\mathbf{r},t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r},t) = \tfrac{1}{2}B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c}B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained
from direct application of Newton's laws:

$$m\ddot{\mathbf{r}} = \frac{q}{c} \dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

Example Lorentz force -- continued

Evaluation of equations :

$$m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0 \qquad \dot{x}(t) = V_0 \sin\left(\frac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0 \qquad \dot{y}(t) = V_0 \cos\left(\frac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{z} = 0 \qquad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \frac{mc}{qB_0}V_0 \cos\left(\frac{qB_0}{mc}t + \phi\right)$$

$$y(t) = y_0 + \frac{mc}{qB_0}V_0 \sin\left(\frac{qB_0}{mc}t + \phi\right)$$

$$z(t) = z_0 + V_{0z}t$$

Example Lorentz force -- continued

Consider formulation with different Gauge: $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{c} B_0 y \right) = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt}(m\dot{y}) + \frac{q}{c} B_0 \dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Does it surprise you that the same equations of motion are obtained with a different Gauge?

How do these two different forms of **A** correspond to the same **B**?

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t)$$

Consider $\quad \mathbf{A}'(\mathbf{r},t)=\mathbf{A}(\mathbf{r},t)+\nabla f(\mathbf{r},t)$

Note that $\nabla\times\mathbf{A}(\mathbf{r},t)=\nabla\times\mathbf{A}'(\mathbf{r},t)$

In our case, $\mathbf{A}(\mathbf{r},t)=\frac{1}{2}B_0\left(-y\hat{\mathbf{x}}+x\hat{\mathbf{y}}\right)$

$$\mathbf{A}'(\mathbf{r},t)=-B_0 y\hat{\mathbf{x}}$$

What is $f(\mathbf{r},t)$?

Now consider formulation of motion with constraints --

Comments on generalized coordinates:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

Lagrange multipliers

$$\text{Lagrangian}: \quad L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\text{Constraints}: \quad f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$$

$$\text{Modified Euler-Lagrange equations}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

Some details --

Lagrangian: $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

Constraints: $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

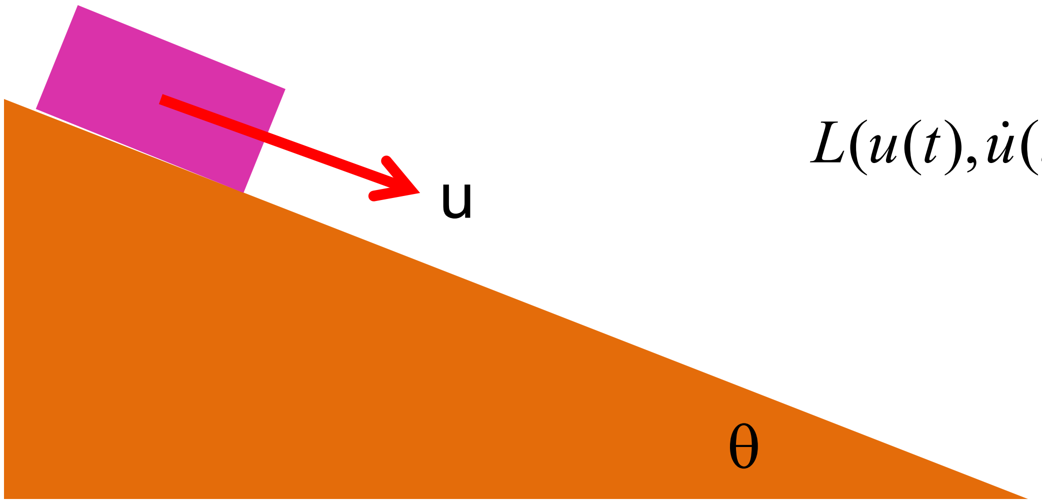
Modified Euler - Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

This amounts to modifying our optimization problem --

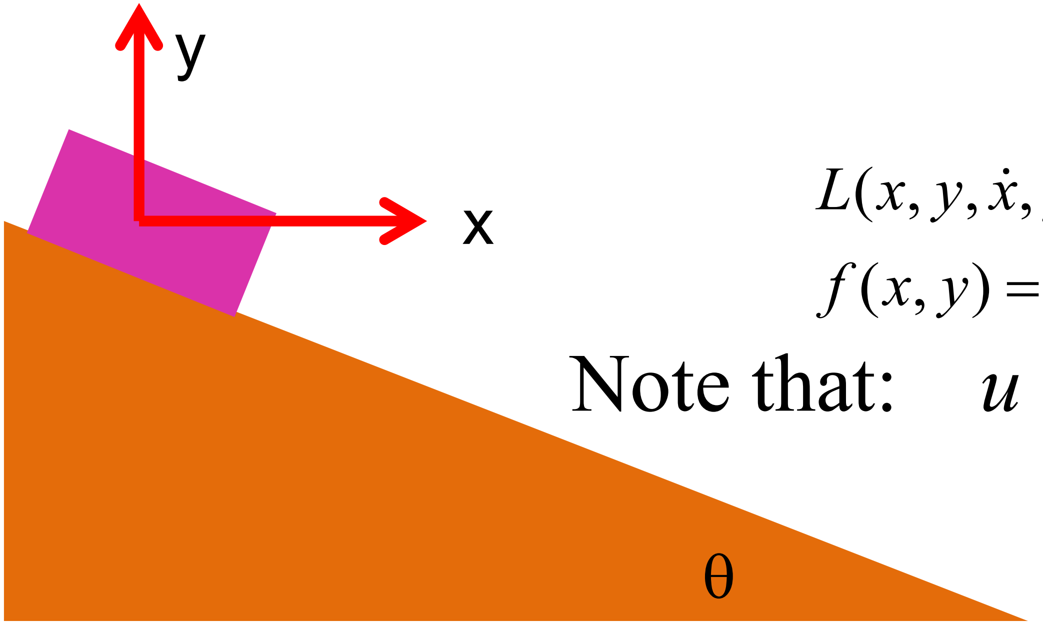
$\delta S = 0$ and for each i: $\delta f_i = 0$

$\Rightarrow \delta W \equiv \delta(S + \sum_i \lambda_i f_i) = 0,$ introducing the new constants λ_i.

Simple example:



$$L(u(t),\dot{u}(t)) = \tfrac{1}{2} m\dot{u}^2 + mgu\sin\theta$$



$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2} m\left(\dot{x}^2+\dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\; x + \cos\theta\; y = 0$$

Note that: $u = x\cos\theta - y\sin\theta$

Case 1:

$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0$$

$$\Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\ x + \cos\theta\ y = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$

$$\sin\theta\ \ddot{x} + \cos\theta\ \ddot{y} = 0$$

$$\Rightarrow \lambda = -mg\cos\theta$$

$$\left(\cos\theta\ \ddot{x} - \sin\theta\ \ddot{y}\right) = g\sin\theta$$

Which method would you use to solve the problem?

Case 1

Case 2

Force of constraint; normal to incline

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right) dt$$

$$\delta S = 0 = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma \right) dt$$

With constraints: $f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma$$

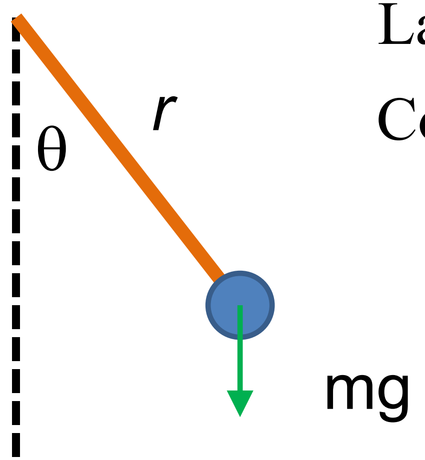
Euler-Lagrange equations with constraints:

$$\text{Lagrangian}: \quad L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\text{Constraints}: \quad f_j = f_j\left(\{q_\sigma(t)\}, t\right) = 0$$

$$\text{Modified Euler-Lagrange equations}: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

Example:



$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

$$\text{Constraints}: \quad f = r - \ell = 0$$

Example continued:

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

$$\text{Constraints}: \quad f = r - \ell = 0$$

$$\frac{d}{dt} m\dot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$\frac{d}{dt} mr^2\dot{\theta} + mgr\sin\theta = 0$$

$$\dot{r} = 0 = \ddot{r} \qquad\qquad r = \ell$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

$$\Rightarrow \quad \lambda = m\ell\dot{\theta}^2 + mg\cos\theta$$

Another example:

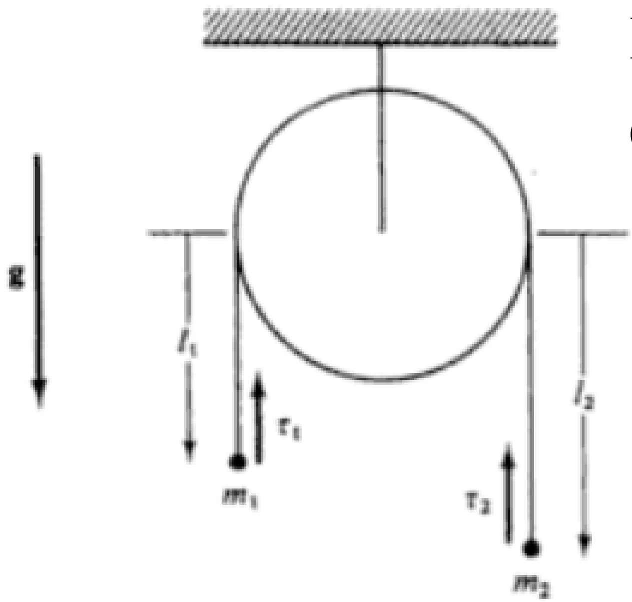


Figure 19.1 Atwood's machine.

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m_1 \dot{\ell}_1^2 + \tfrac{1}{2} m_2 \dot{\ell}_2^2 + m_1 g \ell_1 + m_2 g \ell_2$$

$$\text{Constraints}: \quad f = \ell_1 + \ell_2 - \ell = 0$$

$$\frac{d}{dt} m_1 \dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2 \dot{\ell}_2 - m_2 g + \lambda = 0$$

$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2 m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$